**Tips on writing up induction proofs**

Begin any induction proof by stating precisely, and prominently, the statement (“P(n)”) you plan to prove. A good idea is to put the statement in a display and label it, so that it is easy to spot, and easy to reference; see the sample proofs for examples.

**Induction variable:** $n$ versus $k$. Use the letter $n$ for the variable in the statement of the formula (or proposition, etc.) that you seek to prove (and which, as pointed out above, you should have repeat at the beginning of every induction proof). Use $k$ (or some other letter) for the variable appearing in the induction step. The reason for this distinction is that one can then say something like the following: “Let $k \in \mathbb{Z}_+$ be given, and suppose (1) is true for $n = k$. .... [Proof of induction step goes here] ... Therefore (1) is true for $n = k + 1$.”

**The role of the induction hypothesis:** The induction hypothesis is the case $n = k$ of the statement we seek to prove (“P(k”), and it is what you assume at the start of the induction step. You must get this hypothesis into play at some point during the proof of the induction step—if not, you are doing something wrong. The place where this hypothesis is used is the most crucial step in any induction argument, and you should clearly state, at the appropriate place, when you are using the induction hypothesis (e.g., “By the induction hypothesis we have ...”, or as a parenthetical note “(by induction hypothesis)” in a chain of equations).

**Sample induction proof**

Here is a complete proof of the formula for the sum of the first $n$ integers, that can serve as a model for proofs of similar sum/product formulas.

<table>
<thead>
<tr>
<th>We will prove by induction that, for all $n \in \mathbb{Z}_+$,</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.</td>
</tr>
</tbody>
</table>

**Base case:** When $n = 1$, the left side of (1) is 1, and the right side is $1(1+1)/2 = 1$, so both sides are equal and (1) is true for $n = 1$.

**Induction step:** Let $k \in \mathbb{Z}_+$ be given and suppose (1) is true for $n = k$. Then

$$
\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k + 1) = \frac{k(k+1)}{2} + (k + 1) \quad \text{(by induction hypothesis)}
$$

$$
= \frac{k(k+1) + 2(k+1)}{2} \quad \text{(by algebra)}
$$

$$
= \frac{(k+1)((k+1)+1)}{2} \quad \text{(by algebra)}.
$$

Thus, (1) holds for $n = k + 1$, and the proof of the induction step is complete.

**Conclusion:** By the principle of induction, (1) is true for all $n \in \mathbb{Z}_+$. 

Practice problems: Induction proofs

1. **Induction proofs, type I: Sum/product formulas:** The most common, and the easiest, application of induction is to prove formulas for sums or products of $n$ terms. All of these proofs follow the same pattern. If the formula to prove is not given in the problem, it can usually discovered by evaluating the first few cases.

(a) $\sum_{i=1}^{n} i(i + 1) = \frac{n(n+1)(n+2)}{3}$

(b) $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ (sum of powers of 2)

(c) $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ ($r \neq 1$) (sum of finite geometric series)

(d) $\sum_{i=0}^{n} i!i = (n + 1)! - 1.$

(e) Find a formula for $\sum_{i=1}^{n} \frac{1}{i(i+1)}$, and prove this formula by induction.

(f) Find a formula for $\prod_{i=2}^{n} (1 - \frac{1}{i})$ for $n \geq 2$, and prove this formula by induction.

2. **Induction proofs, type II: Inequalities:** A second general type of application of induction is to prove inequalities involving a natural number $n$. These proofs also tend to be on the routine side; in fact, the algebra required is usually very minimal, in contrast to some of the summation formulas.

In some cases the inequalities don’t “kick in” until $n$ is large enough. By checking the first few values of $n$ one can usually quickly determine the first $n$-value, say $n_0$, for which the inequality holds. Induction with $n = n_0$ as base case can then be used to show that the inequality holds for all $n > n_0$.

(a) $2^n > n$

(b) $2^n \geq n^2$ ($n \geq 4$)

(c) $n! > 2^n$ ($n \geq 4$)

(d) $(1 - x)^n \geq 1 - nx$ ($0 < x < 1$)

(e) $(1 + x)^n \geq 1 + nx$ ($x > 0$)