FOUNDATIONS OF NONSTANDARD ANALYSIS

A Gentle Introduction to Nonstandard Extensions

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1. Introduction

There are many introductions to nonstandard analysis, (some of which are listed in the References) so why write another one? All of the existing introductions have one or more of the following features: (A) heavy use of logical formalism right from the start; (B) early introduction of set theoretic apparatus in excess of what is needed for most applications; (C) dependence on an explicit construction of the nonstandard model, usually by means of the ultrapower construction.

All three of these features have negative consequences. The early use of logical formalism or set theoretic structures is often uncomfortable for mathematicians who do not have a background in logic, and this can effectively deter them from using nonstandard methods. The explicit use of a particular nonstandard model makes the foundations too specific and inflexible, and often inhibits the free use of the ideas of nonstandard analysis. In this exposition we intend to avoid these disadvantages. The readers for whom we have written are experienced mathematicians (including advanced students) who do not necessarily have any background in or even tolerance for symbolic logic. We hope to convince such readers that nonstandard methods are accessible and that the small amount of logical notation which turns out to be useful in applying them is actually simple and natural. Of course readers who do have a background in logic may also find this approach useful.

We give a natural, geometric definition of nonstandard extension in Section 2; no logical formulas are used and there are no set theoretic structures. In Section 3 we introduce logical notation of the kind that all students of mathematics encounter, and we carefully show how it can be used without difficulty to obtain useful facts about nonstandard extensions. In Section 4 we extend the concept of nonstandard extension to mathematical settings in which there may be several basic sets (such as the vector space setting, where there is a field F and a vector space V). In Sections 5 and 6 we show how these ideas can be used to introduce nonstandard extensions in which sets and other objects of higher type can be handled, as is certainly necessary for applications of nonstandard methods in such areas as abstract analysis and topology. However, we do this in stages; in particular, in Section 5 we indicate how to deal with nonstandard extensions in a simple setting where a limited amount of set theoretic apparatus has been introduced. Such limited frameworks for nonstandard analysis are nonetheless adequate for essentially all applications. Section 6 treats the full superstructure apparatus which has become one of the standard ways of formulating nonstandard analysis and which is frequently used in the literature. In Section 7 we briefly discuss saturation properties of nonstandard extensions.

In several places we introduce specific nonstandard extensions using the ultraproduct construction, and we explore the meaning of certain basic concepts (such as *internal set*) in these concrete settings. (See the last parts of Sections 2, 4, 5, and 7.) Our experience shows that it is often helpful at the beginning to have such explicit nonstandard extensions at hand. As noted above, however, we think it is limiting to become dependent on such a construction and we encourage readers to adopt the more flexible axiomatic approach as quickly as possible.

In writing this exposition we have benefitted greatly from conversations with Lou van den Dries about the best ways to present ideas from model theory to the general mathematical public. His ideas are presented in [5] and, with Chris Miller, in [6], and our treatment obviously depends heavily on that work. We have also freely used many ideas from other expositions of nonstandard analysis (listed among the References) and from the other Chapters in this book. To all these authors we express our sincere appreciation, and we recommend their writings to the reader who finishes this exposition with a desire to learn more about how nonstandard methods can be applied.

2. Nonstandard Extensions

The starting point of nonstandard analysis is the construction and use of an ordered field ${}^*\mathbb{R}$ which is a proper extension of the usual ordered field $\mathbb{R}$ of real numbers, and which satisfies *all of the properties of* $\mathbb{R}$ (in a sense that we will soon make precise). We refer to ${}^*\mathbb{R}$ as a field of *nonstandard real numbers*, or as a field of *hyperreal numbers*. Because the ordered field

$\mathbb{R}$ is Dedekind complete, it follows that the extension field ${}^*\mathbb{R}$ will necessarily have among its new elements both infinitesimal and infinite numbers. These new numbers play a fundamental role in nonstandard analysis, which was created by Abraham Robinson [14] in order to make reasoning with infinitesimals rigorous. (An element a of ${}^*\mathbb{R}$ is *finite* if there exists $r \in \mathbb{R}$ such that $-r \leq a \leq r$ in ${}^*\mathbb{R}$; otherwise a is *infinite*; a is *infinitesimal* if $-r \leq a \leq r$ holds for every positive $r \in \mathbb{R}$. In some places a finite number is called *limited*.)

It is easily seen that a proper extension field ${}^*\mathbb{R}$ of $\mathbb{R}$ cannot satisfy literally *all* the properties of $\mathbb{R}$. For example it cannot be Dedekind complete. (The set of finite numbers in ${}^*\mathbb{R}$ cannot have a least upper bound s, because then $s-1$ would be a smaller upper bound.) The challenge was to establish a clear and consistent foundation for reasoning with infinitesimals, that captured the known heuristic arguments as much as possible. This was accomplished by Abraham Robinson in the 1960s. The purpose of this paper is to describe the essential features of the resulting frameworks without getting bogged down in technicalities of formal logic and without becoming dependent on an explicit construction of a specific field ${}^*\mathbb{R}$.

We usually think of $\mathbb{R}$ as being equipped with additional structure, in the form of distinguished sets, relations, and functions; we include whatever objects play a role in the mathematical problems we are considering. For example, these will normally include the set $\mathbb{N}$ of natural numbers (non-negative integers) and often such functions as the sine, the cosine, exponentiation, and the like. When we say (as we did above) that ${}^*\mathbb{R}$ *satisfies all of the properties of* $\mathbb{R}$, we mean (in part) that each of these additional sets, relations, and functions on $\mathbb{R}$ will have a counterpart on ${}^*\mathbb{R}$, and that the entire system of counterpart objects will satisfy an appropriate set of conditions. For example, if we are thinking of $\mathbb{N}$ as one of the given sets, then ${}^*\mathbb{R}$ contains a discrete set ${}^*\mathbb{N}$ which is the counterpart to $\mathbb{N}$. The conditions that we impose on ${}^*\mathbb{R}$ and ${}^*\mathbb{N}$ will imply that $\mathbb{N}$ is an initial segment of ${}^*\mathbb{N}$, and that the elements of ${}^*\mathbb{N} \setminus \mathbb{N}$ are infinite numbers in ${}^*\mathbb{R}$. Moreover, for every positive number r in ${}^*\mathbb{R}$ there will be a unique $N \in {}^*\mathbb{N}$ which is the hyperinteger part of r, in the sense that $N \leq r < N+1$. We will refer to ${}^*\mathbb{N}$ as the set of *nonstandard natural numbers* or as the set of *hypernatural numbers*.

The presence of infinitesimal and infinite numbers allows us to give elegant and useful characterizations of many important mathematical concepts, and this phenomenon is the basis for a large part of the impact of nonstandard analysis. For example, suppose $(s_n)_{n\in\mathbb{N}}$ is a sequence of real numbers and t is a real number. Then one can prove the following characterization of the limit concept:

$$s_n \to t \text{ as } n \to \infty \iff s_N \approx t \text{ for all infinite } N \in {}^*\mathbb{N}.$$

(For any two numbers $s, t \in {}^*\mathbb{R}$ we write $s \approx t$ to mean that the difference $s - t$ is infinitesimal.) Using such a characterization allows many heuristic arguments about limits to be made precise; for example, it becomes easy to give elementary algebraic proofs of the basic facts about the algebra of limits.

Note that in the characterization of the limit condition given above, we used the expression s_N where N was an element of ${}^*\mathbb{N}$. This requires explanation, since s_n was originally given only for $n \in \mathbb{N}$. We think of the sequence as a function $s\colon \mathbb{N} \to \mathbb{R}$ and we regard this function as part of the basic apparatus with which $\mathbb{R}$ is initially equipped. Therefore, it has a counterpart on ${}^*\mathbb{R}$, which will be a function defined on ${}^*\mathbb{N}$ and having values in ${}^*\mathbb{R}$. It is this function that we have in mind when we write s_N for $N \in {}^*\mathbb{N}$.

We are now ready to give a formal description of the properties we require our nonstandard real field ${}^*\mathbb{R}$ to satisfy. For the moment we will only consider *first order* structure on $\mathbb{R}$. Therefore we will not yet be considering the higher order objects of analysis, such as measures, Banach spaces, and the like. We start out in this limited way for pedagogical reasons, to make the task of mastering the fundamental language of nonstandard analysis easier for beginners. (Later on, in Section 5 and especially in Section 6, we will add the machinery of higher type objects which is needed for the full range of arguments in nonstandard analysis.)

We consider $\mathbb{R}$ as being equipped with *all possible* first order properties (*i.e.* sets and relations) and functions. We do this in order to have a foundation which is as flexible as possible and which provides any object we might need later in handling specific mathematical problems. In order to make our basic definition simpler technically, we handle functions by means of their graphs. Therefore, we take the point of view that our additional structure on $\mathbb{R}$ consists of the collection of all possible subsets of every Cartesian power $\mathbb{R}^n$, as n ranges over the integers ≥ 0.

Next we give the key definition. In it we give a precise description of the properties that must be preserved by passage to the nonstandard extension. The requirements are simple and natural, and they have a distinctly geometric character. (Strictly speaking we are defining here a *first order* concept of nonstandard extension; the definition will be suitably modified below when we add higher order objects to our setting.)

2.1. Definition. [Nonstandard Extension of a Set] *Let $\mathbb{X}$ be a non-empty set. A* **nonstandard extension of** $\mathbb{X}$ *consists of a mapping that assigns a set ${}^*A$ to each $A \subseteq \mathbb{X}^m$ for all $m \geq 0$, such that ${}^*\mathbb{X}$ is non-empty and the following conditions are satisfied for all $m, n \geq 0$:*

(E1) The mapping preserves Boolean operations on subsets of $\mathbb{X}^m$:

if $A \subseteq \mathbb{X}^m$*, then* ${}^*A \subseteq ({}^*\mathbb{X})^m$*; if* $A, B \subseteq \mathbb{X}^m$*, then* ${}^*(A \cap B) = ({}^*A \cap {}^*B)$*,* ${}^*(A \cup B) = ({}^*A \cup {}^*B)$*, and* ${}^*(A \setminus B) = ({}^*A) \setminus ({}^*B)$*.*

(E2) The mapping preserves basic diagonals:
if $1 \leq i < j \leq m$ *and* $\Delta = \{(x_1, \ldots, x_m) \in \mathbb{X}^m \mid x_i = x_j\}$ *then* ${}^*\Delta = \{(x_1, \ldots, x_m) \in ({}^*\mathbb{X})^m \mid x_i = x_j\}$*.*

(E3) The mapping preserves Cartesian products:
if $A \subseteq \mathbb{X}^m$ *and* $B \subseteq \mathbb{X}^n$*, then* ${}^*(A \times B) = {}^*A \times {}^*B$*. (We regard* $A \times B$ *as a subset of* $\mathbb{X}^{m+n}$*.)*

(E4) The mapping preserves projections that omit the final coordinate: let π *denote projection of* $n+1$*-tuples on the first* n *coordinates; if* $A \subseteq \mathbb{X}^{n+1}$ *then* ${}^*(\pi(A)) = \pi({}^*A)$*.*

While this definition is reasonably elegant and can be comprehended rather easily, there is certainly some work to be done before we can exploit it. For example, suppose we have a nonstandard extension of $\mathbb{R}$. How do we prove that the subset ${}^*\mathbb{N}$ of ${}^*\mathbb{R}$ has the properties that were claimed above? (Namely, that $\mathbb{N}$ is an initial segment of ${}^*\mathbb{N}$, that the elements of ${}^*\mathbb{N} \setminus \mathbb{N}$ are infinite numbers in ${}^*\mathbb{R}$, and that for every positive number r in ${}^*\mathbb{R}$ there is a unique $N \in {}^*\mathbb{N}$ which satisfies $N \leq r < N+1$.) Moreover, ${}^*\mathbb{R}$ is supposed to be an ordered field extension of $\mathbb{R}$, and even this does not seem to be directly guaranteed by the conditions in the definition.

We first turn to a series of elementary arguments which prove some of the most basic properties of nonstandard extensions, especially those having to do with the handling of functions. Not only are the results important, but the arguments illustrate how one can derive information from conditions (E1) – (E4). Near the end of the Section we continue this theme by means of a set of Exercises.

2.2. Proposition. *For each* $n \geq 0$*,* ${}^*(\mathbb{X}^n) = ({}^*\mathbb{X})^n$ *and* ${}^*\emptyset = \emptyset$*.*

Proof. The first equation follows from repeated use of (E3) and the second equation follows from (E1); note that ${}^*\emptyset = {}^*(\emptyset \setminus \emptyset) = {}^*\emptyset \setminus {}^*\emptyset = \emptyset$. □

2.3. Proposition. *If* $A \subseteq \mathbb{X}^m$ *is non-empty, then* ${}^*A$ *is also non-empty. Therefore, for any* $A, B \subseteq \mathbb{X}^m$*,* ${}^*A = {}^*B \iff A = B$*.*

Proof. For ease of notation we consider only the case $m = 2$. Let π_2 and π_3 be the projections defined by $\pi_2(x, y) = x$ and $\pi_3(x, y, z) = (x, y)$ respectively. If $A \subseteq \mathbb{X}^2$ is non-empty, then $\mathbb{X} = \pi_2(\pi_3(\mathbb{X} \times A))$. Using (E4) we get ${}^*\mathbb{X} = \pi_2(\pi_3({}^*\mathbb{X} \times {}^*A))$. Since ${}^*\mathbb{X}$ is non-empty, it follows that ${}^*A$ must also be non-empty. The second statement follows from the first and (E1). □

2.4. Proposition. *For all* $A, B \subseteq \mathbb{X}^m$*,* $A \subseteq B \iff {}^*A \subseteq {}^*B$*.*

Proof. Suppose $A \subseteq B$. Then $A = A \cap B$, so by (E1) we have ${}^*A = {}^*(A \cap B) = {}^*A \cap {}^*B$ and hence ${}^*A \subseteq {}^*B$. The reverse implication follows by a similar argument and Proposition 2.3. □

2.5. Proposition. *For each $x \in \mathbb{X}$, ${}^*\{x\}$ has exactly one element.*

Proof. By Proposition 2.3, ${}^*\{x\}$ has at least one element. Let $\Delta = \{(u, u) \mid u \in \mathbb{X}\}$, and note that $\{x\} \times \{x\} = \{(x, x)\} \subseteq \Delta$. Using (E3) and (E2) we get ${}^*\{x\} \times {}^*\{x\} \subseteq {}^*\Delta = \{(u, u) \mid u \in {}^*\mathbb{X}\}$, from which it follows that ${}^*\{x\}$ has exactly one element. □

Propositions 2.3 and 2.5 allow us to introduce an embedding of $\mathbb{X}$ into ${}^*\mathbb{X}$ which is canonically associated with the given nonstandard extension. After introducing this embedding, we prove that it is fully compatible with the operation of forming n-tuples, and hence with Cartesian products.

2.6. Notation. For each $x \in \mathbb{X}$, we let ${}^*x$ denote the unique element of the set ${}^*\{x\}$. For each $x = (x_1, \ldots, x_n) \in \mathbb{X}^n$ we let ${}^*x = ({}^*x_1, \ldots, {}^*x_n)$. Note that this gives two usages for an expression of the form ${}^*\beta$; if β is an element of $\mathbb{X}$, then ${}^*\beta$ is defined in this paragraph, while if β is a subset of some Cartesian power $\mathbb{X}^m$, then ${}^*\beta$ is the subset of $({}^*\mathbb{X})^m$ which is provided by the given nonstandard extension.

2.7. Definition. *An element of $({}^*\mathbb{X})^n$ is called* **standard** *if it is of the form ${}^*x$ for some $x \in \mathbb{X}^n$. It follows that an element of $({}^*\mathbb{X})^n$ is standard if and only if all of its coordinates are standard elements of ${}^*\mathbb{X}$.*

2.8. Proposition. *For each $x_1, \ldots, x_n \in \mathbb{X}$,*

$${}^*\{(x_1, \ldots, x_n)\} = \{({}^*x_1, \ldots {}^*x_n)\}.$$

Proof. ${}^*\{(x_1, \ldots, x_n)\} = {}^*(\{x_1\} \times \cdots \times \{x_n\}) = {}^*\{x_1\} \times \cdots \times {}^*\{x_n\} = \{{}^*x_1\} \times \cdots \times \{{}^*x_n\} = \{({}^*x_1, \ldots, {}^*x_n)\}$.

2.9. Proposition. *For each $A \subseteq \mathbb{X}^m$ and $x_1, \ldots, x_m \in \mathbb{X}$,*

$$(x_1, \ldots, x_m) \in A \iff ({}^*x_1, \ldots, {}^*x_m) \in {}^*A.$$

Proof. Using Propositions 2.4 and 2.8 note that $(x_1, \ldots, x_m) \in A \iff \{(x_1, \ldots, x_m)\} \subseteq A \iff {}^*\{(x_1, \ldots, x_m)\} \subseteq {}^*A \iff \{({}^*x_1, \ldots, {}^*x_m)\} \subseteq {}^*A \iff ({}^*x_1, \ldots, {}^*x_m) \in {}^*A$. □

2.10. Remark. The map taking $x \in \mathbb{X}$ to ${}^*x$ is an embedding of $\mathbb{X}$ into ${}^*\mathbb{X}$. Without loss of generality we may assume that $\mathbb{X}$ is a subset of ${}^*\mathbb{X}$ and that ${}^*x = x$ for all $x \in \mathbb{X}$. When this additional condition is satisfied, the given nonstandard extension is truly an extension mapping, in the strong sense that for all $A \subseteq \mathbb{X}^m$, $({}^*A) \cap \mathbb{X}^m = A$ (and therefore, in particular, $A \subseteq {}^*A$).

Justification. For $x, y \in \mathbb{X}$ we have: ${}^*x = {}^*y \iff {}^*\{x\} = {}^*\{y\} \iff \{x\} = \{y\} \iff x = y$, so this map is an embedding. Therefore we may follow the conventional practice of "identifying" ${}^*x$ with x for all $x \in \mathbb{X}$. The precise way to do this is to construct an isomorphic nonstandard extension as follows: let $\mathbb{Y}$ be a set and $h\colon {}^*\mathbb{X} \to \mathbb{Y}$ a bijection, chosen together so that $\mathbb{X} \subseteq \mathbb{Y}$ and $x = h({}^*x)$ for all $x \in \mathbb{X}$. For each $m \geq 0$ and each $A \subseteq \mathbb{X}^m$, let $\Theta(A)$ be the subset of $\mathbb{Y}^m$ defined by

$$\Theta(A) = \{(h(u_1), \ldots, h(u_m)) \mid (u_1, \ldots, u_m) \in {}^*A\}.$$

It is a straightforward exercise using the previous Propositions to show that the set mapping Θ is a nonstandard extension. It is easily seen that $\mathbb{X} \subseteq \mathbb{Y} = \Theta(\mathbb{X})$ and $\Theta(\{x\}) = \{x\}$ for all $x \in \mathbb{X}$, from which it follows that Θ has the extra properties we wanted to achieve. The facts given in the second sentence of this Remark follow immediately using Proposition 2.9. Note that Θ is isomorphic to the original nonstandard extension in a natural sense. □

When we established the framework of nonstandard extensions, we stated briskly that we would handle functions by means of their graphs. Now we must show that this actually works.

2.11. Proposition. *Suppose $A \subseteq \mathbb{X}^m$ and $B \subseteq \mathbb{X}^n$, and let $f\colon A \to B$ be a function; take $\Gamma \subseteq \mathbb{X}^{m+n}$ to be the graph of f. Then ${}^*\Gamma$ is the graph of a function from ${}^*A$ to ${}^*B$.*

Proof. For ease of notation we treat only the case $m = n = 1$. Let π denote the projection defined by $\pi(x, y) = x$. The key properties of Γ which reflect the fact that it is the graph of a function from A to B are the following: $\Gamma \subseteq A \times B$; $\pi(\Gamma) = A$; and

$$(\Gamma \times \Gamma) \cap \{(x, y, u, v) \in \mathbb{X}^4 \mid x = u\} \subseteq \{(x, y, u, v) \in \mathbb{X}^4 \mid y = v\}.$$

The first of these statements expresses the fact that the domain of the function is contained in A and the range is contained in B. The second statement expresses that the domain of the function is A. The third (displayed) statement expresses the fact that Γ is the graph of a function.

Using conditions (E1) – (E4) we conclude: ${}^*\Gamma \subseteq {}^*A \times {}^*B$; $\pi({}^*\Gamma) = {}^*A$; and

$$({}^*\Gamma \times {}^*\Gamma) \cap \{(x,y,u,v) \in ({}^*\mathbb{X})^4 \mid x = u\} \subseteq \{(x,y,u,v) \in ({}^*\mathbb{X})^4 \mid y = v\}.$$

From these conditions the desired statements about ${}^*\Gamma$ follow immediately. □

2.12. Notation. Suppose $A \subseteq \mathbb{X}^m$ and $B \subseteq \mathbb{X}^n$, and let $f\colon A \to B$ be a function; take $\Gamma \subseteq \mathbb{X}^{m+n}$ to be the graph of f. We denote by ${}^*f$ the function from ${}^*A$ to ${}^*B$ whose graph is ${}^*\Gamma$.

2.13. Proposition. *If f is the identity function on $A \subseteq \mathbb{X}^m$, then ${}^*f$ is the identity function on ${}^*A$.*

Proof. If f is the identity function on $A \subseteq \mathbb{X}^m$, then the graph Γ of f is given by the following definition:

$$\Gamma = \{(x_1,\dots,x_m,y_1,\dots,y_m) \in \mathbb{X}^{2m} \mid x_1 = y_1,\dots,x_m = y_m\} \cap (A \times A).$$

This set is the intersection of $A \times A$ with m diagonal subsets of $\mathbb{X}^{2m}$,

$$\Gamma = \Delta_1 \cap \dots \cap \Delta_m \cap (A \times A),$$

where for each $1 \le j \le m$ we define

$$\Delta_j = \{(x_1,\dots,x_m,y_1,\dots,y_m) \in \mathbb{X}^{2m} \mid x_j = y_j\}.$$

Therefore, using (E1) – (E3) we have

$${}^*\Gamma = \{(x_1,\dots,x_m,y_1,\dots,y_m) \in ({}^*\mathbb{X})^{2m} \mid x_1 = y_1,\dots,x_m = y_m\} \cap ({}^*A \times {}^*A).$$

Since ${}^*\Gamma$ is the graph of ${}^*f$, this proves the desired result. □

2.14. Proposition. *Suppose $A \subseteq \mathbb{X}^m$ and $B \subseteq \mathbb{X}^n$, and let $f\colon A \to B$ be a function. For all $(x_1,\dots,x_n) \in A$,*

$$({}^*f)({}^*x_1,\dots,{}^*x_n) = {}^*(f(x_1,\dots,x_n)).$$

Proof. Take $x_1,\dots,x_n \in A$ and let $y = f(x_1,\dots,x_n)$, so $(x_1,\dots,x_n,y) \in \Gamma$ where Γ is the graph of f. From Proposition 2.9 we get $({}^*x_1,\dots,{}^*x_n,{}^*y) \in {}^*\Gamma$, so that $({}^*f)({}^*x_1,\dots,{}^*x_n) = {}^*y$. □

2.15. Proposition. [Permuting and Identifying Variables] *Suppose σ is any function from $\{1,\dots,m\}$ into $\{1,\dots,n\}$. Given $A \subseteq \mathbb{X}^m$ define*

$$B = \{(x_1,\dots,x_n) \in \mathbb{X}^n \mid (x_{\sigma(1)},\dots,x_{\sigma(m)}) \in A\}.$$

Then

$${}^*B = \{(x_1,\dots,x_n) \in ({}^*\mathbb{X})^n \mid (x_{\sigma(1)},\dots,x_{\sigma(m)}) \in {}^*A\}.$$

Proof. For ease of notation we consider the case where $A \subseteq \mathbb{X}^3$ is given and $B \subseteq \mathbb{X}^2$ is defined by

$$B = \{(x, y) \in \mathbb{X}^2 \mid (y, x, y) \in A\}.$$

Introduce $C \subseteq \mathbb{X}^5$ by the definition

$$C = \{(x, y, u, v, w) \in \mathbb{X}^5 \mid u = y \wedge v = x \wedge w = y \wedge (u, v, w) \in A\}.$$

Evidently B is the result of projecting C onto the first two coordinates. Moreover, C is the intersection of three diagonal subsets of $\mathbb{X}^5$ and the set $\mathbb{X}^2 \times A$. Therefore, it follows using conditions (E1) – (E4) that

$${}^*C = \{(x, y, u, v, w) \in ({}^*\mathbb{X})^5 \mid u = y \wedge v = x \wedge w = y \wedge (u, v, w) \in {}^*A\}$$

and that ${}^*B$ is the result of projecting ${}^*C$ onto the first two coordinates. The desired result follows immediately. □

2.16. Proposition. *Condition (E4) in Definition 2.1 holds for all projections π from m-tuples to n-tuples, where $n \leq m$. (By calling π a projection we mean that there exists a sequence $1 \leq i(1) < \ldots < i(n) \leq m$ such that π is defined by*

$$\pi(x_1, \ldots, x_m) = (x_{i(1)}, \ldots, x_{i(n)}).)$$

That is, if $A \subseteq \mathbb{X}^m$, then ${}^*(\pi(A)) = \pi({}^*A)$.

Proof. Let π be as described in the statement of the Proposition. Let σ be a permutation of $\{1, \ldots, m\}$ so that $\sigma(i(j)) = j$ for all $j = 1, \ldots, n$. Let $A \subseteq \mathbb{X}^m$ be given and define

$$B = \{(x_1, \ldots, x_m) \in \mathbb{X}^m \mid (x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \in A\}.$$

It is routine to check that $\pi(A)$ is the same as the result of projecting B onto the first n coordinates. Condition (E4) (applied $m - n$ times) therefore implies that ${}^*(\pi(A))$ is the result of projecting ${}^*B$ onto the first n coordinates. Proposition 2.15 implies that

$${}^*B = \{(x_1, \ldots, x_m) \in ({}^*\mathbb{X})^m \mid (x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \in {}^*A\}.$$

Hence $\pi({}^*A)$ is the same as the result of projecting ${}^*B$ onto the first n coordinates. Therefore ${}^*(\pi(A)) = \pi({}^*A)$. □

2.17. Proposition. *Let $A \subseteq \mathbb{X}^{m+n}$ and $a = (a_1, \ldots, a_m) \in \mathbb{X}^m$. Define*

$$A(a) = \{(x_1, \ldots, x_n) \in \mathbb{X}^n \mid (a_1, \ldots, a_m, x_1, \ldots, x_n) \in A\}$$

and similarly

$$({}^*A)({}^*a) = \{(x_1, \ldots, x_n) \in ({}^*\mathbb{X})^n \mid ({}^*a_1, \ldots, {}^*a_m, x_1, \ldots, x_n) \in {}^*A\}$$

Then ${}^*(A(a)) = ({}^*A)({}^*a)$.

Proof. For ease of notation we consider only the case $m = n = 1$. Let π denote the projection defined by $\pi(x, y) = y$. Note that

$$A(a) = \pi(A \cap (\{a\} \times \mathbb{X})).$$

Therefore, using conditions (E1) and (E3), and Proposition 2.16 we have

$$^*(A(a)) = \pi(^*A \cap (^*\{a\} \times {}^*\mathbb{X}))$$

$$= \pi(^*A \cap (\{^*a\} \times {}^*\mathbb{X})) = (^*A)(^*a).$$

□

2.18. Proposition. *Suppose $A \subseteq \mathbb{X}^m, B \subseteq \mathbb{X}^n$ and $C \subseteq \mathbb{X}^p$; let $f\colon A \to B$ and $g\colon B \to C$ be functions. Then $^*(g \circ f) = (^*g) \circ (^*f)$.*

Proof. For ease of notation we treat only the case $m = n = p = 1$. Let Γ_f be the graph of f and Γ_g the graph of g, and let Γ be the graph of the composition $g \circ f$. Let π be the projection defined by letting $\pi(x, y, u, v) = (x, v)$. Consider the set $A \subseteq \mathbb{X}^4$ defined by

$$A = \{(x, y, u, v) \in \mathbb{X}^4 \mid y = u\} \cap (\Gamma_f \times \Gamma_g).$$

Evidently $\Gamma = \pi(A)$. The desired result follows immediately from Proposition 2.16. □

Next we give some Exercises which continue the themes developed above. The reader is advised to solve them, as much as possible using the methods of this Section. They will be easier to solve once the machinery of logical notation is developed, as it will be in the next Section. However, especially for readers who have no previous experience with logic, working these Exercises at this point will bring significant benefits. Most of all, such effort will cause the reader to appreciate the advantages of logical notation and to understand how simple are the few technical ideas that it embodies.

2.19. Exercise. Condition (E2) holds for all diagonal sets $\Delta \subseteq \mathbb{X}^n$. By a diagonal set we mean that there is an equivalence relation E on $\{1, \ldots, n\}$ such that $\Delta = \{(x_1, \ldots, x_n) \in \mathbb{X}^n \mid x_i = x_j \text{ whenever } iEj\}$. For every such Δ,

$$^*\Delta = \{(x_1, \ldots, x_n) \in (^*\mathbb{X})^n \mid x_i = x_j \text{ whenever } iEj\}.$$

If A is a subset of $\mathbb{X}^m$, then $\{(^*a_1, \ldots, {}^*a_m) \mid (a_1, \ldots, a_m) \in A\}$ is a subset of $^*A$, by Proposition 2.9. Indeed, $\{(^*a_1, \ldots, {}^*a_m) \mid (a_1, \ldots, a_m) \in A\}$ is precisely the set of standard elements of $^*A$. The next two Exercises explore the extent to which $\{(^*a_1, \ldots, {}^*a_m) \mid (a_1, \ldots, a_m) \in A\}$ is a *proper* subset of $^*A$.

2.20. Exercise. If A is a finite subset of $\mathbb{X}^m$, then

$$^*A = \{(^*x_1, \ldots, ^*x_m) \mid (x_1, \ldots, x_m) \in A\}.$$

In particular, if A is finite, then $^*A$ is finite and has the same cardinality as A, and all of its elements are standard.

2.21. Definition. *A nonstandard extension of* $\mathbb{X}$ *is called* **proper** *if for every infinite subset* A *of* $\mathbb{X}$, $^*A$ *contains a nonstandard element.*

2.22. Exercise. Suppose our nonstandard extension is proper. Then, for any infinite set $A \subseteq \mathbb{X}^m$, $^*A$ has a nonstandard element.

2.23. Exercise. Let $A \subseteq \mathbb{X}^m$ and suppose $f \colon A \to \mathbb{X}^n$ is a function.
(a) If $B \subseteq A$, then $^*(f(B)) = (^*f)(^*B)$.
(b) If $C \subseteq \mathbb{X}^n$, then $^*(f^{-1}(C)) = (^*f)^{-1}(^*C)$.
(c) If $B \subseteq A$, then $^*(f|B) = (^*f)|(^*B)$.

2.24. Exercise. For $j = 1, \ldots, n$ let $f_j \colon \mathbb{X}^m \to \mathbb{X}$ be a function, and let $f = (f_1, \ldots, f_n) \colon \mathbb{X}^m \to \mathbb{X}^n$ be the function with $f_1, \ldots, f_n$ as its coordinates. Then $^*f = (^*f_1, \ldots, ^*f_n)$.

2.25. Exercise. Suppose $A \subseteq \mathbb{X}^m$ and $B \subseteq \mathbb{X}^n$, and let $f \colon A \to B$ be a function.
(a) f is injective $\iff$ $^*f$ is injective.
(b) f is surjective $\iff$ $^*f$ is surjective.
(c) If f is a bijection and its inverse is g, then $^*g$ is the inverse of $^*f$.

2.26. Exercise. Consider a nonstandard extension of $\mathbb{R}$. The set $^*\mathbb{R}$ is equipped with binary functions $^*+$ and $^*\times$ and with a binary relation $^*<$. Equipped with this additional structure, $^*\mathbb{R}$ is an ordered field.

2.27. Exercise. Expand all proofs in this Section so that they are fully general and cover all cases of the results being proved.

We conclude this Section by using the ultraproduct construction to prove the existence of proper nonstandard extensions. (See Definition 2.21.)

Let J be any infinite set and let $\mathcal{U}$ be an ultrafilter on J. Consider an indexed family $(A_j \mid j \in J)$ of non-empty sets. We define the ultraproduct of the sets $(A_j \mid j \in J)$ with respect to the ultrafilter $\mathcal{U}$. This will be denoted $\Pi_{\mathcal{U}}(A_j \mid j \in J)$ or simply $\Pi_{\mathcal{U}} A_j$. To define the ultraproduct, consider the ordinary Cartesian product ΠA_j of the given family of sets; this is the set of all functions α which are defined on J and which satisfy $\alpha(j) \in A_j$ for all $j \in J$. We define a relation $\sim$ on ΠA_j by

$$\alpha \sim \beta \iff \{j \in J \mid \alpha(j) = \beta(j)\} \in \mathcal{U}.$$

This is an equivalence relation, as can be proved easily using basic properties of ultrafilters. For each $\alpha \in \Pi A_j$ let $[\alpha]$ denote the equivalence class of α under $\sim$. The *ultraproduct* $\Pi_{\mathcal{U}} A_j$ is then defined to be the set of all equivalence classes $[\alpha]$ as α ranges over ΠA_j:

$$\Pi_{\mathcal{U}}(A_j \mid j \in J) := \{[\alpha] \mid \alpha \in \Pi(A_j \mid j \in J)\}.$$

If the sets $(A_j \mid j \in J)$ are all equal to the same set A, then the ultraproduct $\Pi_{\mathcal{U}}(A_j \mid j \in J)$ is called an *ultrapower* of A and it is denoted $A^J/_{\mathcal{U}}$.

2.28. Theorem. [Existence of Nonstandard Extensions] *Each nonempty set $\mathbb{X}$ has a proper nonstandard extension, in which the set ${}^*\mathbb{X}$ may be taken to be an ultrapower of $\mathbb{X}$ with respect to a countably incomplete ultrafilter.*

Proof. Let J be any infinite index set and let $\mathcal{U}$ be any countably incomplete ultrafilter on J. This means that there exists a sequence $(F_k)_{k\in\mathbb{N}}$ of sets in $\mathcal{U}$ whose intersection $\bigcap\{F_k \mid k \in \mathbb{N}\}$ is empty. There exists such an ultrafilter on each infinite index set J. Moreover if J is countable and $\mathcal{U}$ is any nonprincipal ultrafilter on J, then it is easy to see that $\mathcal{U}$ must be countably incomplete.

The underlying set ${}^*\mathbb{X}$ of our nonstandard extension will be the ultrapower $\mathbb{X}^I/_{\mathcal{U}}$ defined above. Therefore each element of ${}^*\mathbb{X}$ is an equivalence class $[\alpha]$ for some function $\alpha\colon J \to \mathbb{X}$.

Given $m \geq 0$ and $A \subseteq \mathbb{X}^m$, we define ${}^*A \subseteq ({}^*\mathbb{X})^m$ by:

$${}^*A = \{([\alpha_1], \dots, [\alpha_m]) \mid \{j \in J \mid (\alpha_1(j), \dots, \alpha_m(j)) \in A\} \in \mathcal{U}\}.$$

In this definition, $\alpha_1, \dots, \alpha_m$ range over the set $\mathbb{X}^J$ of all functions from J into $\mathbb{X}$.

We need to show this mapping satisfies conditions (E1) – (E4) in Definition 2.1.

(E1) Fix $m \geq 0$ and let $A, B \subseteq \mathbb{X}^m$. It is immediate that ${}^*A$ is a subset of $({}^*\mathbb{X})^m$. Let $\alpha_1, \dots, \alpha_m$ be functions from J to $\mathbb{X}$ and set

$$F = \{j \in J \mid (\alpha_1(j), \dots, \alpha_m(j)) \in A\}$$

$$G = \{j \in J \mid (\alpha_1(j), \dots, \alpha_m(j)) \in B\}.$$

Using properties of the ultrafilter, it is easy to prove

$$([\alpha_1], \dots, [\alpha_m]) \in {}^*(A \cap B) \iff (F \cap G) \in \mathcal{U} \iff F \in \mathcal{U} \wedge G \in \mathcal{U}$$

$$\iff ([\alpha_1], \dots, [\alpha_m]) \in {}^*A \text{ and } ([\alpha_1], \dots, [\alpha_m]) \in {}^*B.$$

Similarly

$$([\alpha_1], \dots, [\alpha_m]) \in {}^*(\mathbb{X}^m \setminus A) \Longleftrightarrow J \setminus F \in \mathcal{U} \Longleftrightarrow F \notin \mathcal{U}$$
$$\Longleftrightarrow ([\alpha_1], \dots, [\alpha_m]) \in ({}^*\mathbb{X}^m) \setminus {}^*A.$$

This suffices to prove condition (E1).

(E2) For simplicity of notation we consider the basic diagonal subset of $\mathbb{X}^2$ given by

$$\Delta = \{(x, y) \in \mathbb{X}^2 \mid x = y\}.$$

Then

$$([\alpha], [\beta]) \in {}^*\Delta \Longleftrightarrow \{j \in J \mid \alpha(j) = \beta(j)\} \in \mathcal{U}$$
$$\Longleftrightarrow \alpha \sim \beta \Longleftrightarrow [\alpha] = [\beta].$$

This shows that ${}^*\Delta$ is the desired diagonal subset of $({}^*\mathbb{X})^2$.

(E3) The fact that Cartesian products are preserved by this mapping is immediate from the definition and an argument similar to the proof of (E1).

(E4) For simplicity of notation we consider only a subset A of $\mathbb{X}^2$ and the projection $\pi(x, y) = x$ onto the first coordinate. Let B be the projection of A under π. Given functions $\alpha, \beta\colon J \to \mathbb{X}$, let $F = \{j \in J \mid (\alpha(j), \beta(j)) \in A\}$ and $G = \{j \in J \mid \alpha(j) \in B\}$. If $([\alpha], [\beta]) \in {}^*A$ then $F \in \mathcal{U}$ and $F \subseteq G$, so that also $G \in \mathcal{U}$ and hence $[\beta] \in {}^*B$. Conversely, suppose $[\beta] \in {}^*B$ so that $G \in \mathcal{U}$. Define $\alpha(j)$ for $j \in G$ by choosing it so that $(\alpha(j), \beta(j)) \in A$. For $j \notin G$ define $\alpha(j)$ arbitrarily in $\mathbb{X}$. For this pair of functions α, β we have $G \subseteq F$ so $F \in \mathcal{U}$ and therefore $([\alpha], [\beta]) \in {}^*A$. This proves (E4).

To finish the proof we must prove that this nonstandard extension is proper. Let A be an infinite subset of $\mathbb{X}$. Consider a sequence $(F_k)_{k\in\mathbb{N}}$ of sets in $\mathcal{U}$ whose intersection $\bigcap\{F_k \mid k \in \mathbb{N}\}$ is empty. Without loss of generality we may assume $F_0 = J$ and $F_n \supseteq F_{n+1}$ for all $n \in \mathbb{N}$. This allows us to define $d(j)$ for all $j \in J$ to be the largest $k \in \mathbb{N}$ for which $j \in F_k$. Note that for all $n \in \mathbb{N}$ and all $j \in J$, $d(j) = n$ if and only if $j \in F_n \setminus F_{n+1}$. Choose a sequence $(a_k)_{k\in\mathbb{N}}$ out of A which has no repetitions. Define $\alpha\colon J \to A$ by setting $\alpha(j) = a_{d(j)}$ for all $j \in J$. It remains to show that $[\alpha]$ is not standard. Indeed, if $[\alpha]$ were equal to the standard element ${}^*a$ for some $a \in \mathbb{X}$, then the set $F = \{j \in J \mid \alpha(j) = a\}$ would be an element of $\mathcal{U}$. (See Exercise 2.29.) However, the construction of α ensures that $F = F_n \setminus F_{n+1}$ for some $n \in \mathbb{N}$, and therefore F is not an element of $\mathcal{U}$. □

We conclude this Section with a few Exercises about ultrapower nonstandard extensions.

2.29. Exercise. Let J be an index set and $\mathcal{U}$ an ultrafilter on J, and consider the ultrapower nonstandard extension of $\mathbb{X}$ that is constructed in the proof of Theorem 2.28.

(a) For each $a \in \mathbb{X}$, ${}^*a = [\alpha]$, where $\alpha: J \to \mathbb{X}$ is the constant function with $\alpha(j) = a$ for all $j \in J$.

(b) Let $f: \mathbb{X}^m \to \mathbb{X}$ be a function. Let $\alpha_1, \dots, \alpha_m$ be elements of $\mathbb{X}^J$ and define $\beta \in \mathbb{X}^J$ by setting $\beta(j) = f(\alpha_1(j), \dots, \alpha_m(j))$ for all $j \in J$. Then ${}^*f([\alpha_1], \dots, [\alpha_m]) = [\beta]$. Give a similar description of ${}^*f$ where $f: A \to B$ is any function, with $A \subseteq \mathbb{X}^m$, and $B \subseteq \mathbb{X}^n$.

2.30. Exercise. Let J be an index set and $\mathcal{U}$ an ultrafilter on J. Suppose $A \subseteq \mathbb{X}^m$ and consider the set ${}^*A$ defined in the proof of Theorem 2.28 above. There is a natural way to identify ${}^*A$ with the ultrapower $A^J/_{\mathcal{U}}$. (Hint: if $\alpha_1, \dots, \alpha_m$ are functions from J to $\mathbb{X}$, then $\alpha = (\alpha_1, \dots, \alpha_m)$ may be regarded as a function from J into $\mathbb{X}^m$, and every such function arises in this way. If $\{j \in J \mid \alpha(j) \in A\} \in \mathcal{U}$, then there exist functions $\beta_1, \dots, \beta_m$ from J into $\mathbb{X}$ such that (i) for all $j \in J$, $(\beta_1(j), \dots, \beta_m(j)) \in A$ and (ii) $\beta_i \sim \alpha_i$ for all $i = 1, \dots, m$.)

2.31. Exercise. Let $\mathcal{U}$ be a nonprincipal ultrafilter on the index set $\mathbb{N}$ and let ${}^*\mathbb{R}$ be the ultrapower nonstandard extension of $\mathbb{R}$ that is defined in the proof of Theorem 2.28 above.

(a) Let $\alpha: \mathbb{N} \to \mathbb{R}$ be a sequence which converges to $+\infty$; the element $[\alpha]$ of ${}^*\mathbb{R}$ is positive infinite. Give some specific examples of such infinite elements of ${}^*\mathbb{R}$ and compare them with respect to the ordering ${}^*<$.

(b) Let $\beta: \mathbb{N} \to \mathbb{R}$ be a sequence which converges to 0 from above; the element $[\beta]$ is a positive infinitesimal in ${}^*\mathbb{R}$. Give some specific examples of such infinitesimal elements of ${}^*\mathbb{R}$ and compare them with respect to the ordering ${}^*<$.

3. Logical Formulas

In this Section we discuss how to use informal and familiar logical notation to streamline our reasoning about nonstandard extensions. Logical notation is often suggestive and transparent when defining or describing sets and functions. Using logical formulas permits us to take advantage of our natural linguistic and logical abilities. Moreover, this turns out to be an ideal framework for bringing out the main properties of nonstandard extensions.

To illustrate this use of formulas, let x, y be variables ranging over nonempty sets A, B respectively, and let $\varphi(x, y)$ and $\psi(x, y)$ denote conditions (formulas) on (x, y) defining subsets Φ and Ψ (respectively) of $A \times B$. We consider certain *logical formulas* that can be built up from $\varphi(x, y)$ and $\psi(x, y)$ (on the left below) and the *sets* that are defined by them (on the right):

$\neg\varphi(x,y)$	defines	the complement of Φ in $A \times B$,
$\varphi(x,y) \vee \psi(x,y)$	defines	the union $\Phi \cup \Psi$,
$\varphi(x,y) \wedge \psi(x,y)$	defines	the intersection $\Phi \cap \Psi$,
$\exists x\, \varphi(x,y)$	defines	the projection $\pi(\Phi)$, where $\pi(x,y) = y$ is the projection onto the second coordinate,
$\forall y\, \varphi(x,y)$	defines	$\{x \in A \mid \{x\} \times B \subseteq \Phi\}$.

Here we are using familiar logical symbols, which have the following meanings:

$\neg$	stands for	the negation, "not",
$\vee$	stands for	the disjunction, "or"
$\wedge$	stands for	the conjunction, "and",
$\exists$	stands for	the existential quantifier, "there exists", and
$\forall$	stands for	the universal quantifier, "for all."

In our use of the quantifiers above, we followed the given restriction that x ranges over A and y ranges over B. This can be made explicit by writing $\exists x \in A\, \varphi(x,y)$ or $\forall y \in B\, \psi(x,y)$ instead of what is written above. In our use of logical formulas we will always have an explicit or implicit understanding about the set over which a given variable ranges.

To illustrate the usefulness of these simple ideas, consider a given function $f\colon A \to B$. The range of f, namely the set $f(A)$, can be defined by the equivalence

$$y \in f(A) \iff \exists x \in A\, [f(x) = y].$$

Let Γ be the graph of f, which is defined as a subset of $A \times B$ by the formula $f(x) = y$. Therefore, this equivalence exhibits the fact that $f(A)$ is the projection of Γ under the projection map π onto the second coordinate. This reduction of *arbitrary* functions to *projections* is used frequently; indeed, we have used it already several times in Section 2, when we used condition (E4) of Definition 2.1 to prove results about functions.

Simple and familiar logical equivalences often capture mathematical facts that seem complicated when they are viewed directly without the use of logical formulas. For example, the familiar equivalence

$$\forall y\, \varphi(x,y) \iff \neg\exists y\, \neg\varphi(x,y)$$

shows that the set defined by $\forall y\, \varphi(x,y)$ can be obtained from Φ by first taking the complement in $A \times B$, then projecting onto the first coordinate, and then taking the complement of that set in A. This technique is particularly useful when dealing with logically complicated notions, such as continuity or differentiability, which we express in the usual way with ϵ's and δ's and quantifiers over them. In such cases we often deal with formulas having more than two variables and with repeated quantifiers.

We will use several additional notational conventions. A formula $\varphi(x, y)$ defining a subset of $A \times B$ will also be viewed sometimes as defining a condition on triples (x, y, z), where z ranges over a non-empty set C; in that case $\varphi(x, y)$ defines a subset of $A \times B \times C$. In such a situation we will indicate the formula also as $\varphi(x, y, z)$ to show that we are thinking of this formula as defining a subset of $A \times B \times C$ instead of just $A \times B$. Here we are making a distinction between the appearance of the formula itself (in which the variable z does not occur) and the notation we use for referring to the formula in a proof or other discussion. This is similar to the situation in algebra where one routinely regards a polynomial $p(x, y)$ as a polynomial in three variables x, y, z in which all monomials containing z are taken to have coefficient 0. For logical formulas the general convention is that when we use notation such as $\varphi(x_1, \dots, x_n)$ to refer to a formula, then the variables $x_1, \dots, x_n$ must be distinct and they must *include* all of the variables that occur in the formula in a way that makes them free for substitution. The other variables in the formula, all of which are bound by quantifiers, need not be included in this list. We also sometime denote a formula by writing φ or ψ without any list of variables, when it is not important to name the variables that may be free for substitution. The context will determine which notation we are using.

We use the implication sign $\rightarrow$, as in $\varphi(x, y) \rightarrow \psi(x, y)$, to abbreviate the formula $(\neg\varphi(x, y)) \vee \psi(x, y)$. We use the equivalence symbol $\leftrightarrow$, as in $\varphi(x, y) \leftrightarrow \psi(x, y)$, to abbreviate $[\varphi(x, y) \rightarrow \psi(x, y)] \wedge [\psi(x, y) \rightarrow \varphi(x, y)]$.

Now let us consider the particular logical formulas that we will use in working with nonstandard extensions. For the moment, all of our variables will range over $\mathbb{X}$. For each set $A \subseteq \mathbb{X}^m$ we will regard $(x_1, \dots, x_m) \in A$ as a formula, in which $x_1, \dots, x_m$ are variables ranging over $\mathbb{X}$; we do not require these variables to be distinct. Sometimes we will write this formula in the equivalent form $A(x_1, \dots, x_m)$, if this fits more smoothly with the usual mathematical role of the set A. In a few situations this formula is written in other ways: for example, if A corresponds to a linear ordering $<$, in the sense that A is the set of pairs (a, b) which satisfy the ordering condition $a < b$, then it is natural to write the formula $x < y$ as synonymous with $(x, y) \in A$. All of this is quite familiar usage in mathematics. Moreover, in the formula $(x_1, \dots, x_n) \in A$ we can replace some or all of the variables x_j by specific elements of $\mathbb{X}$.

If $f \colon A \rightarrow B$ is a function, where $A \subseteq \mathbb{X}^m$ and $B \subseteq \mathbb{X}^n$, then we will also make use of the formulas $f(x_1, \dots, x_m) = (y_1, \dots, y_n)$. If Γ is the graph of f, then this formula is equivalent to the formula $(x_1, \dots, x_m, y_1, \dots, y_n) \in \Gamma$.

This corresponds to what we did in the previous Section, handling functions by means of their graphs.

Building on the basic formulas discussed in the previous paragraph, we construct more complicated formulas using quantifiers (with variables ranging over the set $\mathbb{X}$) and the logical connectives that are discussed above: $\neg, \vee, \wedge, \rightarrow$, and $\leftrightarrow$. We will refer to these logical formulas as *formulas over* $\mathbb{X}$. (To be precise, the logical formulas we are using here are *first order formulas*. This reflects the fact that the quantifiers we use range over elements of $\mathbb{X}$, and we do not have any quantifiers ranging over subsets of $\mathbb{X}$ or other higher type objects based on $\mathbb{X}$.) To be precise, we have the following definition by induction:

3.1. Definition. [**Formulas Over** $\mathbb{X}$] *Let* $\mathbb{X}$ *be a non-empty set. The set of* **formulas over** $\mathbb{X}$ *is the smallest set of logical formulas which satisfies the following closure conditions. (We let* x *and* $x_1, \ldots, x_m, y_1, \ldots, y_n$ *stand for arbitrary variables, which need not be distinct.)*

(i) For each set $A \subseteq \mathbb{X}^m$, $(x_1, \ldots, x_m) \in A$ *is a formula over* $\mathbb{X}$*;*

(ii) for each function $f\colon A \rightarrow B$*, where* $A \subseteq \mathbb{X}^m$ *and* $B \subseteq \mathbb{X}^n$,

$$f(x_1, \ldots, x_m) = (y_1, \ldots, y_n)$$

is a formula over $\mathbb{X}$*;*

(iii) if $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ *is a formula over* $\mathbb{X}$ *and* $a_1, \ldots, a_n \in \mathbb{X}$*, then* $\varphi(x_1, \ldots, x_m, a_1, \ldots, a_n)$ *is a formula over* $\mathbb{X}$*;*

(iv) if φ *and* ψ *are formulas over* $\mathbb{X}$*, then* $\neg\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$, $\exists x\, \varphi$*, and* $\forall x\, \varphi$ *are formulas over* $\mathbb{X}$.

Note that functions appear in formulas over $\mathbb{X}$ only through their graphs. This is less restrictive than it may seem at first. For example, suppose f, g, and h are functions from $\mathbb{X}$ into itself, and we want to express the condition that h is the composition of f and g. This can be done using the following formula

$$\forall x \forall y\, [h(x) = y \leftrightarrow \exists z\, (g(x) = z \wedge f(z) = y)]$$

which is a formula over $\mathbb{X}$. Suppose $< \subseteq \mathbb{X}^2$ is an ordering relation on $\mathbb{X}$, and we want to express the condition that $f(x) < g(x)$ holds for all elements x of $\mathbb{X}$. This can be done using the following formula over $\mathbb{X}$:

$$\forall x \forall y \forall z\, [(f(x) = y \wedge g(x) = z) \rightarrow y < z].$$

In this way we see how statements involving the composition of functions and the substitution of functions in predicates can be expressed using formulas over $\mathbb{X}$.

If we consider a formula over $\mathbb{X}$ syntactically, as a string of symbols, then it is important to distinguish two different ways in which variables can be used. The *free* variables are the ones for which values can be substituted; all other occurrences of variables are *bound*, meaning that their use is controlled by the occurence of quantifiers in the formula. The best way to make this precise is to give the following inductive definition of free variables in a formula φ:

(i) all variables occurring within a basic formula $(x_1, \dots, x_m) \in A$ or $f(x_1, \dots, x_m) = (y_1, \dots, y_n)$ are free variables;

(ii) the free variables in $\neg\varphi$ are the same as the free variables in φ;

(iii) the free variables in $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \rightarrow \psi$, or $\varphi \leftrightarrow \psi$ are the free variables in φ together with the free variables in ψ;

(iv) the free variables in $\exists x\, \varphi$, and $\forall x\, \varphi$ are the free variables in φ that are distinct from x.

If φ is a formula whose free variables are among $x_1, \dots, x_m$, we indicate this fact by writing the formula as $\varphi(x_1, \dots, x_m)$; when establishing this notation for the first time we require that $x_1, \dots, x_m$ be distinct. We then will indicate the result of substituting other variables or functional expressions $t_1, \dots, t_m$ for $x_1, \dots, x_m$ respectively by writing the result of the substitutions in the form $\varphi(t_1, \dots, t_m)$; in such a situation we do not require that the substituted expressions $t_1, \dots, t_m$ be distinct.

A *sentence* is a logical formula with no free variables. It makes a definite true-or-false statement about the structures to which it refers.

Now we discuss how formulas over $\mathbb{X}$ can be used in connection with nonstandard extensions of $\mathbb{X}$. Consider a specific nonstandard extension of $\mathbb{X}$, based on the set ${}^*\mathbb{X}$. We will regard this nonstandard extension as fixed for the rest of this Section. Since ${}^*\mathbb{X}$ is also a (non-empty) set, we also have the class of formulas over ${}^*\mathbb{X}$. We will now see that there is an important connection between the formulas over $\mathbb{X}$ and (some of the) formulas over ${}^*\mathbb{X}$. We will normally use the convention that lower case variables such as $x_1, \dots, x_n$ range over $\mathbb{X}$ and upper case variables such as $X_1, \dots, X_n$ range over ${}^*\mathbb{X}$. We will *never* mix the two types of variables in the same formula. More generally, all of the formulas we consider will either be formulas over $\mathbb{X}$ or they will be formulas over ${}^*\mathbb{X}$.

3.2. Definition. [**$*$-Transform of a Formula Over** $\mathbb{X}$] *Fix a nonstandard extension of the set $\mathbb{X}$. Let $\varphi(x_1, \dots, x_n)$ be a formula over $\mathbb{X}$. The* **$*$-transform** *of $\varphi(x_1, \dots, x_n)$ is a formula over ${}^*\mathbb{X}$, written ${}^*\varphi(X_1, \dots, X_n)$, which is defined inductively by the following conditions:*

Basis cases:

(i) Let $\{\sigma(1), \dots, \sigma(m)\} \subseteq \{1, \dots, n\}$ and $A \subseteq \mathbb{X}^m$; the $*$-transform of the basic formula

$$(x_{\sigma(1)}, \dots, x_{\sigma(m)}) \in A$$

is the formula

$$(X_{\sigma(1)}, \dots, X_{\sigma(m)}) \in {}^*A;$$

similarly, the $*$-transform of

$$f(x_{\sigma(1)},\dots,x_{\sigma(k)}) = (x_{\sigma(k+1)},\dots,x_{\sigma(m)})$$

is

$$^*f(X_{\sigma(1)},\dots,X_{\sigma(k)}) = (X_{\sigma(k+1)},\dots,X_{\sigma(m)});$$

(ii) if $\varphi(x_1,\dots,x_n,y_1,\dots,y_p)$ is a basic formula (as treated in (i)) and $a_1,\dots,a_p \in \mathbb{X}$, then the $*$-transform of $\varphi(x_1,\dots,x_n,a_1,\dots,a_p)$ is $^*\varphi(X_1,\dots,X_n,{}^*a_1,\dots,{}^*a_p)$;

Induction cases: Let φ and ψ be formulas over $\mathbb{X}$;

(iii) the $*$-transform of the negation $\neg\varphi$ is $\neg{}^*\varphi$;

(iv) the $*$-transform of the disjunction $\varphi \vee \psi$ is $^*\varphi \vee {}^*\psi$;

(v) the $*$-transform of the conjunction $\varphi \wedge \psi$ is $^*\varphi \wedge {}^*\psi$;

(vi) if x is a variable ranging over $\mathbb{X}$, then the $*$-transform of the quantified formula $\exists x\,\varphi$ is $\exists X\,{}^*\varphi$;

(vii) if x is a variable ranging over $\mathbb{X}$, then the $*$-transform of the quantified formula $\forall x\,\varphi$ is $\forall X\,{}^*\varphi$.

While this definition may look complicated, it is merely the precise formulation of a simple idea: constructing the $*$-transform of a formula $\varphi(x_1,\dots,x_n)$ over $\mathbb{X}$ requires the following steps:

(a) Find all of the sets $A \subseteq \mathbb{X}^m$ that occur in $\varphi(x_1,\dots,x_n)$ in basic formulas, and replace each such set by its counterpart $^*A$ over $^*\mathbb{X}$; similarly, replace each function $f\colon A \to B$ by $^*f$ and replace each element a of $\mathbb{X}$ by $^*a$, and

(b) replace every variable x in $\varphi(x_1,\dots,x_n)$, including the ones that are used with quantifiers, by a corresponding variable X which ranges over $^*\mathbb{X}$.

For example, suppose Γ is a subset of $\mathbb{X}^2$. The sentence over $\mathbb{X}$ given by

$$\forall x\forall y\forall z\,[[(x,y)\in\Gamma \wedge (x,z)\in\Gamma] \to y = z] \wedge \forall x\exists y\,[(x,y)\in\Gamma]$$

expresses the condition that Γ is the graph of a function from $\mathbb{X}$ to $\mathbb{X}$. The $*$-transform of this sentence is given by

$$\forall X\forall Y\forall Z\,[[(X,Y)\in{}^*\Gamma \wedge (X,Z)\in{}^*\Gamma] \to Y = Z] \wedge \forall X\exists Y\,[(X,Y)\in{}^*\Gamma].$$

This is a sentence over $^*\mathbb{X}$, meaning in particular that the variables X, Y, Z range over $^*\mathbb{X}$. This sentence expresses the condition that $^*\Gamma$ is the graph of a function from $^*\mathbb{X}$ to $^*\mathbb{X}$.

From Proposition 2.11 we know that these two sentences are equivalent, and this is no accident. This is an instance of the Transfer Principle, which we prove next. The Transfer Principle is a flexible and useful result which expresses nearly everything that one needs to know about nonstandard extensions. In particular, it gives precise meaning to the statement "*the nonstandard extension of $\mathbb{X}$ possesses all of the properties that $\mathbb{X}$ does.*"

3.3. Theorem. [Transfer Principle] *Let $\mathbb{X}$ be a non-empty set and consider a fixed nonstandard extension of $\mathbb{X}$.*

(a) Let $\varphi(x_1,\dots,x_m)$ be a formula over $\mathbb{X}$ and let ${}^*\varphi(X_1,\dots,X_m)$ be its $*$-transform. Suppose $B \subseteq \mathbb{X}^m$ is the set defined by $\varphi(x_1,\dots,x_m)$:

$$B = \{(x_1,\dots,x_m) \in \mathbb{X}^m \mid \varphi(x_1,\dots,x_m) \text{ is true in } \mathbb{X}\}.$$

Then ${}^*B$ is the set defined by ${}^*\varphi(X_1,\dots,X_m)$:

$${}^*B = \{(X_1,\dots,X_m) \in ({}^*\mathbb{X})^m \mid {}^*\varphi(X_1,\dots,X_m) \text{ is true in } {}^*\mathbb{X}\}.$$

(b) Let φ be any sentence over $\mathbb{X}$, and let ${}^*\varphi$ be its $*$-transform. Then

$$\varphi \text{ is true in } \mathbb{X} \Longleftrightarrow {}^*\varphi \text{ is true in } {}^*\mathbb{X}.$$

Proof. We prove (a) by induction on the syntactic complexity of formulas over $\mathbb{X}$. In other words we structure our proof so that it follows the same path as the inductive definition of the $*$-transform.

Before giving the inductive proof, we prove that (a) implies (b). Suppose φ is a sentence over $\mathbb{X}$ and ${}^*\varphi$ is its $*$-transform, a sentence over ${}^*\mathbb{X}$. Let A be the set defined by φ, so that ${}^*A$ is the set defined by ${}^*\varphi$, according to the statement above (which we are using in the case $n = 0$, where the formula treated does not have any variables that are free for substitution). Evidently A is either $\mathbb{X}^0$ or $\emptyset$, according to whether φ is true in $\mathbb{X}$ or not. Similarly ${}^*A$ is either $({}^*\mathbb{X})^0$ or $\emptyset$, according to whether or not ${}^*\varphi$ is true in ${}^*\mathbb{X}$ or not. Proposition 2.2 implies that either $A = \mathbb{X}^0$ and ${}^*A = ({}^*\mathbb{X})^0$ must both hold, or $A = \emptyset$ and ${}^*A = \emptyset$ must both hold. The equivalence of φ and ${}^*\varphi$ follows immediately.

Now we turn to the inductive proof of (a). For the basis step we must consider formulas $\varphi(x_1,\dots,x_n)$ of the form $(x_{\sigma(1)},\dots,x_{\sigma(m)}) \in A$, where $\{\sigma(1),\dots,\sigma(m)\} \subseteq \{1,\dots,n\}$ and $A \subseteq \mathbb{X}^m$. Then ${}^*\varphi(X_1,\dots,X_n)$ is $(X_{\sigma(1)},\dots,X_{\sigma(m)}) \in {}^*A$. Let B be the set of all $(x_1,\dots,x_n) \in \mathbb{X}^n$ for which $(x_{\sigma(1)},\dots,x_{\sigma(m)}) \in A$ is true. Proposition 2.15 states that ${}^*B$ is the set of all $(X_1,\dots,X_n) \in ({}^*\mathbb{X})^n$ for which $(X_{\sigma(1)},\dots,X_{\sigma(m)}) \in {}^*A$ is true. This is what we needed to prove.

More generally, in the basis step we must also take into account the possibility that one or more variables in $(x_{\sigma(1)},\dots,x_{\sigma(m)}) \in A$ are replaced by specific elements of $\mathbb{X}$. This is handled using Propositions 2.17 and 2.15.

The induction steps are handled using the conditions in Definition 2.1 directly. The logical connectives are handled using condition (E1). Existential quantifiers are handled using the strengthening of (E4) that is given in Proposition 2.16; the stronger form of (E4) is needed in case the existentially quantified variable is not the last variable in the given list. Finally, the duality between universal and existential quantifiers means that universal quantifiers can be handled as a combination of negations and existential quantifiers. □

We remark that the Transfer Principle exactly captures the content of the definition of nonstandard extension. That is, if the Transfer Principle holds and if the equality relation = is given its usual interpretation in the nonstandard extension, then conditions (E1) – (E4) must be true. Proving this is an exercise in the use of logical formulas. Usually the Transfer Principle is explicitly included in the definition of nonstandard extension. We have delayed our discussion of the Transfer Principle in order to avoid heavy use of logical formulas at the beginning of the exposition and to permit introducing logical notation in a natural and convincing way.

To illustrate the usefulness of this result, let us treat some of the Exercises from Section 2. First consider Exercise 2.23. For ease of notation, assume $m = n = 1$. Let A, B, C and f be as given there. The set $f(B)$ is defined by the equivalence

$$x \in f(B) \Longleftrightarrow \exists y\, [y \in B \wedge f(y) = x].$$

Therefore, the Transfer Principle gives us that the equivalence

$$X \in {}^*(f(B)) \Longleftrightarrow \exists Y\, [Y \in {}^*B \wedge {}^*f(Y) = X]$$

holds in ${}^*\mathbb{X}$. But the formula on the right side of this equivalence defines $({}^*f)({}^*B)$, so we have the equality needed for part (a) of the Exercise.

For part (b) we use the equivalence

$$x \in f^{-1}(C) \Longleftrightarrow \exists y\, [y \in C \wedge f(x) = y]$$

and for part (c) we use the equivalence

$$(f|B)(x) = y \Longleftrightarrow [f(x) = y \wedge x \in B].$$

In both cases the Transfer Principle gives us immediately what is needed.

Now consider Exercise 2.24. For simplicity take $m = n = 2$. The function f is characterized by the equivalence

$$[f(u,v) = (x,y)] \Longleftrightarrow [f_1(u,v) = x \wedge f_2(u,v) = y]$$

where u, v, x, y are variables ranging over $\mathbb{X}$. The Transfer Principle yields that the equivalence

$$[({}^*f)(U,V) = (X,Y)] \Longleftrightarrow [({}^*f_1)(U,V) = X \wedge ({}^*f_2)(U,V) = Y]$$

holds in ${}^*\mathbb{X}$. This implies ${}^*f = ({}^*f_1, {}^*f_2)$ as desired.

Next we treat Exercise 2.25. For ease of notation we consider only the case $m = n = 1$. Suppose $A \subseteq \mathbb{X}$ and $B \subseteq \mathbb{X}$, and let $f\colon A \to B$ be a function. For part (a), we note that f is injective if and only if the sentence

$$\forall x \forall y \forall z\, [[f(x) = z \wedge f(y) = z] \to x = y]$$

is true in $\mathbb{X}$. The $*$-transform of this sentence is

$$\forall X \forall Y \forall Z\, [[{}^*f(X) = Z \wedge {}^*f(Y) = Z] \to X = Y].$$

This sentence holds in ${}^*\mathbb{X}$ if and only if the function ${}^*f$ is injective. Therefore the Transfer Principle gives the desired result immediately. Similar arguments using other simple sentences will easily give parts (b) and (c).

Finally we treat Exercise 2.26. Each of the axioms for ordered fields can be expressed as a first-order sentence in which the quantifiers range over the underlying set of the field. For example, the statement that every non-zero element of $\mathbb{R}$ has a multiplicative inverse is expressed by the following sentence over $\mathbb{R}$:

$$\forall x \exists y\, [\neg x = 0 \to x \times y = 1].$$

The $*$-transform of this sentence is

$$\forall X \exists Y\, [\neg X = {}^*0 \to X \; {}^*{\times}\; Y = {}^*1].$$

By the Transfer Principle, this sentence is true in ${}^*\mathbb{R}$. Since ^{*0} is the additive identity and ^{*1} is the multiplicative identity in ${}^*\mathbb{R}$, as is shown in a similar way using other sentences over $\mathbb{R}$, it follows that every non-zero element of ${}^*\mathbb{R}$ has a multiplicative inverse. Similar arguments complete the Exercise.

4. Nonstandard Extensions of Multisets

In many parts of mathematics it is customary to encounter not just a single set, but several sets which are interacting in some way. For example, a vector space over the real field consists of the set $\mathbb{R}$ together with the underlying set $\mathbb{W}$ of the vector space. Among the objects which are included in this vector space setting is the operation of scalar multiplication, which is a function from $\mathbb{R} \times \mathbb{W}$ to $\mathbb{W}$. If $\mathbb{W}$ is a normed space, it is convenient to add the dual space $\mathbb{W}'$ as a third set. One operation which involves all three of these sets is the pairing $\langle w, f\rangle := f(w)$, considered as a function from $\mathbb{W} \times \mathbb{W}'$ into $\mathbb{R}$.

It is therefore natural to extend our concept of nonstandard extension to this kind of setting. Fortunately it is easy to do, requiring nothing more than a more elaborate notation. We lay out the details in this Section, but we omit proofs since they are so close to the ones which we gave in Sections 2 and 3. This material will be required in the next two Sections when we develop frameworks for introducing higher type objects into the foundations of nonstandard analysis.

Fix a non-empty index set I. The objects we consider here consist of families of (non-empty) sets indexed over I. We will refer to them as *multisets* or as *many sorted sets*, when the specific reference to I is omitted, and as *I-sets* when I needs to be mentioned.

4.1. Definition. *An* **I-set** $\mathbb{X}_I$ *is an indexed family* $(\mathbb{X}_i)_{i\in I}$ *of sets. A* **sort** *of the I-set* $\mathbb{X}_I$ *is one of the sets* $\mathbb{X}_i$*, where* $i \in I$*. We say that* $\mathbb{X}_I$ *is* **non-empty** *if* $\mathbb{X}_i$ *is non-empty for every* $i \in I$*.*

We now establish some notation for dealing with I-sets. We will let letters such as α, β, γ stand for finite sequences taken from I. Usually we will write α for the sequence $\alpha(1), \ldots, \alpha(m)$; m will be called the *length* of α and we will also denote the length by $|\alpha|$. Similarly we will normally understand that n is the length of the sequence β and p is the length of γ.

Given such a finite sequence α from I and given an I-set $\mathbb{X}_I$, we consider the Cartesian product of the sorts of $\mathbb{X}_I$ that are indexed by the coordinates of α; our notation for this Cartesian product is the following:

$$\mathbb{X}^\alpha = \mathbb{X}_{\alpha(1)} \times \cdots \times \mathbb{X}_{\alpha(m)}.$$

We consider the I-set $\mathbb{X}_I$ as equipped with all possible subsets of every Cartesian product $\mathbb{X}^\alpha$, where α ranges over all finite sequences from the index set I. In particular, this includes the graph of every function from one such Cartesian product $\mathbb{X}^\alpha$ to another Cartesian product $\mathbb{X}^\beta$. If $f\colon \mathbb{X}^\alpha \to \mathbb{X}^\beta$ is such a function, then its graph is a subset of $\mathbb{X}^\alpha \times \mathbb{X}^\beta$. Note that this product is also a Cartesian product of sorts. Indeed, $\mathbb{X}^\alpha \times \mathbb{X}^\beta = \mathbb{X}^\gamma$, where γ is the concatenation of α and β: $\gamma = \alpha(1), \ldots, \alpha(m), \beta(1), \ldots, \beta(n)$; $|\gamma| = m + n$.

Now we are ready to give the definition of *nonstandard extension* for I-sets. This results from a straightforward modification of the concept of nonstandard extension introduced Definition 2.1 for single sets. A nonstandard extension of an I-set $\mathbb{X}_I = (\mathbb{X}_i)_{i\in I}$ will be another non-empty I-set $({}^*\mathbb{X}_i)_{i\in I}$. For each finite sequence α from I, we will use the notation $({}^*\mathbb{X})^\alpha$ for the Cartesian product

$${}^*\mathbb{X}_{\alpha(1)} \times \cdots \times {}^*\mathbb{X}_{\alpha(m)}.$$

4.2. Definition. [Nonstandard Extension of a Multiset] *Let* $\mathbb{X}_I$ *be a non-empty I-set. A* **nonstandard extension of** $\mathbb{X}_I$ *is a mapping which assigns a set* ${}^*A$ *to each* $A \subseteq \mathbb{X}^\alpha$ *for all finite sequences* α *from* I*, such that*

${}^*\mathbb{X}_i$ is non-empty for all $i \in I$ and the following conditions are satisfied for all finite sequences α, β from I:

(M1) The mapping preserves Boolean operations on subsets of $\mathbb{X}^\alpha$: if $A \subseteq \mathbb{X}^\alpha$, then ${}^*A \subseteq ({}^*\mathbb{X})^\alpha$; if $A, B \subseteq \mathbb{X}^\alpha$, then ${}^*(A \cap B) = ({}^*A \cap {}^*B)$, ${}^*(A \cup B) = ({}^*A \cup {}^*B)$, and ${}^*(A \setminus B) = ({}^*A) \setminus ({}^*B)$.

(M2) The mapping preserves basic diagonals: suppose $1 \leq i < j \leq m = |\alpha|$ and suppose $\alpha(i) = \alpha(j)$; if

$$\Delta = \{(x_1, \ldots, x_m) \in \mathbb{X}^\alpha \mid x_i = x_j\}$$

then ${}^*\Delta = \{(x_1, \ldots, x_n) \in ({}^*\mathbb{X})^\alpha \mid x_i = x_j\}$.

(M3) The mapping preserves Cartesian products: if $A \subseteq \mathbb{X}^\alpha$ and $B \subseteq \mathbb{X}^\beta$, then ${}^*(A \times B) = {}^*A \times {}^*B$.

(M4) The mapping preserves projections that omit the final coordinate: suppose α has length $n+1$ and let π be projection of $n+1$-tuples on the first n coordinates; if $A \subseteq \mathbb{X}^\alpha$, then ${}^*(\pi(A)) = \pi({}^*A)$.

For the rest of this Section we fix a nonstandard extension of $\mathbb{X}_I$, based on the non-empty I-set $({}^*\mathbb{X}_i)_{i \in I}$.

We now follow exactly the same path of Propositions and Exercises as in Sections 2 and 3. In order to be clear about what is intended, we give the results in a precisely worded form, modified appropriately for I-sets. It is routine to modify the arguments given in Sections 2 and 3 for this new setting, and we therefore omit all proofs here.

4.3. Proposition. *For each finite sequence α of elements of I, ${}^*(\mathbb{X}^\alpha) = ({}^*\mathbb{X})^\alpha$ and ${}^*\emptyset = \emptyset$.*

4.4. Proposition. *If $A \subseteq \mathbb{X}^\alpha$ is non-empty, then ${}^*A$ is also non-empty. Therefore, for any $A, B \subseteq \mathbb{X}^\alpha$, ${}^*A = {}^*B \iff A = B$.*

4.5. Proposition. *For all $A, B \subseteq \mathbb{X}^\alpha$, $A \subseteq B \iff {}^*A \subseteq {}^*B$.*

4.6. Proposition. *For each $i \in I$ and each $x \in \mathbb{X}_i$, ${}^*\{x\}$ has exactly one element.*

4.7. Notation. For each $i \in I$ and each $x \in \mathbb{X}_i$, we let ${}^*x$ denote the unique element of the set ${}^*\{x\}$. For each $x = (x_1, \ldots, x_m) \in \mathbb{X}^\alpha$ we let ${}^*x = ({}^*x_1, \ldots, {}^*x_m)$.

4.8. Definition. *An element of $({}^*\mathbb{X})^\alpha$ is called* **standard** *if it is of the form ${}^*x$ for some $x \in \mathbb{X}^\alpha$. It follows that an element of $({}^*\mathbb{X})^\alpha$ is standard if and only if all of its coordinates are standard elements of the appropriate sorts ${}^*\mathbb{X}_{\alpha(j)}$.*

4.9. Proposition. *For each $(x_1, \ldots, x_m) \in \mathbb{X}^\alpha$,*

$${}^*\{(x_1, \ldots, x_m)\} = \{({}^*x_1, \ldots {}^*x_m)\}.$$

4.10. Proposition. *For each* $A \subseteq \mathbb{X}^\alpha$ *and* $(x_1, \ldots, x_m) \in \mathbb{X}^\alpha$*,*

$$(x_1, \ldots, x_m) \in A \Longleftrightarrow ({}^*x_1, \ldots, {}^*x_m) \in {}^*A.$$

4.11. Proposition. *Suppose* $A \subseteq \mathbb{X}^\alpha$ *and* $B \subseteq \mathbb{X}^\beta$*, and let* $f\colon A \to B$ *be a function; take* $\Gamma \subseteq \mathbb{X}^\alpha \times \mathbb{X}^\beta$ *to be the graph of* f*. Then* ${}^*\Gamma$ *is the graph of a function from* ${}^*A$ *to* ${}^*B$*.*

4.12. Notation. Suppose $A \subseteq \mathbb{X}^\alpha$ and $B \subseteq \mathbb{X}^\beta$, and let $f\colon A \to B$ be a function; take Γ to be the graph of f. We denote by ${}^*f$ the function from ${}^*A$ to ${}^*B$ whose graph is ${}^*\Gamma$.

4.13. Proposition. *If* f *is the identity function on* $A \subseteq \mathbb{X}^\alpha$*, then* ${}^*f$ *is the identity function on* ${}^*A$*.*

4.14. Proposition. *Suppose* $A \subseteq \mathbb{X}^\alpha$ *and* $B \subseteq \mathbb{X}^\beta$*, and let* $f\colon A \to B$ *be a function. For all* $(x_1, \ldots, x_m) \in A$*,*

$$({}^*f)({}^*x_1, \ldots, {}^*x_m) = {}^*(f(x_1, \ldots, x_m)).$$

4.15. Proposition. **[Permuting and Identifying Variables]** *Suppose* α, β *are finite sequences from* I*, with* $m = |\alpha|$ *and* $n = |\beta|$*. Suppose* σ *is any function from* $\{1, \ldots, m\}$ *into* $\{1, \ldots, n\}$*. Assume* $\beta(\sigma(j)) = \alpha(j)$ *for all* $j = 1, \ldots, m$*. Given* $A \subseteq \mathbb{X}^\alpha$ *define*

$$B = \{(x_1, \ldots, x_n) \in \mathbb{X}^\beta \mid (x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \in A\}.$$

Then

$${}^*B = \{(x_1, \ldots, x_n) \in ({}^*\mathbb{X})^\beta \mid (x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \in {}^*A\}.$$

4.16. Proposition. *Condition (M4) in Definition 4.2 holds for all projections* π*.*

4.17. Proposition. *Let* $A \subseteq \mathbb{X}^\gamma$ *and* $a = (a_1, \ldots, a_m) \in \mathbb{X}^\alpha$*, where* γ *is the sequence obtained by putting* β *after* α*. Define*

$$A(a) = \{(x_1, \ldots, x_n) \in \mathbb{X}^\beta \mid (a_1, \ldots, a_m, x_1, \ldots, x_n) \in A\}$$

and similarly

$$({}^*A)({}^*a) = \{(x_1, \ldots, x_n) \in ({}^*\mathbb{X})^\beta \mid ({}^*a_1, \ldots, {}^*a_m, x_1, \ldots, x_n) \in {}^*A\}$$

Then ${}^*(A(a)) = ({}^*A)({}^*a)$*.*

4.18. Proposition. *Suppose* $A \subseteq \mathbb{X}^\alpha, B \subseteq \mathbb{X}^\beta$ *and* $C \subseteq \mathbb{X}^\gamma$*; let* $f\colon A \to B$ *and* $g\colon B \to C$ *be functions. Then* ${}^*(g \circ f) = ({}^*g) \circ ({}^*f)$.

4.19. Exercise. Condition (M2) holds for all diagonal sets $\Delta \subseteq \mathbb{X}^\alpha$.

4.20. Exercise. If A is a finite subset of $\mathbb{X}^\alpha$, then ${}^*A = \{({}^*x_1, \dots, {}^*x_m) \mid (x_1, \dots, x_m) \in A\}$. In particular, ${}^*A$ is finite and has the same cardinality as A, and all of its elements are standard.

4.21. Definition. *A nonstandard extension of* $\mathbb{X}_I$ *is called* **proper** *if for every* $i \in I$ *and every infinite subset* A *of* $\mathbb{X}_i$*,* ${}^*A$ *contains a nonstandard element.*

4.22. Exercise. Suppose our nonstandard extension is proper. Then, for any infinite set $A \subseteq \mathbb{X}^\alpha$, ${}^*A$ has a nonstandard element.

4.23. Exercise. Let $A \subseteq \mathbb{X}^\alpha$ and suppose $f\colon A \to \mathbb{X}^\beta$ is a function.
(a) If $B \subseteq A$, then ${}^*(f(B)) = ({}^*f)({}^*B)$.
(b) If $C \subseteq \mathbb{X}^\beta$, then ${}^*(f^{-1}(C)) = ({}^*f)^{-1}({}^*C)$.
(c) If $B \subseteq A$, then ${}^*(f|B) = ({}^*f)|({}^*B)$.

4.24. Exercise. For $j = 1, \dots, n$ let $f_j\colon \mathbb{X}^\alpha \to \mathbb{X}_{\beta(j)}$ be a function, and let $f = (f_1, \dots, f_n)\colon \mathbb{X}^\alpha \to \mathbb{X}^\beta$ be the function with $f_1, \dots, f_n$ as its coordinates. Then ${}^*f = ({}^*f_1, \dots, {}^*f_n)$.

4.25. Exercise. Suppose $A \subseteq \mathbb{X}^\alpha$ and $B \subseteq \mathbb{X}^\beta$, and let $f\colon A \to B$ be a function.
(a) f is injective $\Longleftrightarrow$ ${}^*f$ is injective.
(b) f is surjective $\Longleftrightarrow$ ${}^*f$ is surjective.
(c) if f is a bijection and its inverse is g, then ${}^*g$ is the inverse of ${}^*f$.

Next we introduce logical formulas in order to state the Transfer Principle for nonstandard extensions of I-sets. Let $\mathbb{X}_I$ be a fixed I-set. For each $i \in I$ we will make use of variables that range over the sort $\mathbb{X}_i$; no other variables will be used in formulas over the I-set $\mathbb{X}_I$. If necessary, we will indicate that a variable ranges over the sort $\mathbb{X}_i$ by including i as a superscript in the name of the variable; thus x^i, y^i, x^i_j all denote variables that range over $\mathbb{X}_i$. However, we will usually omit such superscripts and let the context determine the sort over which a given variable ranges.

For each finite sequence α from I and for each set $A \subseteq \mathbb{X}^\alpha$ we will regard $(x_1, \dots, x_m) \in A$ as a formula; $x_1, \dots, x_m$ are variables with the property that for each $j = 1, \dots, m$ the variable x_j ranges over the sort $\mathbb{X}_{\alpha(j)}$. As before, we do not require these variables to be distinct. If $f\colon A \to B$ is a function, where $A \subseteq \mathbb{X}^\alpha$ and $B \subseteq \mathbb{X}^\beta$, then we also take $f(x_1, \dots, x_m) = (y_1, \dots, y_n)$ to be a formula, where each variable x_i ranges over the sort $\mathbb{X}_{\alpha(i)}$ and each y_j ranges over $\mathbb{X}_{\beta(j)}$. Moreover, in these basic formulas we can replace some or all of the variables by specific elements of the sorts over which they range.

We construct more complicated formulas using quantifiers (with variables ranging over the sorts of $\mathbb{X}_I$) and the logical connectives $\neg, \vee, \wedge, \rightarrow$, and $\leftrightarrow$. We will refer to these logical formulas as *formulas over* $\mathbb{X}_I$. It is left to the reader to formulate a precise definition of this set of formulas similar to Definition 3.1.

Now we discuss how formulas over $\mathbb{X}_I$ can be used in connection with nonstandard extensions of $\mathbb{X}_I$. Consider a specific nonstandard extension of $\mathbb{X}_I$, based on the I-set $({}^*\mathbb{X}_i)_{i\in I}$. We will regard this nonstandard extension as fixed for the rest of this Section. Since $({}^*\mathbb{X}_i)_{i\in I}$ is also an I-set, we also have the class of formulas over $({}^*\mathbb{X}_i)_{i\in I}$. As before, we will see that there is an important connection between the formulas over $\mathbb{X}_I$ and (some of the) formulas over $({}^*\mathbb{X}_i)_{i\in I}$. We will again use the convention that lower case variables such as x^i range over specific sorts $\mathbb{X}_i$ and the corresponding upper case variables X^i range over the corresponding sort ${}^*\mathbb{X}_i$ of the nonstandard extension. As noted above, however, we will not always include the superscript and will let the natural context determine the role of the variables as much as possible.

4.26. Definition. [**$*$-Transform of a Formula Over $\mathbb{X}_I$**] *Consider a given nonstandard extension of the I-set $\mathbb{X}_I$. Let $\varphi(x_1,\dots,x_n)$ be a formula over $\mathbb{X}_I$. The* **$*$-transform** *of $\varphi(x_1,\dots,x_n)$ is a formula over $({}^*\mathbb{X}_i)_{i\in I}$, written ${}^*\varphi(X_1,\dots,X_n)$, which is defined inductively by the following conditions:*

Basis cases:

(i) Let $\{\sigma(1),\dots,\sigma(m)\} \subseteq \{1,\dots,n\}$ and $A \subseteq \mathbb{X}^\alpha$, with $|\alpha| = m$; the $*$-transform of the basic formula

$$(x_{\sigma(1)},\dots,x_{\sigma(m)}) \in A$$

is the formula

$$(X_{\sigma(1)},\dots,X_{\sigma(m)}) \in {}^*A;$$

similarly, the $*$-transform of

$$f(x_{\sigma(1)},\dots,x_{\sigma(k)}) = (x_{\sigma(k+1)},\dots,x_{\sigma(m)})$$

is

$${}^*f(X_{\sigma(1)},\dots,X_{\sigma(k)}) = (X_{\sigma(k+1)},\dots,X_{\sigma(m)});$$

(ii) suppose $\varphi(x_1,\dots,x_n,y_1,\dots,y_p)$ is a basic formula (as treated in (i)), where the variables x_i range over sort $\mathbb{X}_{\beta(i)}$ for all $i = 1,\dots,n$ and the variables y_j range over sort $\mathbb{X}_{\gamma(j)}$ for all $j = 1,\dots,p$; if $a_j \in \mathbb{X}_{\gamma(j)}$

for all $j = 1, \dots, p$, *then the* $*$*-transform of* $\varphi(x_1, \dots, x_n, a_1, \dots, a_p)$ *is* ${}^*\varphi(X_1, \dots, X_n, {}^*a_1, \dots, {}^*a_p)$*;*

Induction cases: Let φ *and* ψ *be formulas over* $\mathbb{X}$*;*

(iii) the $*$*-transform of the negation* $\neg\varphi$ *is* $\neg{}^*\varphi$*;*

(iv) the $*$*-transform of the disjunction* $\varphi \vee \psi$ *is* ${}^*\varphi \vee {}^*\psi$*;*

(v) the $*$*-transform of the conjunction* $\varphi \wedge \psi$ *is* ${}^*\varphi \wedge {}^*\psi$*;*

(vi) if x *is a variable ranging over a sort* $\mathbb{X}_i$*, then the* $*$*-transform of the quantified formula* $\exists x\, \varphi$ *is* $\exists X\, {}^*\varphi$*, in which* X *ranges over* ${}^*\mathbb{X}_i$*;*

(vii) if x *is a variable ranging over a sort* $\mathbb{X}_i$*, then the* $*$*-transform of the quantified formula* $\forall x\, \varphi$ *is* $\forall X\, {}^*\varphi$*, in which* X *ranges over* ${}^*\mathbb{X}_i$*.*

As before, this definition captures a simple idea: constructing the $*$-transform of a formula $\varphi(x_1, \dots, x_n)$ over $\mathbb{X}_I$ requires the following steps:

(a) Find all of the sets $A \subseteq \mathbb{X}^\alpha$ that occur in $\varphi(x_1, \dots, x_n)$ in basic formulas, and replace each such set by its counterpart ${}^*A$ over $({}^*\mathbb{X}_i)_{i\in I}$; similarly, replace each function $f\colon A \to B$ by ${}^*f$ and replace each element a of a sort $\mathbb{X}_i$ by ${}^*a$, and

(b) replace every variable x^i in $\varphi(x_1, \dots, x_n)$, including the ones that are used with quantifiers, by a corresponding variable X^i which ranges over ${}^*\mathbb{X}_i$.

4.27. Theorem. [Transfer Principle for I**-sets]** *Let* $\mathbb{X}_I$ *be an* I*-set and consider a fixed nonstandard extension of* $\mathbb{X}_I$*, based on the* I*-set* $({}^*\mathbb{X}_i)_{i\in I}$*.*

(a) Let $\varphi(x_1, \dots, x_m)$ *be a formula over* $\mathbb{X}_I$*; let* $\alpha(j)$ *be the index of the sort over which* x_j *ranges, for each* $j = 1, \dots, m$*; let* ${}^*\varphi(X_1, \dots, X_m)$ *be the* $*$*-transform of this formula. Suppose* $B \subseteq \mathbb{X}^\alpha$ *is the set defined by* $\varphi(x_1, \dots, x_m)$*:*

$$B = \{(x_1, \dots, x_m) \in \mathbb{X}^\alpha \mid \varphi(x_1, \dots, x_m) \text{ is true in } \mathbb{X}_I\}.$$

Then ${}^*B$ *is the set defined by* ${}^*\varphi(X_1, \dots, X_m)$*:*

$${}^*B = \{(X_1, \dots, X_m) \in ({}^*\mathbb{X})^\alpha \mid \varphi(X_1, \dots, X_m) \text{ is true in } ({}^*\mathbb{X}_i)_{i\in I}\}.$$

(b) Let φ *be any sentence over* $\mathbb{X}_I$*, and let* ${}^*\varphi$ *be its* $*$*-transform. Then*

$$\varphi \text{ is true in } \mathbb{X}_I \iff {}^*\varphi \text{ is true in } ({}^*\mathbb{X}_i)_{i\in I}.$$

4.28. Theorem. [Existence of Nonstandard Extensions] *Each nonempty* I*-set* $\mathbb{X}_I$ *has a proper nonstandard extension, in which the sets* $({}^*\mathbb{X}_i)_{i\in I}$ *may be taken to be ultrapowers of the sets* $(\mathbb{X}_i)_{i\in I}$ *with respect to a fixed countably incomplete ultrafilter.*

Proof. Let J be any infinite index set and let $\mathcal{U}$ be any countably incomplete ultrafilter on J. For each $i \in I$ let ${}^*\mathbb{X}_i$ be the ultrapower $\mathbb{X}_i^J/_{\mathcal{U}}$. For each finite sequence α from I and each set $A \subseteq \mathbb{X}^\alpha$, define ${}^*A$ by

$$ {}^*A = \{([\gamma_1], \dots, [\gamma_m]) \mid \{j \in J \mid (\gamma_1(j), \dots, \gamma_m(j)) \in A\} \in \mathcal{U}\}. $$

In this definition, for each $k = 1, \dots m$ we let γ_k range over the set $\mathbb{X}_{\alpha(k)}^J$ of all functions from J into $\mathbb{X}_{\alpha(k)}$, so that $[\gamma_k]$ denotes a typical element of the ultrapower $\mathbb{X}_{\alpha(k)}^J/\mathcal{U}$ The proof that this defines a proper nonstandard extension of $\mathbb{X}_I$ is similar to the proof of Theorem 2.28, and we leave the details to the reader as an Exercise. $\Box$

5. Nonstandard Extensions of the Multiset $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$

In this Section we will use the methods developed in Section 4 to give an indication of how to introduce higher type objects into the framework of nonstandard analysis. First we consider a non-empty set $\mathbb{X}$ and the collection $\mathcal{P}(\mathbb{X})$ of all subsets of $\mathbb{X}$. We regard this as a multiset $(\mathbb{X}_0, \mathbb{X}_1)$ indexed over a set of two elements, with $\mathbb{X}_0 = \mathbb{X}$ and $\mathbb{X}_1 = \mathcal{P}(\mathbb{X})$.

Consider an arbitrary nonstandard extension of $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$, which we denote as $({}^*\mathbb{X}, {}^*\mathcal{P}(\mathbb{X}))$. Let E be the restriction of the membership relation $\in$ to $\mathbb{X}$ and $\mathcal{P}(\mathbb{X})$:

$$ E = \{(x, A) \in \mathbb{X} \times \mathcal{P}(\mathbb{X}) \mid x \in A\}. $$

As usual, write $\mathcal{P}({}^*\mathbb{X})$ for the collection of all subsets of ${}^*\mathbb{X}$.

5.1. Remark. Without loss of generality we may assume the given nonstandard extension satisfies the following conditions:

(a) $\mathbb{X} \subseteq {}^*\mathbb{X}$ and ${}^*x = x$ for all $x \in \mathbb{X}$;

(b) ${}^*\mathcal{P}(\mathbb{X}) \subseteq \mathcal{P}({}^*\mathbb{X})$ and ${}^*E$ is the restriction of the usual membership relation to ${}^*\mathbb{X} \times {}^*\mathcal{P}(\mathbb{X})$:

$$ {}^*E = \{(x, Y) \in {}^*\mathbb{X} \times {}^*\mathcal{P}(\mathbb{X}) \mid x \in Y\}. $$

Justification. We show that every nonstandard extension of the multiset $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$ is isomorphic to a nonstandard extension which satisfies (a) and (b). First we carry out a step like the one in the justification of Remark 2.10. As done there, let $\mathbb{Y}$ be a suitable set and $h\colon {}^*\mathbb{X} \to \mathbb{Y}$ a bijection, chosen so that $\mathbb{X} \subseteq \mathbb{Y}$ and $h({}^*x) = x$ for all $x \in \mathbb{X}$. Moreover, given $Y \in {}^*\mathcal{P}(\mathbb{X})$, define

$$ \Phi(Y) = \{h(x) \mid x \in {}^*\mathbb{X} \text{ and } (x, Y) \in {}^*E\}, $$

which is a subset of $\mathbb{Y}$. It is easy to check that Φ is a 1-1 map on ${}^*\mathcal{P}(\mathbb{X})$. Finally, we define the new nonstandard extension of $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$ to ensure that the pair (h, Φ) of bijections is an isomorphism of nonstandard extensions. That is, in the new nonstandard extension we map each set $A \subseteq \mathbb{X}^m \times \mathcal{P}(\mathbb{X})^n$ to the set

$$\{(h(x_1), \ldots, h(x_m), \Phi(Y_1), \ldots, \Phi(Y_n)) \mid (x_1, \ldots, x_m, Y_1, \ldots, Y_n) \in {}^*A\}.$$

It is routine to check that this new mapping is a nonstandard extension of $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$ and that it satisfies conditions (a) and (b). □

In this kind of situation it is often convenient to suppress the explicit use of the maps h and Φ; rather we may follow a customary abuse of notation and "identify" ${}^*x$ with x for each $x \in \mathbb{X}$. With this understanding, the definition of $\Phi(Y)$ for each $Y \in {}^*\mathcal{P}(\mathbb{X})$ becomes

$$\Phi(Y) = \{x \in {}^*\mathbb{X} \mid (x, Y) \in {}^*E\}.$$

We may then identify Y with the subset $\Phi(Y)$ of ${}^*\mathbb{X}$ defined in this way. This is particularly convenient when (as later in this Section) the nonstandard extension has been constructed in an explicit way, such as we do here using the ultrapower construction.

For the rest of this Section we assume that we have a nonstandard extension of $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$ which satisfies (a) and (b) in Remark 5.1.

Condition (b) in Remark 5.1 ensures that the elements of ${}^*\mathcal{P}(\mathbb{X})$ are ordinary subsets of ${}^*\mathbb{X}$, and that the $*$-transform of any formula φ over $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$ is well behaved with respect to the membership relation. Suppose x is a variable ranging over $\mathbb{X}$ and y is a variable ranging over $\mathcal{P}(\mathbb{X})$, and suppose $x \in y$ occurs in φ. Recall that $x \in y$ is equivalent to the basic formula $(x, y) \in E$; the process of forming the $*$-transform will replace this basic formula by $(X, Y) \in {}^*E$, which is equivalent to $X \in Y$ according to condition (b). In other words, in forming the $*$-transform we may simply replace basic formulas of the form $x \in y$ by $X \in Y$, when the nonstandard extension satisfies (b).

Consider a subset A of $\mathbb{X}$. It can be considered either as a subset of $\mathbb{X}$ or as an element of $\mathcal{P}(\mathbb{X})$. Accordingly, there are two possible interpretations of the expression ${}^*A$: let us temporarily write $*(A)$ for the set which the given nonstandard extension assigns to A, and reserve the notation ${}^*A$ (as in paragraph 4.7) to denote the unique element of the set $*(\{A\})$ which this nonstandard extension assigns to $\{A\}$. Both of these are subsets of ${}^*\mathbb{X}$. Fortunately they are equal when we adopt the normalization described in Remark 5.1, as we now prove.

5.2. Proposition. *For each $A \subseteq \mathbb{X}$, we have $*(A) = {}^*A$.*

Proof. Fix $A \subseteq \mathbb{X}$ and let E be the restriction of the membership relation as above. Evidently we have that the sentence

$$\forall x \in \mathbb{X}\,[(x, A) \in E \leftrightarrow x \in A]$$

is true in our basic structure $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$. By the Transfer Principle (Theorem 4.27), we conclude that

$$\forall X \in {}^*\mathbb{X}\,[(X, {}^*A) \in *(E) \leftrightarrow X \in *(A)]$$

holds in the nonstandard extension. Using condition (b) in Remark 5.1, we see that

$$\forall X \in {}^*\mathbb{X}\,[X \in {}^*A \leftrightarrow X \in *(A)]$$

holds in the nonstandard extension. This proves $*(A) = {}^*A$ since both are subsets of ${}^*\mathbb{X}$. □

Next we introduce one of the key distinctions in nonstandard analysis: the distinction between *internal* and *external* subsets of ${}^*\mathbb{X}$.

5.3. Definition. [Internal Subset of ${}^*\mathbb{X}$] *A subset A of ${}^*\mathbb{X}$ is* **internal** *if it is an element of ${}^*\mathcal{P}(\mathbb{X})$; A is* **external** *if it is not internal.*

We note that it is only internal subsets of ${}^*\mathbb{X}$ that are referred to within the $*$-transform of a logical formula over $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$. That is, all variables in such a formula either range over ${}^*\mathbb{X}$ itself, or they range over ${}^*\mathcal{P}(\mathbb{X})$. If φ is a logical sentence over $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$ and ${}^*\varphi$ is its $*$-transform, it follows that we get the same truth value for ${}^*\varphi$ in the nonstandard extension $({}^*\mathbb{X}, {}^*\mathcal{P}(\mathbb{X}))$ as in the multiset $({}^*\mathbb{X}, \mathcal{P}({}^*\mathbb{X}))$. The same is true for logical formulas over $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$ into which we have substituted elements of ${}^*\mathbb{X}$ for all the free first order variables and *internal* subsets of ${}^*\mathbb{X}$ for all the free set variables. (This need *not* be true if we substitute *external* subsets of ${}^*\mathbb{X}$ for one or more of the free set variables in ${}^*\varphi$ and interpret it in $({}^*\mathbb{X}, \mathcal{P}({}^*\mathbb{X}))$.)

The next result is an easy consequence of the Transfer Principle, but it is a key tool for handling internal sets.

5.4. Theorem. [Internal Definition Principle] *Let $\varphi(x, x_1, \ldots, x_m, y_1, \ldots, y_n)$ be a formula over the multiset $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$. Suppose the variables x and x_j range over $\mathbb{X}$ for each j and the variable y_k ranges over $\mathcal{P}(\mathbb{X})$ for each k. Let $a_1, \ldots, a_m \in {}^*\mathbb{X}$ and let $A_1, \ldots, A_n$ be internal subsets of ${}^*\mathbb{X}$. Let B be the subset of ${}^*\mathbb{X}$ defined by ${}^*\varphi(X, a_1, \ldots, a_m, A_1, \ldots, A_n)$:*

$$B = \{X \in {}^*\mathbb{X} \mid {}^*\varphi(X, a_1, \ldots, a_m, A_1, \ldots, A_n) \text{ is true in } ({}^*\mathbb{X}, {}^*\mathcal{P}(\mathbb{X}))\}.$$

Then B is internal.

Proof. Apply the Transfer Principle (Theorem 4.27) to the sentence

$$\forall x_1 \ldots \forall x_m \forall y_1 \ldots \forall y_n \exists z \forall x \,[x \in z \leftrightarrow \varphi(x, x_1, \ldots, x_m, y_1, \ldots, y_n)],$$

which is true in the $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$; therefore the sentence

$$\forall X_1 \ldots \forall X_m \forall Y_1 \ldots \forall Y_n \exists Z \forall X \,[X \in Z \leftrightarrow {}^*\varphi(X, X_1, \ldots, X_m, Y_1, \ldots, Y_n)]$$

is true in the nonstandard extension $({}^*\mathbb{X}, {}^*\mathcal{P}(\mathbb{X}))$. Note that the variables X and $X_1, \ldots, X_m$ range over ${}^*\mathbb{X}$ and $Y_1, \ldots, Y_n$ are restricted to range over ${}^*\mathcal{P}(X)$. Substituting a_j for X_j for each $j = 1, \ldots, m$ and A_k for Y_k for each $k = 1, \ldots, n$ gives the desired result. Note that the substitution of A_k for Y_k is permitted only because A_k is assumed to be an internal subset of ${}^*\mathbb{X}$. This is a key aspect of the Internal Definition Principle. □

5.5. Exercise. Let A, B be internal subsets of ${}^*\mathbb{X}$.

(i) Every Boolean combination of A, B is internal.

(ii) If $f\colon \mathbb{X} \to \mathbb{X}$ is any function, then the sets $({}^*f)(A)$ and $({}^*f)^{-1}(A)$ are internal.

(iii) Every standard element of ${}^*\mathcal{P}(\mathbb{X})$ is an internal subset of ${}^*\mathbb{X}$.

Now we return to the setting in which $\mathbb{X} = \mathbb{R}$. Consider the linear ordering $<$ as a subset of $\mathbb{R}^2$ and the graphs Γ_+ and $\Gamma_\times$ of the functions $+$ and $\times$ as subsets of $\mathbb{R}^3$. By Proposition 4.11, ${}^*\Gamma_+$ and ${}^*\Gamma_\times$ are subsets of $({}^*\mathbb{R})^3$ which are graphs of functions from $({}^*\mathbb{R})^2$ to ${}^*\mathbb{R}$. For ease of notation, we will follow the customary practice of dropping the $*$ and denoting these functions as $+$ and $\times$. Using condition (a) in Remark 5.1 and Proposition 4.14, it follows that $+$ and $\times$ on $({}^*\mathbb{R})^2$ are extensions of the original functions $+$ and $\times$ on $\mathbb{R}^2$. Similarly, the relation $<$ on ${}^*\mathbb{R}$ is an extension of the given linear ordering $<$ on $\mathbb{R}$. Also, for any $A \subseteq \mathbb{R}$, A is easily seen to be a subset of ${}^*A$ using similar reasoning. The following Exercise can be fairly easily solved using the ideas above, especially including the Transfer Principle (Theorem 4.27) and the Internal Definition Principle (Theorem 5.4).

5.6. Exercise. (i) $({}^*\mathbb{R}, +, \times, <)$ is an ordered field extension of the ordered field $(\mathbb{R}, +, \times, <)$.

(ii) $\mathbb{N}$ is an initial segment of ${}^*\mathbb{N}$ and the elements of ${}^*\mathbb{N} \setminus \mathbb{N}$ are infinite numbers in ${}^*\mathbb{R}$.

(iii) For every positive $r \in {}^*\mathbb{R}$ there exists a unique $N \in {}^*\mathbb{N}$ such that $N \leq r < N + 1$.

(iv) If A is a non-empty internal subset of ${}^*\mathbb{R}$ which is bounded above in ${}^*\mathbb{R}$, then A has a least upper bound in ${}^*\mathbb{R}$; this need not be true if A is external.

(v) For each $N \in {}^*\mathbb{N}$, let $\{0, 1, \ldots, N\}$ denote the set of $M \in {}^*\mathbb{N}$ which satisfy $0 \leq M \leq N$; the set $\{0, 1, \ldots, N\}$ is internal.

(vi) For each $r < s$ in ${}^*\mathbb{R}$ let $[r, s]$ denote the set of all $t \in {}^*\mathbb{R}$ such that $r \le t \le s$; the set $[r, s]$ is internal.

(vii) The set ${}^*\mathbb{N} \setminus \mathbb{N}$ is **not** an internal subset of ${}^*\mathbb{R}$.

(viii) The set of infinitesimal elements of ${}^*\mathbb{R}$ is **not** an internal subset of ${}^*\mathbb{R}$.

(ix) The set of finite elements of ${}^*\mathbb{R}$ is **not** an internal subset of ${}^*\mathbb{R}$.

(x) (Overspill Principle) Let A be an internal subset of ${}^*\mathbb{R}$; if A contains arbitrarily large finite numbers, then it also contains an infinite positive number.

(xi) (Underspill Principle) Let A be an internal subset of ${}^*\mathbb{R}$; if A contains arbitrarily small positive infinite numbers, then it also contains a positive finite number.

We briefly indicate an expansion of this approach which allows treatment of internal functions between internal subsets of ${}^*\mathbb{X}$. To introduce such functions we consider the multiset $(\mathbb{X}, \mathcal{P}(\mathbb{X}), \mathcal{P}(\mathbb{X} \times \mathbb{X}))$; if $A, B \subseteq \mathbb{X}$ and $f\colon A \to B$ is a function, then we regard f as an element of this multiset by considering its graph Γ_f, which is an element of the third sort $\mathcal{P}(\mathbb{X} \times \mathbb{X}))$.

Consider an arbitrary nonstandard extension of $(\mathbb{X}, \mathcal{P}(\mathbb{X}), \mathcal{P}(\mathbb{X} \times \mathbb{X}))$, which we denote as $({}^*\mathbb{X}, {}^*\mathcal{P}(\mathbb{X}), {}^*\mathcal{P}(\mathbb{X} \times \mathbb{X}))$. Expanding on the discussion in Remark 5.1, we may pass to an isomorphic nonstandard extension which satisfies the following three conditions. We use the notation

$$E_1 = \{(x, A) \in \mathbb{X} \times \mathcal{P}(\mathbb{X}) \mid x \in A\};$$

$$E_2 = \{(x, y, A) \in \mathbb{X} \times \mathbb{X} \times \mathcal{P}(\mathbb{X} \times \mathbb{X}) \mid (x, y) \in A\}.$$

(a) $\mathbb{X} \subseteq {}^*\mathbb{X}$ and ${}^*x = x$ for all $x \in \mathbb{X}$;

(b) ${}^*\mathcal{P}(\mathbb{X}) \subseteq \mathcal{P}({}^*\mathbb{X})$ and ${}^*E_1$ is the restriction of the usual membership relation to ${}^*\mathbb{X} \times {}^*\mathcal{P}(\mathbb{X})$:

$${}^*E_1 = \{(x, Y) \in {}^*\mathbb{X} \times {}^*\mathcal{P}(\mathbb{X}) \mid x \in Y\}.$$

(c) ${}^*\mathcal{P}(\mathbb{X} \times \mathbb{X}) \subseteq \mathcal{P}({}^*\mathbb{X} \times {}^*\mathbb{X})$ and ${}^*E_2$ is the restriction of the usual ordered pairs membership relation to ${}^*\mathbb{X} \times {}^*\mathbb{X} \times {}^*\mathcal{P}(\mathbb{X} \times \mathbb{X})$:

$${}^*E_2 = \{(x, y, Y) \in {}^*\mathbb{X} \times {}^*\mathbb{X} \times {}^*\mathcal{P}(\mathbb{X} \times \mathbb{X}) \mid (x, y) \in Y\}.$$

Internal subsets of ${}^*\mathbb{X}$ are handled as was done earlier in this Section. Similarly, we call a subset of ${}^*\mathbb{X} \times {}^*\mathbb{X}$ *internal* if it is an element of ${}^*\mathcal{P}(\mathbb{X} \times \mathbb{X})$. If A, B are subsets of ${}^*\mathbb{X}$ and $f\colon A \to B$ is any function, we say that f is an *internal* function if its graph Γ_f is an internal subset of ${}^*\mathbb{X} \times {}^*\mathbb{X}$.

5.7. Exercise. Consider the multiset $(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R} \times \mathbb{R})$ and a nonstandard extension of it $({}^*\mathbb{R}, {}^*\mathcal{P}(\mathbb{R}), {}^*\mathcal{P}(\mathbb{R} \times \mathbb{R}))$, which has been normalized so that conditions (a), (b), and (c) above are satisfied. We adopt the notation described just before Exercise 5.6.

(i) If f is an internal function between subsets of ${}^*\mathbb{R}$, then the domain and range of f are internal subsets of ${}^*\mathbb{R}$.

(ii) If f is an internal function between subsets of ${}^*\mathbb{R}$ and A is an internal set contained in the domain of f, then the restriction of f to A is internal.

(iii) If f, g are internal functions between subsets of ${}^*\mathbb{R}$ and the domain of g contains the range of f, then the composition $g \circ f$ is internal.

(iv) Suppose $f\colon {}^*\mathbb{N} \to {}^*\mathbb{R}$ is an internal function; there exists a unique internal function $F\colon {}^*\mathbb{N} \to {}^*\mathbb{R}$ such that $F(0) = f(0)$ and for all $n \in {}^*\mathbb{N}$, $F(n+1) = F(n) + f(n+1)$.

5.8. Remark. Consider the setting of part (iv) in the previous Exercise. The function F can be viewed as the result of summing the values of f over initial segments of ${}^*\mathbb{N}$, and this is a useful idea in many applications of nonstandard analysis. For obvious reasons, it is customary to denote $F(n)$ for all $n \in {}^*\mathbb{N}$ (including nonstandard n) by

$$\sum_{i=0}^{n} f(i).$$

Such hyperfinite sums appear, for example, in the nonstandard approach to measure and integration.

5.9. Exercise. Suppose $f\colon {}^*\mathbb{N} \to {}^*\mathbb{R}$ and $g\colon {}^*\mathbb{N} \to {}^*\mathbb{R}$ are internal functions, and $c \in {}^*\mathbb{R}$. Consider the notation introduced in the previous Remark.

(i) For all $n \in {}^*\mathbb{N}$, $\sum_{i=0}^{n}(f(i) + g(i)) = \sum_{i=0}^{n} f(i) + \sum_{i=0}^{n} g(i)$.

(ii) For all $n \in {}^*\mathbb{N}$, $\sum_{i=0}^{n} c \cdot f(i) = c \cdot \sum_{i=0}^{n} f(i)$.

5.10. Exercise. Consider the multiset which has three sorts, $\mathbb{X}, \mathcal{P}(\mathbb{X})$, and $\mathcal{P}(\mathcal{P}(\mathbb{X}))$, and develop the ideas of this Section in that context. The nonstandard extension should be modified so that not only is ${}^*\mathcal{P}(\mathbb{X})$ a subset of $\mathcal{P}({}^*\mathbb{X})$, but also ${}^*\mathcal{P}(\mathcal{P}(\mathbb{X}))$ is a subset of $\mathcal{P}(\mathcal{P}({}^*\mathbb{X}))$, and so that the modified nonstandard extension preserves the restriction of the membership relation between $\mathbb{X}$ and $\mathcal{P}(\mathbb{X})$, as well as between $\mathcal{P}(\mathbb{X})$ and $\mathcal{P}(\mathcal{P}(\mathbb{X}))$. In this way all elements of ${}^*\mathcal{P}(\mathbb{X})$ and ${}^*\mathcal{P}(\mathcal{P}(\mathbb{X}))$ can be handled as sets in a canonical way.

The setting described in Exercise 5.10 provides a framework in which nonstandard methods can be applied to subsets of $\mathbb{X}$ as well as to collections of subsets of $\mathbb{X}$. This would be a suitable framework for applying nonstandard methods to the study of topologies on $\mathbb{X}$, for example, with the collection of open sets being an element of the third sort. This would also allow us to consider "internal topologies" on ${}^*\mathbb{X}$. These are internal collections $\mathcal{T}$

of (necessarily internal) subsets of ${}^*\mathbb{X}$ which satisfy the $*$-transform of the formula expressing the familiar defining conditions satisfied by topologies.

If we take the point of view of this Section to its natural limit, we get the type theoretic formulation of nonstandard analysis that Abraham Robinson used in his book [14]. Although this framework did not catch on at the time, that is likely due to the heavily formal presentation in [14] rather than to any essential disadvantages of this point of view. For an example of the use of such a framework for an important application of nonstandard methods, see the nonstandard proof due to van den Dries and Wilkie of Gromov's Theorem about groups of polynomial growth. (See [7], pages 356–363; they present their nonstandard extension explicitly as an ultrapower.)

We conclude this Section by discussing the nature of internal sets in the ultrapower nonstandard extensions that are constructed in the proofs of Theorems 2.28 and 4.28. We restrict our attention to nonstandard extensions of $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$, where $\mathbb{X}$ is any nonempty set. Let J be any infinite index set and let $\mathcal{U}$ be an ultrafilter on J. Let $({}^*\mathbb{X}, {}^*\mathcal{P}(\mathbb{X}))$ be the ultrapower nonstandard extension of $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$ that is constructed in the proof of Theorem 4.28.

Let a be any element of $\mathbb{X}$. We follow the customary practice of identifying a with the corresponding standard element ${}^*a$ of ${}^*\mathbb{X}$. As discussed in Exercise 2.29, this means we are identifying a with the equivalence class $[\alpha]$, where α is the constant function defined by $\alpha(j) = a$ for all $j \in J$.

Consider the effect of the normalization that is discussed in Remark 5.1. This mainly hinges on the behavior of the mapping Φ that is defined there. Let Y be an arbitrary element of ${}^*\mathcal{P}(\mathbb{X})$ and consider

$$\Phi(Y) = \{x \in {}^*\mathbb{X} \mid (x, Y) \in {}^*E\}.$$

In this setting Y is an equivalence class $[F]$ where F is a function from J into $\mathcal{P}(\mathbb{X})$; in other words, F is an indexed family of subsets of $\mathbb{X}$. Taking into account the definition of ${}^*E$ leads to the equation

$$\Phi([F]) = \{[\alpha] \mid \{j \in J \mid \alpha(j) \in F(j)\} \in \mathcal{U}\}.$$

Therefore, a subset A of ${}^*\mathbb{X}$ is internal (in the normalized version of the ultrapower nonstandard extension $({}^*\mathbb{X}, {}^*\mathcal{P}(\mathbb{X}))$ if and only if there is an indexed family $F\colon J \to \mathcal{P}(\mathbb{X})$ of subsets of $\mathbb{X}$ such that for all $[\alpha] \in {}^*\mathbb{X}$:

$$[\alpha] \in A \Longleftrightarrow \{j \in J \mid \alpha(j) \in F(j)\} \in \mathcal{U}.$$

5.11. Exercise. Consider the ultrapower nonstandard extension discussed in the preceding paragraphs.

(a) Let $F\colon J \to \mathcal{P}(\mathbb{X})$ be an indexed family of subsets of $\mathbb{X}$ with the property that $F(j)$ is nonempty for each $j \in J$. The internal subset of ${}^*\mathbb{X}$

determined as above by F can be identified with the ultraproduct $\Pi_{\mathcal{U}}(F(j) \mid j \in J)$.

(b) Every non-empty internal subset of ${}^*\mathbb{X}$ can be represented as described in (a).

6. Superstructures

In this Section we will explain a setting for nonstandard analysis which was introduced in [13] by Robinson and Zakon. This framework gives a convenient way to apply nonstandard methods to essentially any part of mathematics. Much of the research literature of nonstandard analysis is expressed in terms of the framework that is explained here. The essential ideas in this Section are just an easy elaboration of what was done in the previous Section.

In order to use nonstandard extensions effectively, they must be applicable to mathematical systems which contain objects of higher type, such as spaces of functions, collections of sets (such as filters), systems of open sets in a topological space, and the like. Such objects occur in essentially every part of mathematics, and our framework must accomodate them in a smooth way. Experience has shown that a convenient way to accomplish this is to introduce the *superstructure* based on a given set S of elementary mathematical objects. (In most applications it is natural to take $S = \mathbb{R}$ or $S = \mathbb{N}$. We will always assume that S contains $\mathbb{N}$ as a subset. The choice of S is otherwise somewhat arbitrary and depends on the mathematical problems that are being considered.) The elements of this superstructure are precisely the mathematical objects that can be obtained from S in a finite number of steps, where in each step we form all sets of the previously constructed objects and add each of these sets as a new object in its own right.

If T is a set, we write $\mathcal{P}(T)$ for the *power set* of T, which is the collection of all subsets of T.

6.1. Definition. **[Superstructure]** *Fix a set S such that $\mathbb{N} \subseteq S$. The* **superstructure based on** *S is the family of sets $(\mathbb{V}_k(S))_{k\in\mathbb{N}}$ defined by the following induction on k:*

$$\mathbb{V}_0(S) = S; \qquad \mathbb{V}_{k+1}(S) = \mathbb{V}_k(S) \cup \mathcal{P}(\mathbb{V}_k(S)).$$

This system is an $\mathbb{N}$-set in the terminology of Section 4; we denote it by $\mathbb{V}(S)$. An element of the union $\bigcup_{k=0}^{\infty} \mathbb{V}_k(S)$ is called an **object in** *$\mathbb{V}(S)$. The* **rank** *of an object a in $\mathbb{V}(S)$ is the smallest k for which $a \in \mathbb{V}_k(S)$.*

Note that

$$S = \mathbb{V}_0(S) \subseteq \mathbb{V}_1(S) \subseteq \mathbb{V}_2(S) \subseteq \dots$$

and hence also

$$\mathbb{V}_j(S) \in \mathbb{V}_k(S) \ \text{ whenever } j < k.$$

When we interpret the membership relation $\in$ in $\mathbb{V}(S)$, we treat members of S as having no elements. Note that the objects of rank ≥ 1 in $\mathbb{V}(S)$ are precisely the sets in $\mathbb{V}(S)$, and the basic objects (elements of S) are the objects of rank 0. The empty set $\emptyset$ has rank 1. If b is an object in $\mathbb{V}(S)$ of rank ≥ 1 and a is an element of b, then a is also an object in $\mathbb{V}(S)$. Note also that when a, b are objects in $\mathbb{V}(S)$, $a \in b$ always implies that the rank of a is strictly less than the rank of b.

We assume that the reader is familiar with a small amount of naive set theory. In particular, we form basic pairs within $\mathbb{V}(S)$ using the familiar definition $\langle x, y\rangle = \{\{x, y\}, \{x\}\}$. Note that the rank of $\langle x, y\rangle$ is $r + 2$ where r is the larger of the ranks of x, y. For each $n \geq 2$ we define the ordered n-tuple $(x_1, \ldots, x_n)$ to be the set $\{\langle i, x_i\rangle \mid i = 1, \ldots, m\}$. Recall that we require $\mathbb{N}$ to be a subset of the basic set S of the superstructure $\mathbb{V}(S)$; therefore $(x_1, \ldots, x_n)$ is an object in $\mathbb{V}(S)$ whenever $x_1, \ldots, x_n$ are objects in $\mathbb{V}(S)$. Moreover, the rank of $(x_1, \ldots, x_n)$ is $r + 3$ if r is the maximum of the ranks of $x_1, \ldots, x_n$. If A is a set of rank k in $\mathbb{V}(S)$ and $n \geq 2$, then A^n, taken to be the set of ordered n-tuples of elements of A, will be a set in $\mathbb{V}(S)$ and its rank will be $k + 3$. Note that this is independent of n. Similar remarks can be made about mixed Cartesian products.

We now want to develop a suitable concept of *nonstandard extension* for superstructures, based on regarding a superstructure as an $\mathbb{N}$-set and using the tools from Section 4. However, as in Section 5 some additional considerations arise from the fact that the sorts $\mathbb{V}_k(S)$ are not just independent sets but rather have a high degree of interrelation. We will work with nonstandard extensions that have been normalized in a way similar to that discussed in Remark 5.1. In the superstructure setting it is convenient to change perspective slightly and to work with rank preserving embeddings between superstructures.

Suppose $\mathbb{V}(S)$ and $\mathbb{V}(T)$ are superstructures and $F\colon \mathbb{V}(S) \to \mathbb{V}(T)$ is any rank preserving function. Let α be a finite sequence from $\mathbb{N}$ and suppose $A \subseteq \mathbb{V}(S)^\alpha = \mathbb{V}_{\alpha(1)}(S) \times \ldots \times \mathbb{V}_{\alpha(m)}(S)$. For large enough $k \in \mathbb{N}$ the set A is a set in $\mathbb{V}_k(S)$ so $F(A)$ is a well defined set in $\mathbb{V}(T)$. (Here $F(A)$ is the value of the function F at A, not to be confused with $\{F(a) \mid a \in A\}$.) It turns out that a good approach to defining nonstandard extensions of superstructures is to define ${}^*A$ to be $F(A)$ for every such A.

6.2. Definition. [Nonstandard Extension of a Superstructure] *Let $\mathbb{V}(S), \mathbb{V}(T)$ be superstructures and let $F\colon \mathbb{V}(S) \to \mathbb{V}(T)$ be a rank preserving function. Consider the mapping defined by letting ${}^*A = F(A)$ for each $A \subseteq \mathbb{V}(S)^\alpha$, where α is any finite sequence from $\mathbb{N}$. We say F is a* **nonstandard**

extension of $\mathbb{V}(S)$ (as a superstructure) *if $T = {}^*S$ and the following conditions are satisfied: (Note that ${}^*\mathbb{V}_k(S) \subseteq \mathbb{V}_k({}^*S)$ for each $k \in \mathbb{N}$, because $T = {}^*S$ and F is rank preserving.)*

(S1) This mapping is a nonstandard extension of $\mathbb{V}(S)$ (considered as the multiset $(\mathbb{V}_k(S))_{k\in\mathbb{N}}$ indexed by $\mathbb{N}$) in the sense of Definition 4.2.

(S2) If $a \in S$, then ${}^*a = a$; in particular $S \subseteq {}^*S$.

(S3) This mapping preserves the membership relation:

For each $k \in \mathbb{N}$ let E_k be the usual membership relation restricted to $\mathbb{V}_k(S)$,

$$E_k = \{(a, b) \in \mathbb{V}_k(S)^2 \mid a \in b\};$$

we require that ${}^*E_k$ is the restriction of the usual membership relation to ${}^*\mathbb{V}_k(S)$,

$${}^*E_k = \{(x, y) \in ({}^*\mathbb{V}_k(S))^2 \mid x \in y\}.$$

(S4) The nonstandard universe is transitive:

For each $k \in \mathbb{N}$, if $a \in {}^*\mathbb{V}_{k+1}(S)$ and $b \in a$, then $b \in {}^*\mathbb{V}_k(S)$.

The extra conditions (S2) – (S4) have a normalizing effect on the nonstandard extension and make it easier to work with. Moreover, if $F\colon \mathbb{V}(S) \to \mathbb{V}(T)$ satisfies only condition (S1), then the extra conditions (S2) – (S4) can be achieved by a series of simple modifications to F which are like those used in the justification of Remark 5.1.

For the remainder of this Section assume that $F\colon \mathbb{V}(S) \to \mathbb{V}({}^*S)$ satisfies the conditions in Definition 6.2. We will explore a few consequences of the Definition and then proceed to introduce some of the main ideas through which nonstandard extensions of superstructures are applied.

We will use the notation ${}^*\mathbb{V}(S)$ for the $\mathbb{N}$-set $({}^*\mathbb{V}_k(S))_{k\in\mathbb{N}}$. Note that ${}^*\mathbb{V}(S)$ is contained in the superstructure $\mathbb{V}({}^*S)$. This means that every object in ${}^*\mathbb{V}(S)$ is a mathematical object of the usual kind, and that it lies at some finite level of higher type objects over the set ${}^*S$. When working with the sets in ${}^*\mathbb{V}(S)$ from the *outside* so to speak, this means that we can regard them as ordinary mathematical objects, to which all of the usual mathematical concepts can be applied. In particular, we can speak of the cardinality (finite, infinite, countable, uncountable, etc) of each set A in ${}^*\mathbb{V}(S)$. This plays a useful role in many applications of nonstandard analysis. When needed for clarity, we will refer to the *external* cardinality of A when we are making use of this point of view.

The set theoretic nature of superstructures means we need to be careful when interpreting the definition of nonstandard extension and when applying the Transfer Principle. An element of $\mathbb{V}_k(S) \setminus V_0(S)$ is simultaneously (1) an element of the sort $V_k(S)$ and (2) a subset of possibly many different Cartesian products of sorts $\mathbb{V}(S)^\alpha$. Under each of these interpretations there is a separate definition of the expression ${}^*a$. In all of the cases under

(2), ${}^*a$ is taken to be the unique element $F(a)$ by definition. In case (1), we know that ${}^*a$ is the unique element of $F(\{a\})$, as defined in paragraph 4.7 and justified by Proposition 4.6. But in fact there is no ambiguity, as follows from condition (S3) and a proof like that given for Proposition 5.2. All of these interpretations of the notation ${}^*a$ refer to the same object.

Suppose x, y are variables ranging over $\mathbb{V}_k(S)$ of the kind that occur in applications of the Transfer Principle. In this context it is permissible to use $x \in y$ as a basic formula, since it is equivalent to the basic formula $(x, y) \in E_k$. The $*$-transform of $(x, y) \in E_k$ is defined to be $(X, Y) \in {}^*E_k$, where X, Y are variables ranging over ${}^*\mathbb{V}_k(S)$. However, condition (S3) implies that this is equivalent to $X \in Y$. In other words, if we use $x \in y$ in a formula φ over $\mathbb{V}(S)$, and we want to construct the $*$-transform of φ in order to apply the Transfer Principle, then we simply modify the formula $x \in y$ to the formula $X \in Y$. Precisely the same thing is true when the variables x, y do not necessarily have the same rank.

When applying the Transfer Principle to nonstandard extensions of superstructures, it is common to use *bounded* quantifiers. These are relativized quantifiers of the form $\forall x \in a$ and $\exists x \in a$, where a is a set in $\mathbb{V}(S)$. Here we can take x to be a variable ranging over $\mathbb{V}_k(S)$, where k is chosen to be at least as large as the ranks of the elements of a. It is easy to interpret these quantifiers in terms of the ones we have been using, and thus determine what to do with them when applying the Transfer Principle. For example, consider a formula of the form $\forall x \in a\, \varphi$; this is equivalent to $\forall x \in \mathbb{V}_k(S)\, [x \in a \rightarrow \varphi]$. The $*$-transform of this formula is $\forall X \in {}^*\mathbb{V}_k(S)\, [X \in {}^*a \rightarrow {}^*\varphi]$, which is in turn equivalent to $\forall X \in {}^*a\, [{}^*\varphi]$. (Here we used the fact that all elements of ${}^*a$ are in ${}^*\mathbb{V}_k(S)$.) Similarly we can take the $*$-transform of $\exists x \in a\, \varphi$ to be $\exists X \in {}^*a\, [{}^*\varphi]$.

It is also possible to use bounded quantifiers in which two variables appear: $\forall x \in y$ and $\exists x \in y$. Recall that x and y must be variables that range over specific levels of $\mathbb{V}(S)$; say x ranges over $\mathbb{V}_k(S)$. Then $\exists x \in y\, \varphi$ is equivalent to $\exists x \in \mathbb{V}_k(S)\, [x \in y \wedge \varphi]$, which is a formula we already know how to handle. Similarly we rewrite $\forall x \in y\, \varphi$ as $\forall x \in \mathbb{V}_k(S)\, [x \in y \rightarrow \varphi]$.

Formulas such as $z = \{\{x, y\}, \{x\}\}$ and $z = (x_1, \dots, x_n)$ can easily be expressed in superstructures using simple logical formulas in which only the membership relation $\in$ and bounded quantifiers occur. Therefore, when constructing the $*$-transform of a formula in which such basic formulas occur, they are unchanged except for the fact that the variables $x, y, z, x_1, \dots, x_n$ are modified to range over the sorts ${}^*\mathbb{V}_k(S)$ for appropriate k.

Finally, if a, b are subsets of $\mathbb{V}_k(S)$ and $f\colon a \rightarrow b$ is a function, the condition $f(x) = y$ can also be expressed using a formula over $\mathbb{V}(S)$ in which only the membership relation $\in$ and bounded quantifiers are used. (As in the previous paragraph, x and y can also be replaced by ordered tuples.) Indeed, if we take x, y to be variables ranging over the sort $\mathbb{V}_k(S)$, then $f(x) = y$ is expressed by

$$\exists z \in \mathbb{V}_{k+3}(S)\,[z = (x, y) \wedge z \in f]$$

which is a bounded formula over $\mathbb{V}(S)$. The $*$-transform of this formula is equivalent to

$$\exists Z \in {}^*\mathbb{V}_{k+3}(S)\,[Z = (X, Y) \wedge Z \in {}^*f]$$

and this is a formula over ${}^*\mathbb{V}(S)$ which expresses the condition ${}^*f(X) = Y$.

6.3. Exercise. Let $a_1, \ldots, a_n$ be in $\mathbb{V}(S)$.

(i) ${}^*\{a_1, \ldots, a_n\} = \{{}^*a_1, \ldots, {}^*a_n\}$.

(ii) ${}^*(a_1, \ldots, a_n) = ({}^*a_1, \ldots, {}^*a_n)$.

6.4. Exercise. Let $a, b, c, d, a_1, \ldots, a_n, f, r$ be sets in $\mathbb{V}(S)$.

(i) $a \in b \Longleftrightarrow {}^*a \in {}^*b$.

(ii) $a = b \Longleftrightarrow {}^*a = {}^*b$.

(iii) $a \subseteq b \Longleftrightarrow {}^*a \subseteq {}^*b$.

(iv) ${}^*(a \cup b) = {}^*a \cup {}^*b$, ${}^*(a \cap b) = {}^*a \cap {}^*b$, and ${}^*(a \setminus b) = ({}^*a) \setminus ({}^*b)$.

(v) ${}^*(a_1 \times \ldots \times a_n) = {}^*a_1 \times \ldots \times {}^*a_n$.

(vi) f is a function from a to $b \Longleftrightarrow {}^*f$ is a function from ${}^*a$ to ${}^*b$.

(vii) r is a relation on $a \times b \Longleftrightarrow {}^*r$ is a relation on ${}^*a \times {}^*b$; if these conditions are true, and if c is the domain of r (projection on the first coordinate) and d is the range of r (projection on the second coordinate), then ${}^*c$ is the domain of ${}^*r$ and ${}^*d$ is the range of ${}^*r$.

The arguments needed to solve these Exercises are simple applications of the Transfer Principle (Theorem 4.27).

In the next definition we introduce one of the most important concepts in the superstructure framework; this is a natural extension of what was done in Section 5 (Definition 5.3):

6.5. Definition. [Internal Object in $\mathbb{V}({}^*S)$] *An object in $\mathbb{V}({}^*S)$ is* **internal** *if there exists $k \in \mathbb{N}$ such that $a \in {}^*\mathbb{V}_k(S)$. Therefore, the collection of internal objects is transitive: a internal and $b \in a$ implies b internal. An object in $\mathbb{V}({}^*S)$ is* **external** *if it is not internal.*

6.6. Proposition. *Let a be in $\mathbb{V}({}^*S)$; a is internal if and only if it is an element of some standard set in $\mathbb{V}({}^*S)$. That is, a is internal if there exists b in $\mathbb{V}(S)$ such that $a \in {}^*b$.*

Proof. If a is internal, then $a \in {}^*\mathbb{V}_k(S)$ by definition, so a is an element of a standard set. Conversely, suppose $b \in \mathbb{V}(S)$ and $a \in {}^*b$. This implies b is a non-empty set so there exists $k \in \mathbb{N}$ with $b \subseteq \mathbb{V}_k(S)$. But then $a \in {}^*b \subseteq {}^*\mathbb{V}_k(S)$, and we are done. □

Suppose φ is a logical formula over $\mathbb{V}(S)$ and let ${}^*\varphi$ be its $*$-transform. Observe that each quantified variable in ${}^*\varphi$ is restricted to range over the elements of ${}^*\mathbb{V}_k(S)$ for some k. Therefore the quantified variables in ${}^*\varphi$ range only over *internal* elements. This means that if the free variables of ${}^*\varphi$ are taken to stand for internal objects in $\mathbb{V}({}^*S)$, then we will get the same truth value if we interpret ${}^*\varphi$ in the full superstructure $\mathbb{V}({}^*S)$ as if we evaluate it in the nonstandard extension ${}^*\mathbb{V}(S)$. External objects simply do not enter into the picture when we evaluate whether or not ${}^*\varphi$ is true in $\mathbb{V}({}^*S)$.

The following result is an easy consequence of the Transfer Principle (Theorem 4.27) applied to nonstandard extensions of superstructures. Nonetheless, it is an important tool in applications of nonstandard methods.

6.7. Theorem. **[Internal Definition Principle]** *Let $\varphi(x_1, \dots, x_m, y_1, \dots, y_n)$ be a formula over $\mathbb{V}(S)$. Suppose the variable x_j ranges over $\mathbb{V}_{\alpha(j)}(S)$ for each j and the variable y_k ranges over $\mathbb{V}_{\beta(k)}(S)$ for each k. Let $a_1, \dots, a_n$ be internal objects in $\mathbb{V}({}^*S)$, with $a_j \in {}^*\mathbb{V}_{\alpha(j)}(S)$ for each j. Let b be the set in $\mathbb{V}({}^*S)$ defined by ${}^*\varphi(X_1, \dots, X_m, a_1, \dots, a_n)$:*

$$b = \{(X_1, \dots, X_m) \in {}^*\mathbb{V}(S)^\alpha \mid {}^*\varphi(X_1, \dots, X_m, a_1, \dots, a_n) \textit{ holds in } {}^*\mathbb{V}(S)\}.$$

Then b is internal.

Proof. This proof follows the same line of argument as the proof of Theorem 5.4. □

6.8. Remark. Note the requirement in the Internal Definition Principle that the objects $a_1, \dots, a_n$ are internal. This is very important. A very common mistake when using nonstandard analysis is to misapply the Internal Definition Principle in a situation where some of the objects $a_1, \dots, a_n$ are external.

6.9. Exercise. Let $a_1, \dots, a_m$ be internal objects in $\mathbb{V}({}^*S)$.

(i) $\{a_1, \dots, a_n\}$ is internal.

(ii) $(a_1, \dots, a_n\rangle)$ is internal.

(iii) Every standard set in $\mathbb{V}({}^*S)$ is internal.

6.10. Exercise. Let $a, b, a_1, \dots, a_n, f, r$ be internal sets in $\mathbb{V}({}^*S)$.

(i) Every Boolean combination of a, b is internal.

(ii) Every element of a is internal.

(iii) $a_1 \times \dots \times a_n$ is internal.

(iv) If r is a relation on $a \times b$, then the domain of r and the range of r are internal.

(v) The union of all members of a and the intersection of all members of a are both internal.

(vi) The collection of all internal subsets of a is internal.

An important example of an internal concept is the notion of *hyperfinite* set. These are internal sets in $\mathbb{V}({}^*S)$ which obey all the formally expressible properties of finite sets. As a result, they can be handled using ideas of combinatorial and discrete mathematics. However, when viewed from outside, they may be infinite sets, and may share many qualitative features of continuous objects of mathematics. Many important applications of nonstandard analysis depend on the use of hyperfinite sets. The definition of *hyperfinite* is also a model for the introduction of many interesting concepts for internal sets.

6.11. Definition. [Hyperfinite Sets] *Let F_k be the collection of all finite sets in $\mathbb{V}_k(S)$. A set a in $\mathbb{V}({}^*S)$ is* **hyperfinite** *(equivalently* $*$**-finite***) if $a \in {}^*F_k$ for some $k \in \mathbb{N}$.*

6.12. Remark. Note that according to Proposition 6.6, every hyperfinite set is internal, since ${}^*F_k$ is a standard set for each k.

6.13. Notation. See the discussion before Exercise 5.6 for an explanation of the meaning of the relation $<$ and the functions $+$ and $\times$ on ${}^*\mathbb{R}$. Given $N \in {}^*\mathbb{N}$, we write $\{0, 1, \ldots, N\}$ for the set $\{a \in {}^*\mathbb{N} \mid 0 \leq a \leq N\}$. Note that $\{0, 1, \ldots, N\}$ is an infinite set if N is an infinite number in ${}^*\mathbb{N}$.

6.14. Exercise. (i) Every finite set in ${}^*\mathbb{V}(S)$ is hyperfinite.

(ii) For each $N \in \mathbb{N}$ the set $\{0, 1, \ldots, N\}$ is hyperfinite; ${}^*\mathbb{N}$ is not hyperfinite.

(iii) If a is a set in $\mathbb{V}(S)$ and ${}^*a$ is hyperfinite, then a is a finite set and ${}^*a = \{{}^*b \mid b \in a\}$.

6.15. Exercise. A set a in $\mathbb{V}({}^*S)$ is hyperfinite if and only if there exists an internal bijection between a and $\{0, 1, \ldots, N-1\}$ for some $N \in \mathbb{N}$. This N, if it exists, is unique.

6.16. Definition. *If a is a hyperfinite set in $\mathbb{V}({}^*S)$, the unique $N \in \mathbb{N}$ such that there exists an internal bijection between a and $\{0, 1, \ldots, N-1\}$ is called* **the internal cardinality of** *a.*

6.17. Exercise. Let $a, b, a_1, \ldots, a_n, f, r$ be hyperfinite sets in $\mathbb{V}({}^*S)$.

(i) Every Boolean combination of a, b is hyperfinite.

(ii) $a_1 \times \ldots \times a_n$ is hyperfinite.

(iii) If r is a relation on $a \times b$, then the domain of r and the range of r are hyperfinite.

(iv) If every member of a is a hyperfinite set, then the union of all members of a and the intersection of all members of a are both hyperfinite.

(v) The collection of all internal subsets of a is hyperfinite.

(vi) Every internal subset of a is hyperfinite, and its internal cardinality is $\leq$ the internal cardinality of a.

(vii) Suppose S contains $\mathbb{R}$; if a is a hyperfinite subset of ${}^*\mathbb{R}$ and N is the internal cardinality of a, then there is an internal increasing bijection from $\{0, 1, \ldots, N-1\}$ onto a.

6.18. Exercise. Suppose S contains $\mathbb{R}$. Let A be the set of all (α, N) where $N \in {}^*\mathbb{N}$ and α is an internal function from $\{0, 1, \ldots, N\}$ into ${}^*\mathbb{R}$. See part (iv) of Exercise 5.7 and Exercise 5.9.

(i) A is an internal set;

(ii) there is a unique internal function $\Sigma \colon A \to {}^*\mathbb{R}$ such that for all $(\alpha, N) \in A$

$$\Sigma(\alpha, N) = \sum_{k=0}^{N} \alpha(k).$$

6.19. Remark. If a is an internal set in ${}^*\mathbb{V}(S)$, we let ${}^*\mathcal{P}(a)$ denote the set of all internal subsets of a. By part (vi) of Exercise 6.10 we know that ${}^*\mathcal{P}(a)$ is an internal set in ${}^*\mathbb{V}(S)$; it is called the *internal power set of a*. Using part (i) of the same Exercise, we see that ${}^*\mathcal{P}(a)$ is closed under finite Boolean operations. Since it is an internal set, this implies that ${}^*\mathcal{P}(a)$ is actually closed under hyperfinite unions and intersections. Such internal Boolean algebras of sets are very important in the construction of Loeb measures.

6.20. Exercise. Let A be a set in $\mathbb{V}(S)$. Then $({}^*A, {}^*\mathcal{P}(A), {}^*\mathcal{P}(A \times A))$ is a nonstandard extension of $(A, \mathcal{P}(A), \mathcal{P}(A \times A))$, and it satisfies the normalizing assumptions (b) and (c) given above just before Exercise 5.7.

For a more complete discussion of superstructures and more complete proofs of many facts about their nonstandard extensions, the reader may consult the textbook [8]; see also [1] [2] [4] [9] [11] [13] and [15].

A completely different set theoretic foundation for nonstandard analysis, *Internal Set Theory (IST)*, was introduced by Nelson in [12]. It is based on nonstandard models for the full ZFC axioms for the foundations of mathematics. (ZFC = Zermelo Fraenkel axioms for set theory with the Axiom of Choice.)

7. Saturation

For most applications, especially those in topology and abstract analysis, it is necessary to work with nonstandard extensions which satisfy richness conditions stronger than nontriviality or properness. (See Definitions 2.21

and 4.21.) The most useful of the extra hypotheses are the *saturation* conditions, which were carried over from model theory to nonstandard analysis by Luxemburg [11].

For this Section we fix a superstructure $\mathbb{V}(S)$ and a nonstandard extension ${}^*\mathbb{V}(S)$ of it. Recall that a family F of sets is said to have the *finite intersection property* if each intersection of a finite subcollection of F is non-empty. We let κ stand for an uncountable cardinal number.

7.1. Definition. *The given nonstandard extension is* κ**-saturated** *if it satisfies the following condition: let $\mathcal{F}$ be a (possibly external) family of internal sets; if $\mathcal{F}$ has (external) cardinality strictly less than κ and $\mathcal{F}$ has the finite intersection property, then the total intersection of $\mathcal{F}$ is non-empty. (The total intersection of $\mathcal{F}$ is the set $\{a \in \mathbb{V}({}^*S) \mid a \in b \text{ for all } b \in \mathcal{F}\}$. Of course this set may be external.)*

The following result gives an alternate formulation of κ-saturation. It is expressed in terms of simultaneous satisfiability of conditions, each expressed by formulas over ${}^*\mathbb{V}(S)$, in which only internal objects are allowed.

7.2. Theorem. *Let ${}^*\mathbb{V}(S)$ be a κ-saturated nonstandard extension of the superstructure $\mathbb{V}(S)$, where κ is an uncountable cardinal number. Let J be an index set of cardinality $< \kappa$. Let a be an internal set in ${}^*\mathbb{V}(S)$. For each $j \in J$, let $\varphi_j(X)$ be a formula over ${}^*\mathbb{V}(S)$, so all objects mentioned in $\varphi_j(X)$ are internal. Further, suppose that the set of formulas $\{\varphi_j(X) \mid j \in J\}$ is finitely satisfied in a; this means that for every finite subset α of J there exists some $c \in a$ (which may depend on α) such that $\varphi_j(c)$ holds in ${}^*\mathbb{V}(S)$ for all $j \in \alpha$. Then there exists $c \in a$ such that $\varphi_j(c)$ holds in ${}^*\mathbb{V}(S)$ simultaneously for all $j \in J$.*

Proof. For each $j \in J$, let f_j be the subset of a that is defined by $\varphi_j(X)$; that is,

$$f_j = \{c \in a \mid \varphi_j(c) \text{ is true in } {}^*\mathbb{V}(S)\}.$$

The Internal Definition Principle implies that each f_j is an internal subset of a. The hypotheses imply that the collection $\{f_j \mid j \in J\}$ has the finite intersection property. Since the cardinality of J is $< \kappa$, the fact that our nonstandard extension is κ-saturated implies that the total intersection $\cap\{f_j \mid j \in J\}$ is nonempty. Any element of this intersection satisfies the conclusion of the Theorem. □

Of special importance for most applications is $\aleph_1$-saturation, where $\aleph_1$ denotes the first uncountable cardinal number. This means that whenever $\mathcal{F}$ is a countable collection of internal sets and $\mathcal{F}$ has the finite intersection property, then the total intersection of $\mathcal{F}$ is non-empty. It is customary in nearly all research articles in nonstandard analysis to assume that the nonstandard extensions being used are at least $\aleph_1$-saturated. It is this hypothesis that ensures, for example, that Loeb measures are σ-additive and that nonstandard hulls of metric spaces are complete. In some areas, especially in topology, an even stronger hypothesis of κ-saturation is needed for many applications. For example, in order to give a smooth treatment of a topological space T using the methods of nonstandard analysis, it is usually necessary to assume that the nonstandard extension is κ-saturated where κ is strictly larger than the number of open subsets of T.

Note that the saturation hypotheses can also be applied to the simpler nonstandard extensions treated in Section 5.

7.3. Proposition. *Assume that the nonstandard extension is κ-saturated. Every infinite internal set in $\mathbb{V}({}^*S)$ has (external) cardinality $\geq \kappa$.*

Proof. Suppose otherwise, that a is an infinite internal set of cardinality strictly less than κ. Let $\mathcal{F}$ be the collection of all sets of the form $a \setminus \{x\}$ as x ranges over a. Then $\mathcal{F}$ is a collection of internal sets, and the cardinality of $\mathcal{F}$ is less than κ. Moreover, $\mathcal{F}$ obviously has the finite intersection property, since a is infinite. But the total intersection of $\mathcal{F}$ is obviously empty; this contradicts the hypothesis that the nonstandard extension is κ-saturated. $\square$

7.4. Proposition. *Assume that the nonstandard extension is κ-saturated. Let a be an internal set in $\mathbb{V}({}^*S)$. Let A be a (possibly external) subset of a such that A has cardinality strictly less than κ. Then there exists a hyperfinite subset b of a such that b contains A as a subset.*

Proof. For each $x \in A$, let F_x denote the set of all hyperfinite subsets of a which contain x as an element. The Internal Definition Principle (Theorem 6.7) yields that each F_x is an internal set in $\mathbb{V}({}^*S)$. Let $\mathcal{F}$ be the collection of all the sets F_x as x ranges over A. Obviously $\mathcal{F}$ has cardinality strictly less than κ. Moreover, $\mathcal{F}$ has the finite intersection property: given finitely many elements $x_1, \ldots, x_n$ from A, the set $\{x_1, \ldots, x_n\}$ is hyperfinite and is an element of F_{x_j} for all $j = 1, \ldots, n$. Since our nonstandard extension is κ-saturated, there exists an object b which is an element of F_x for every $x \in A$. This b is therefore the desired hyperfinite set. $\square$

7.5. Theorem. [Saturated Extensions are Comprehensive] *Assume that the nonstandard extension is κ-saturated. Let a and b be internal sets in $\mathbb{V}({}^*S)$. Let A be a (possibly external) subset of a such that A has cardinality strictly less than κ and suppose that $f\colon A \to b$ is a (possibly external) function. Then there exists an internal function $g\colon a \to b$ such that g is an extension of f. In particular, if $\{c_k \mid k \in \mathbb{N}\}$ is a (possibly external) sequence of elements of b, then there exists an internal function $g\colon {}^*\mathbb{N} \to b$ such that $g(k) = c_k$ for all $k \in \mathbb{N}$.*

Proof. For each $x \in A$ let F_x be the set of all internal functions $g\colon a \to b$ which satisfy $g(x) = f(x)$. The Internal Definition Principle (Theorem 6.5) implies that each F_x is internal. Let $\mathcal{F}$ be the collection of all F_x as x ranges over A. Obviously $\mathcal{F}$ has cardinality strictly less than κ. Moreover, $\mathcal{F}$ has the finite intersection property: given finitely many elements $x_1, \ldots, x_n$ from A, consider the function $g\colon a \to b$ which takes x_j to $f(x_j)$ for $j = 1, \ldots, n$ and which takes all other elements of a to (say) $f(x_1)$. The Internal Definition Principle implies that this function is internal, and it is obviously an element of F_{x_j} for all $j = 1, \ldots, n$. From the fact that our nonstandard model is assumed to be κ-saturated, it follows that there is an object g which is an element of F_x for all $x \in A$. This g is the desired internal function from a to b. □

7.6. Exercise. The following conditions are equivalent:

(a) The nonstandard extension is $\aleph_1$-saturated.

(b) (Countable Comprehension Property) Whenever b is an internal set in $\mathbb{V}({}^*S)$ and $(c_k)_{k\in\mathbb{N}}$ is a (possibly external) sequence of elements of b, then there exists an internal function $g\colon {}^*\mathbb{N} \to b$ such that $g(k) = c_k$ for all $k \in \mathbb{N}$.

7.7. Exercise. Assume the nonstandard extension is $\aleph_1$-saturated and let a be an internal set in $\mathbb{V}({}^*S)$. A (possibly external) subset b of a is a Σ^0_1 set if there exists a sequence $\{c_k \mid k \in \mathbb{N}\}$ of internal subsets of a such that $b = \bigcup\{c_k \mid k \in \mathbb{N}\}$. Similarly, b is a Π^0_1 set if there exists a sequence $\{c_k \mid k \in \mathbb{N}\}$ of internal subsets of a such that $b = \bigcap\{c_k \mid k \in \mathbb{N}\}$. A Σ^0_1 set is sometimes called a *galaxy*, and a Π^0_1 set is sometimes called a *monad* or a *halo*.

(a) Suppose $\{c_k \mid k \in \mathbb{N}\}$ is a sequence of internal subsets of a such that b_1 is the Σ^0_1 set $\bigcup\{c_k \mid k \in \mathbb{N}\}$. If b_1 is internal, then there exists $N \in \mathbb{N}$ such that $b_1 = c_1 \cup \ldots \cup c_N$. (Hint: without loss of generality the sequence $\{c_k \mid k \in \mathbb{N}\}$ is the restriction to $\mathbb{N}$ of an internal increasing sequence $\{c_k \mid k \in {}^*\mathbb{N}\}$ of subsets of a. If $b_1 = \bigcup\{c_k \mid k \in \mathbb{N}\}$ is internal, then the set of $N \in {}^*\mathbb{N}$ such that $b_1 \subseteq c_1 \cup \ldots \cup c_N$ is internal and contains all infinite N. By the Underspill Principle, Exercise 5.6, there exists a finite $N \in \mathbb{N}$ such that $b_1 \subseteq c_1 \cup \ldots \cup c_N$.)

(b) Suppose $\{d_k \mid k \in \mathbb{N}\}$ is a sequence of internal subsets of a such that b_2 is the Π^0_1 set $\bigcap\{d_k \mid k \in \mathbb{N}\}$. If b_2 is internal, then there exists $N \in \mathbb{N}$ such that $b_2 = d_1 \cap \ldots \cap d_N$.

(c) Suppose b_1 is a Σ^0_1 subset of a represented as in (a) and b_2 is a Π^0_1 subset of a represented as in (b). If $b_1 \subseteq b_2$ then there is an internal set e such that $b_1 \subseteq e \subseteq b_2$. In particular, if $b_1 = b_2$, then $b_1 (= b_2)$ is an internal set.

7.8. Exercise. This improves on Proposition 7.3 when $\kappa = \aleph_1$. Assume the nonstandard extension is $\aleph_1$-saturated. Every infinite internal set has (external) cardinality $\geq 2^{\aleph_0}$. (Hint: without loss of generality the infinite internal set is of the form $\{0, 1, \ldots, N\}$ where N is an infinite element of ${}^*\mathbb{N}$. For each standard real number r in the interval $0 < r < 1$ show that there is a smallest element k of ${}^*\mathbb{N}$ such that $N \cdot {}^*r \leq k$. Obviously $1 \leq k \leq N$. Moreover, k is uniquely determined by r.)

It is useful to introduce two additional richness properties of nonstandard extensions:

7.9. Definition. *(a) A nonstandard extension of* $\mathbb{V}(S)$ *is* **polysaturated** *if it is* κ*-saturated for some* κ *greater than or equal to the number of objects in* $\mathbb{V}(S)$.

(b) A nonstandard extension of $\mathbb{V}(S)$ *is an* **enlargement** *if for each set* a *in* $\mathbb{V}(S)$ *there exists a hyperfinite set* $b \subseteq {}^*a$ *such that* b *contains every standard element of* ${}^*a$*; that is, we require* $\{{}^*c \mid c \in a\} \subseteq b \subseteq {}^*a$.

7.10. Exercise. The following conditions are equivalent:

(a) The nonstandard extension is an enlargement.

(b) If $k \in \mathbb{N}$ and $\mathcal{F}$ is a collection of subsets of $\mathbb{V}_k(S)$ with the finite intersection property, then $\bigcap\{{}^*a \mid a \in \mathcal{F}\}$ is nonempty.

(c) If $k \in \mathbb{N}$ and $(L, \leq)$ is a partially ordered set in $\mathbb{V}_k(S)$ which is directed upwards, then there exists $b \in {}^*L$ such that for all $a \in L$, ${}^*a \; {}^*\!\!\leq b$.

7.11. Exercise. Every polysaturated nonstandard extension of $\mathbb{V}(S)$ is an enlargement.

Next we give a proof using ultrapowers of the existence of enlargements.

7.12. Theorem. *Every superstructure* $\mathbb{V}(S)$ *has a nonstandard extension which is an enlargement; it can be taken to be an ultrapower extension of* $\mathbb{V}(S)$ *with respect to a suitably chosen ultrafilter.*

Proof. Let J be the collection of all nonempty finite sets of objects from $\mathbb{V}(S)$. Let $\mathcal{U}$ be an ultrafilter on J such that for each object a in $\mathbb{V}(S)$ the set

$$\{j \mid j \text{ is a finite set of objects from } \mathbb{V}(S) \text{ and } a \in j\}$$

is in $\mathcal{U}$. Such an ultrafilter exists because the collection of all these subsets of J has the finite intersection property. Consider the nonstandard extension of $\mathbb{V}(S)$ constucted as an ultrapower using $\mathcal{U}$ as in the proof of Theorem 4.28. We will show that this is an enlargement of $\mathbb{V}(S)$.

Fix a set a from $\mathbb{V}(S)$. For each $j \in J$ let $F(j) = a \cap j$ and let $b = [F]$ be the element of the nonstandard extension that is determined by F. Note that if a has rank k, then all of the values of F are in $\mathbb{V}_k(S)$ so b is an element of ${}^*\mathbb{V}_k(S)$. Since every $F(j)$ is a finite subset of a, it follows that b

is a hyperfinite subset of $^*a$. It remains to show that every standard element of $^*a$ is an element of b. Let $c \in a$. As in Exercise 2.29 $^*c$ is the equivalence class $[\alpha]$ where α is the constant function with value c at each argument in J. To show that $^*c \in b$ we must show that the set $\{j \in J \mid c \in F(j)\}$ is in $\mathcal{U}$. But this set equals $\{j \in J \mid c \in j\}$, which is in $\mathcal{U}$ by construction. □

7.13. Theorem. *Let $\mathbb{V}(S)$ be a superstructure and let κ be an uncountable cardinal number. There exists a κ-saturated nonstandard extension of $\mathbb{V}(S)$. In particular, there exists a polysaturated nonstandard extension of $\mathbb{V}(S)$.*

Proof. We begin by proving the important fact that if $\mathcal{U}$ is any countably incomplete ultrafilter on an infinite index set, then the ultrapower nonstandard extension of $\mathbb{V}(S)$ constructed as in Theorem 4.28 is necessarily $\aleph_1$-saturated. Since $\mathcal{U}$ is countably incomplete, we may suppose $(F_k)_{k\in\mathbb{N}}$ is a decreasing sequence in $\mathcal{U}$ whose intersection is empty and with $F_0 = J$. Let $\{a_k \mid k \in \mathbb{N}\}$ be a set of internal sets in $\mathbb{V}(^*S)$ with the finite intersection property. We must show that $\bigcap\{a_k \mid k \in \mathbb{N}\}$ is non-empty. Without loss of generality we may suppose that $a_{k+1} \subseteq a_k$ for all $k \in \mathbb{N}$. Therefore there exists $r \in \mathbb{N}$ so that a_k has rank at most r for all $k \in \mathbb{N}$. Since $^*\mathbb{V}(S)$ was obtained by the ultrapower construction using the ultrafilter $\mathcal{U}$, for each $k \in \mathbb{N}$ there is a set function $A_k\colon J \to \mathbb{V}_r(S)$ such that a_k is the equivalence class $[A_k]$. Since $\{a_k \mid k \in \mathbb{N}\}$ has the finite intersection property, for each $k \in \mathbb{N}$ we have

$$\{j \in J \mid A_0(j) \cap \ldots \cap A_k(j) \neq \emptyset\} \in \mathcal{U}.$$

Define G_k for $k \in \mathbb{N}$ as follows: $G_0 = J$ and for $k \geq 1$

$$G_k = F_k \cap \{j \in J \mid A_0(j) \cap \ldots \cap A_k(j) \neq \emptyset\}.$$

Therefore $J = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_k \supseteq \ldots$, $G_k \in \mathcal{U}$ for all $k \in \mathbb{N}$, and $\bigcap\{G_k \mid k \in \mathbb{N}\} = \emptyset$. Therefore we may define $d(j)$ for each $j \in J$ to be the largest $k \in \mathbb{N}$ for which $j \in G_k$. Now we construct $[\alpha]$ in $^*\mathbb{V}(S)$ which is an element of $\bigcap\{a_k \mid k \in \mathbb{N}\}$. Fix $j \in J$ and define $\alpha(j)$ as follows. If $d(j) = 0$ let $\alpha(j)$ be an arbitrary element of $\mathbb{V}_r(S)$. If $d(j) \geq 1$, choose $\alpha(j)$ to be an element of $A_0(j) \cap \ldots \cap A_{d(j)}(j)$, which is guaranteed to be non-empty by the definition of $d(j)$. It is obvious that for each $k \in \mathbb{N}$, $\alpha(j) \in A_k(j)$ holds whenever $d(j) \geq k$ and $d(j) \geq 1$. Therefore $\{j \in J \mid \alpha(j) \in A_k(j)\} \supseteq G_k \in \mathcal{U}$ for $k \geq 1$ and $\{j \in J \mid \alpha(j) \in A_0(j)\} \supseteq G_1 \in \mathcal{U}$. This completes the proof that $[\alpha]$ is an element of $\bigcap\{a_k \mid k \in \mathbb{N}\}$.

We will not give the details of a proof of the general case. The easiest construction of a κ-saturated nonstandard extension of $\mathbb{V}(S)$ for $\kappa > \aleph_1$ is to take the direct limit of a well ordered chain of successive enlargements. The length of the chain should be a regular cardinal number $\geq \kappa$; a chain of length κ^+ will suffice, where κ^+ is the next cardinal number larger than κ. It is also possible, but rather intricate in the case where $\kappa > \aleph_1$, to construct a κ-saturated nonstandard extension in one step as an ultrapower, by choosing the ultrafilter carefully. Details may be found in [8] [9] [11] and [15].

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