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# MODEL THEORY OF NAKANO SPACES

BY

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DISSERTATION

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*To the memory of Alejandro Glusman.*

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# Preface

Historically, the connection between model theory and functional analysis was first made evident by the introduction of Banach space ultraproducts by Bretagnolle, Dacunha-Castelle, and Krivine [6], and of non-standard hulls by Luxemburg [22]. Both constructions went on to have important and plentiful applications in analysis. Krivine also borrowed the concept of quantifier-free stability from model theory in his formulation of what has come to be known as *stability* in the analysis literature. As an application, Krivine and Maurey proved that if $X$ is a stable Banach space, then $X$ contains almost isometric copies of $l_p$ for some $p \in [1, \infty)$ [20]. In [14], Henson worked out a logical framework, positive bounded logic with approximate semantics, to provide a model theoretic foundation for the uses of Banach space ultraproducts. In [16], Henson and Iovino presented a fuller picture of the model theory that could be developed within that framework. More recently, Ben Yaacov, Berenstein, Henson, and Usvyatsov have developed a newer framework, continuous logic for metric structures, which provides an equivalent background for the ultraproduct construction, and which is easier to use. (See [3], [4], and [2].) It is important to note that continuous logic for metric structures, while not a first-order logic, satisfies the analogues of the most important classical theorems of ordinary first-order model theory, including the compactness theorem ([2, Theorem 5.8] and [3, Corollary 2.14]) and the Löwenheim-Skolem theorems ([2, Proposition 7.3] and [3, Facts 2.16 and 2.17]).

Using the positive bounded logic developed by Henson and Iovino, the classical $L_p(\mu)$-Banach lattices have been studied from the model theoretic point of view (see [1]) and their model theory is now well understood. The class of $L_p(\mu)$-Banach lattices over atomless measure spaces is axiomatizable by positive bounded sentences in the signature $\mathcal{L}$ of Banach lattices, and it admits quantifier elimination. Moreover, this class of $L$-structures is model-theoretically stable.

Nakano spaces are rich and interesting generalizations of $L_p$-Banach lattices. The main achievement of this thesis is to show that Nakano spaces have nice model-theoretic properties. We now proceed to define Nakano spaces and then to summarize the contents of this work.

Let $(\Omega, \Sigma, \mu)$ be a measure space, and let $p$ be a $\Sigma$-measurable function from $\Omega$ into a bounded interval of the form $[1, N]$. Let $L_0(\Omega, \Sigma, \mu)$ denote the space of equivalence classes of real-valued measurable functions on $(\Omega, \Sigma, \mu)$ modulo the equivalence relation of equality $\mu$-a.e. Define the convex modular $\Theta_{p(\cdot)} : L_0(\Omega, \Sigma, \mu) \to$

$[0, \infty]$ by setting

$$\Theta_{p(\cdot)}(f) = \int_\Omega |f(\omega)|^{p(\omega)}\, d\mu(\omega).$$

Let $\|\cdot\|_{p(\cdot)} : L_0(\Omega, \Sigma, \mu) \to [0, \infty]$ be defined by

$$\|f\|_{p(\cdot)} = \inf\{\epsilon > 0 \mid \Theta_{p(\cdot)}(f/\epsilon) \leq 1\}.$$

Let

$$L_{p(\cdot)}(\Omega, \Sigma, \mu) = \left\{ f \in L_0(\Omega, \Sigma, \mu) \;\middle|\; \|f\|_{p(\cdot)} < \infty \right\}.$$

We say that $X := L_{p(\cdot)}(\Omega, \Sigma, \mu)$ thus defined is a *Nakano space*. Given a Banach lattice $X$ with norm $\|\cdot\|$ and a convex modular $\Theta$ on $X$, we say that $(X, \Theta)$ is a *modulared Banach lattice* if $\|x\| = \inf\{\epsilon > 0 \mid \Theta(x/\epsilon) \leq 1\}$ for all $x \in X$. If $X := L_{p(\cdot)}(\Omega, \Sigma, \mu)$ and $\Theta_{p(\cdot)}$ is its associated convex modular, then $(X, \Theta_{p(\cdot)})$ is a modulared Banach lattice. We say that $(X, \Theta_{p(\cdot)})$ is a *modulared Nakano space*. Given two modulared Banach lattices $(X, \Theta)$ and $(Y, \Psi)$, we say that they are *isomodularly isomorphic* if there is a map $F : X \to Y$ which is a Banach-lattice isomorphism and satisfies $\Psi(F(x)) = \Theta(x)$ for all $x \in X$.

For $p(\cdot) \in L_0(\Omega, \Sigma, \mu)$ given, we define its *essential range* to be the set

$$\mathcal{R}_{p(\cdot)} = \left\{ q \in [1, \infty) \;\middle|\; \forall \epsilon > 0 \; \mu\left(\left\{\omega \in \Omega \;\middle|\; |p(\omega) - q| < \epsilon\right\}\right) > 0 \right\}.$$

If $X := L_{p(\cdot)}(\Omega, \Sigma, \mu)$ is a Nakano space, we will say that $X$ has *essential range* $\mathcal{R}_{p(\cdot)}$.

There are different signatures in which we regard classes of Nakano spaces: the signature $L$ of Banach lattices and, for each $k \in \mathbb{N}$, a signature $L_k$ obtained by expanding $L$ to include a new sequence $(\Theta^{(n)} \mid n \in \mathbb{N})$ of unary predicate symbols with associated moduli of uniform continuity $(\Delta_k^{(n)} \mid n \in \mathbb{N})$. The interpretation of the new predicate symbol $\Theta^{(n)}$ is the restriction of the convex modular $\Theta_{p(\cdot)}$ described above to the ball of radius n centered at the origin. (We specify the moduli of uniform continuity $\Delta_k^{(n)}$ on page 20.) A Nakano space with essential range $K$ can be regarded as an $L$-structure (if we care only about its Banach lattice structure) or as an $L_k$-structure, for $k \geq 2^{\sup K}$, if we wish to consider it as equipped with a compatible convex modular.

Let us fix $K$ to be a compact subset of $[1, \infty)$. These are the main new results proved in this thesis:

(i) The class $\mathcal{N}_{\subseteq K}$ of Banach lattices $X$ such that $X$ is $L$-isomorphic to a Nakano space with essential range $\subseteq K$ is axiomatizable in $L$, and the class $\mathcal{N}_{\subseteq K}^{\mathrm{mod}}$ of modulared Banach lattices $(X, \Theta)$ such that $(X, \Theta)$ is isomodularly isomorphic to a modulared Nakano space with essential range $\subseteq K$ is axiomatizable in $L_k$, for $k \geq 2^{\sup K}$.

(ii) The class $\mathcal{N}_K$ of Banach lattices $X$ such that $X$ is $L$-isomorphic to a Nakano space with essential range $= K$ is axiomatizable in $L$, and the class $\mathcal{N}_K^{\text{mod}}$ of modulared Banach lattices $(X, \Theta)$ such that $(X, \Theta)$ is isomodularly isomorphic to a modulared Nakano space with essential range $= K$ is axiomatizable in $L_k$, for $k \geq 2^{\sup K}$.

(iii) The class $\mathcal{AN}_K^{\text{mod}}$ of modulared Banach lattices $(X, \Theta)$ such that $(X, \Theta)$ is isomodularly isomorphic to a modulared Nakano space over an atomless measure space having essential range $= K$ is axiomatizable in $L_k$ for $k \geq 2^{\sup K}$. Its theory admits quantifier elimination and is complete. If, in addition, we have that $\inf K > 1$, then $\mathcal{AN}_K^{\text{mod}}$ is model-theoretically stable.

Chapter 1 of this thesis is expository in nature. It contains the relevant background in continuous logic for metric structures. Three good references are Berenstein and Henson's Newton Institute lecture notes [4], Ben Yaacov and Usvyatsov's [3], and the article [2] by Ben Yaacov, Berenstein, Henson, and Usvyatsov.

Chapter 2 contains an exposition of a few results in the more general setting of Orlicz lattices. Section 2.1 consists of well known facts about Orlicz lattices (see [26]) and explains how to think of Orlicz lattices as metric structures. Section 2.2 provides axioms for some classes of Orlicz lattices. Section 2.3 introduces Musielak-Orlicz spaces as examples of Orlicz lattices. There is a representation theorem for Orlicz lattices due to Wnuk (see [25] and [26]) which says that given an Orlicz lattice $(E, \Theta)$, there is a Musielak-Orlicz space $L_\psi(\Omega, \Sigma, \mu)$ such that if $\Psi$ denotes its associated convex modular, $(L_\psi(\Omega, \Sigma, \mu), \Psi)$ is isomodularly isomorphic to $(E, \Theta)$. In the formulation of this result appearing in [13], it is remarked that the representation can be achieved with the additional requirement that the Musielak-Orlicz function $\psi$ be such that $\psi(\cdot, \omega)$ belongs to the closure (in the topology of pointwise convergence) of the convex hull of the set $D$ of all functions $t \mapsto \frac{\Theta(tx)}{\Theta(x)}$, where $x \in E$. In [9], Dacunha-Castelle found classes of Musielak-Orlicz spaces that are closed under ultraproducts. Section 2.4 uses this result to identify universally axiomatizable classes of Musielak-Orlicz spaces. Let $L_\psi(\Omega, \Sigma, \mu)$ be a Musielak-Orlicz space such that for all $\omega \in \Omega$, $\psi(\cdot, \omega)$ belongs to the closure of the convex hull of some fixed set $D$ of Orlicz functions that is closed under dilations. In section 2.5, we show that such a Musielak-Orlicz space can be lattice-embedded in a Musielak-Orlicz function space $L_{\widehat{\psi}}(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu})$ whose Musielak-Orlicz function $\widehat{\psi}$ satisfies the condition that $\psi(\cdot, \omega)$ belongs to $D$ for all $\omega \in \Omega$.

The structure of chapter 3 is as follows. Sections 3.1–3.2 contain requisite background material from [13] and [17], and section 3.3 introduces Nakano spaces. Section 3.4 establishes the closure under ultraproducts of the classes $\mathcal{N}_{\subseteq K}$ and $\mathcal{N}_K$, for $K$ a compact subset of $[1, \infty)$. In section 3.5, a duality map for smooth Nakano spaces is calculated. This duality map plays an important role, along with the classical procedures of *convexification* and *concavification*, in establishing the closure under ultraroots of $\mathcal{N}_K$. This result is proved in sections 3.6–3.8. Let $\mathcal{AN}_K^{\text{mod}}$ denote the class of atomless modulared Banach lattices $(X, \Theta)$ such that $X$

is isomodularly isomorphic to a modulared Nakano space with bounded essential range $= K$. In section 3.9 we establish that the theory of the class $\mathcal{AN}_K^{\mathrm{mod}}$ admits quantifier elimination and is complete. In section 3.10 we show that if $\inf K > 1$, then the class $\mathcal{AN}_K^{\mathrm{mod}}$ is model-theoretically stable.

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# Chapter 1

# Preliminaries: continuous logic

## 1.1 Metric structures

In introducing continuous logic for *metric structures*, we will follow Berenstein and Henson's exposition in [4] very closely, but will make a few *ad hoc* changes to accommodate for the particular classes of structures discussed in this thesis. Essentially the entire content of this chapter is contained in the above mentioned manuscript. (For a different exposition, the reader is referred to [3].)

A *metric structure* $\mathcal{M}$ consists of

(i) A family $((M^{(s)}, d^{(s)}) \mid s \in S)$ of bounded metric spaces.

(ii) A collection of functions

$$F : M^{(s_1)} \times \cdots \times M^{(s_n)} \to M^{(s_0)}.$$

that are uniformly continuous with respect to the canonical metric on $M^{(s_1)} \times \cdots \times M^{(s_n)}$ and the canonical metric on $M^{(s_0)}$. We will refer to them as *operations with range in* $M_{s_0}$.

(iii) For each $M \in \mathbb{N}$, a collection of functions

$$R : M^{(s_1)} \times \cdots \times M^{(s_n)} \to [0, N] \subseteq \mathbb{R}$$

that are uniformly continuous with respect to the canonical metric on $M^{(s_1)} \times \cdots \times M^{(s_n)}$ and the usual metric on $[0, N]$, and to which we will refer as *predicates with range in* $[0, N]$.

Functions whose domain consists only of the empty tuple will be called *constant symbols*, just as in the setting of ordinary model theory.

*Example.* Let $(M, d)$ be a bounded metric space with diameter 1. Then we have only one sort in the metric structure: the sort of $M$. Additionally, we have the predicate $d$.

*Example.* Let $(X, \|\cdot\|)$ be a normed vector space. We can regard it as a metric structure in the following way. The sorts are $B_n := \{x \in X \mid \|x\| \leq n\}$, indexed by $n \in \mathbb{N}$. It comes equipped with distinguished elements $0^n \in B_n$, functions $+_n : B_n \times B_n \to B_{2n}$ (the restrictions of vector addition to bounded balls centered at the origin), $-_n : B_n \to B_n$ (the restrictions of the additive inverse unary operation to balls centered at the origin), and for $r \in \mathbb{R}$, functions $f_r^{(n)} : B_n \to B_(\lfloor r \rfloor + 1)n)$ (the restrictions of scalar multiplication by $r$ to bounded balls centered at the origin), with $n$ ranging over $\mathbb{N}$. The metrics can be thought of as distinguished predicates in the same way as in the preceding example.

A *modulus of uniform continuity* is a function $\Delta : (0,1] \to (0,1]$. If $(M,d)$ and $(M_0,d_0)$ are metric spaces and $f : M \to M_0$ is any function, we say that $\Delta : (0,1] \to (0,1]$ is a *modulus of uniform continuity for* $f$ if for every $\epsilon \in (0,1]$ and every $x, y \in M$ we have

$$d(x,y) < \Delta(\epsilon) \Longrightarrow d_0(f(x), f(y)) \leq \epsilon.$$

Note that $f$ is uniformly continuous if and only if it has a modulus of uniform continuity.

The *signature* $L$ of a metric structure

$$\mathcal{M} = \Big((M^{(s)}, d^{(s)})_{s\in S}; (c_i)_{i\in I}; (F_j)_{j\in J}; (R_k)_{k\in K}\Big)$$

consists of:

(i) The nonempty sort index set $S$;

(ii) Constant symbols $\underline{c}_i$ with arity$(\underline{c}_i) := s_0$ for distinguished elements $c_i \in M^{(s_0)}$;

(iii) Function symbols $f_j$ with arity$(f_j) := (s_1, \ldots, s_n, s_0)$ for distinguished operations

$$F_j : M^{(s_1)} \times \cdots \times M^{(s_n)} \to M^{(s_0)};$$

(iv) Predicate symbols $P_k$ with arity$(P_k) := (s_1, \ldots, s_n)$ and range $I_{P_k} = [0, N]$ for

$$R_k : M^{(s_1)} \times \cdots \times M^{(s_n)} \to [0, N];$$

(v) For each $s \in S$, a constant $N^{(s)} \in \mathbb{N}$ indicating an upper bound for the diameter of $d^{(s)}$;

(vi) For each function symbol $f_j$, a modulus of uniform continuity $\Delta[f_j]$.

(vii) For each predicate symbol $P_k$ with range in $[0, M]$, a modulus of uniform continuity $\Delta[P_k]$.

(viii) A fixed infinite set $V_L$ of sorted variables, usually taken to be countable, unless otherwise necessary.

(ix) For each $s \in S$, a logical symbol $d^{(s)}$ for the metric of the sort $s$.

The *cardinality* of $L$ is the number of symbols in $L$.

*Example.* The signature of Banach lattices consists of:

(i) A sort $s_n$ corresponding to the closed ball $B_n$ of diameter $n$ centered at the origin, for each $n \in \mathbb{N}$.

(ii) Constant symbols $0_n$ (with $\text{arity}(0_n) = s_n$) for each $n \in \mathbb{N}$.

(iii) Function symbols:

- For each $m, n \in \mathbb{N}$ with $m < n$, a function symbol $I_{m,n}$ with $\text{arity}(I_{m,n}) = (s_m, s_n)$ and modulus of uniform continuity $\Delta[I_{m,n}]$ defined by $\Delta[I_{m,n}](\epsilon) = \epsilon$.
- For each $n \in \mathbb{N}$, function symbols $\vee_n$ and $\wedge_n$; both symbols have arity $(s_n, s_n, s_{2n})$ and moduli of uniform continuity $\Delta[\vee_n]$, $\Delta[\wedge_n]$ defined by $\Delta[\vee_n](\epsilon) = \Delta[\wedge_n](\epsilon) = \epsilon/2$.
- For $n \in \mathbb{N}$, function symbols $+_n$ and $-_n$ with arity $(s_n, s_n, s_{2n})$ and moduli of uniform continuity $\Delta[+_n]$ and $\Delta[-_n]$ defined by $\Delta[+_n](\epsilon) = \Delta[-_n](\epsilon) = \epsilon/2$.
- For $n \in \mathbb{N}$ and given $k < \lambda \leq k+1$ (for some $k \in \mathbb{N}$), a function symbol $\lambda_n$ with arity $(s_n, s_{kn})$ and modulus of uniform continuity $\Delta[\lambda_n]$ defined by $\Delta[\lambda_n](\epsilon) = \epsilon/k+1$.

(iv) For $n \in \mathbb{N}$, a predicate symbol $\|\cdot\|^{(n)}$ with arity $(s_n)$ and modulus of uniform continuity $\Delta[\|\cdot\|^{(n)}](\epsilon) = \epsilon$.

(v) For $n \in \mathbb{N}$, a logical symbol $d^{(n)}$ for the predicate $d^{(n)} : B_n \times B_n \to [0, 2n]$ defined by $d_n(f,g) = \|f - g\|$.

## $L$-structures

Given any signature $L$, $\mathcal{M} = \big((M^{(s)}, d^{(s)})_{s \in S}; (c_i)_{i \in I}; (F_j)_{j \in J}; (R_k)_{k \in K}\big)$ is an *$L$-structure* if the following conditions are met:

(i) The sort index set of $L$ coincides with the sort index set $S$ of $\mathcal{M}$.

(ii) If $\text{arity}(c) = (s)$, then $c^{\mathcal{M}} \in M^{(s)}$.

(iii) If $\text{arity}(f) = (s_1, \ldots, s_n, s_0)$, then $\Delta[f]$ is a modulus of uniform continuity for

$$f^{\mathcal{M}} : M^{(s_1)} \times \cdots \times M^{(s_n)} \to M^{(s_0)}.$$

(iv) If $\text{arity}(P) = (s_1, \ldots, s_n)$ and the range of $P$ is $I_P$, then $\Delta[P]$ is a modulus of uniform continuity for

$$P^{\mathcal{M}} : M^{(s_1)} \times \cdots \times M^{(s_n)} \to I_P.$$

(v) For each $s \in S$, $d^{(s)}(x,y) \leq N^{(s)}$ for all $x, y \in M^{(s)}$.

*Example.* In the signature $L$ of Banach lattices, any Banach lattice $X$ can be looked upon as an $L$-structure in the following way: for a fixed $n \in \mathbb{N}$, the sort $s_n$ is interpreted as the ball $B_n$ of radius $n$ centered at the origin; each of the function symbols $+_n$, $-_n$, $\vee_n$, $\wedge_n$, and $\lambda_n$ is interpreted as the restriction to the closed ball $B_n$ of radius $n$ centered at the origin of the corresponding operation on $X$; for $m < n$, the function symbol $I_{m,n}$ is interpreted as the inclusion function from $B_m$ into $B_n$; for a fixed $n \in \mathbb{N}$, the predicate symbol $\|\cdot\|^{(n)}$ is interpreted as the restriction to the ball $B_n$ of the norm on $X$; finally, the interpretation of $d_n$ is the restriction of the metric $d$ on $X$ defined by $d(x,y) = \|x - y\|$ to the ball $B_n$.

## 1.2 Continuous logic for metric structures

### Syntax

Given a signature $L$, we define $L$-*terms* in exactly the same way as in many-sorted first-order logic. Each variable of sort $s$ and each constant symbol of sort $s$ is an $L$-*term with range of sort* $s$. If $f$ is a function symbol with $\text{arity}(f) = (s_1, \ldots, s_k, s_0)$ and $t_1, \ldots, t_k$ are $L$-terms with ranges of sorts $s_1, \ldots, s_k$, respectively, then $f(t_1, \ldots, t_k)$ is an $L$-term of sort $s_0$. All $L$-terms are constructed in this way. The $L$-*atomic formulas* can be of two forms:

(i) $P(t_1, \ldots, t_n)$, where the predicate symbol $P$ has range $I_P$ and arity $(s_1, \ldots, s_n)$ and $t_1, \ldots, t_n$ are $L$-terms with ranges of sorts $s_1, \ldots, s_n$, respectively.

(ii) $d^{(s)}(t_1, t_2)$, where $t_1, t_2$ are $L$-terms with ranges of the same sort $s$.

For the purposes of this work, $n$-ary *connectives* are simply continuous functions $u : [0,\infty)^n \to [0,\infty)$, and sup and inf play the role of the quantifiers $\forall$ and $\exists$.

The class of $L$-*formulas* is the smallest class of expressions satisfying the following conditions:

(i) The atomic formula $P(t_1, ...)$ is an $L$-formula with range in $I_P$. The atomic formula $d^{(s)}(t_1, t_2)$ is an $L$-formula with range in $[0, N^{(s)}]$.

(ii) If $u : [0,\infty)^n \to [0,\infty)$ is a continuous $n$-ary connective and $\varphi_1, \ldots, \varphi_n$ are $L$-formulas with ranges in $I_{\varphi_1}, \ldots, I_{\varphi_n}$, respectively, then

$$u(\varphi_1, \ldots, \varphi_n)$$

is an $L$-formula with range in the interval $u(\prod_{i=1}^n I_{\varphi_i})$.

(iii) If $\varphi$ is an $L$-formula with range in $I_\varphi$ and $x$ is a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are $L$-formulas with range in $I_\varphi$.

*Remark.* Note that $I_\varphi$ is always a compact interval $[a, b] \subseteq [0, \infty)$ (possibly with $a = b$).

Free and bound occurrences of variables in $L$-formulas are defined in a manner similar to how this is done in first-order logic, with the role of quantifiers played by sup and inf.

An $L$-formula is *quantifier free* if it is generated inductively from atomic formulas without using the third rule above. An $L$-sentence is an $L$-formula which has no free variables. When $t$ is a term and the variables occurring in it are among the distinct variables $x_1, \ldots, x_n$, we indicate this by writing $t$ as $t(x_1, \ldots, x_n)$. Likewise, we write an $L$-formula as $\varphi(x_1, \ldots, x_n)$ to indicate that its free variables are among the distinct variables $x_1, \ldots, x_n$.

An $L$-*statement* $E$ is a formal expression of the form $\varphi = \psi$, where $\varphi$ and $\psi$ are $L$-formulas. We say that a statement is *closed* if both $\varphi$ and $\psi$ are sentences. Since any real number $r \geq 0$ is a connective, expressions of the form $\varphi = r$ are statements for any $L$-formula $\varphi$.

## Semantics

Let $L$ be a signature. Let $\mathcal{M}$ be a metric space $L$-structure with underlying (bounded) metric space sorts $(M^{(s)}, d_s^{\mathcal{M}})$ for $s \in S$. If $A = (A^{(s)} \mid s \in S)$ and for each $s \in S$, $A^{(s)} \subseteq M^{(s)}$, we will say that $A$ is a *subset* of $\mathcal{M}$.

Let $A$ be a subset of $\mathcal{M}$. We can extend $L$ to a signature $L(A)$ by adding a new constant symbol (of the appropriate sort) $c(a)$ to $L$ for each $a \in A$. We call $c(a)$ the *name* of $a$ in $L(A)$.

Given an $L$-term $t(x_1, \ldots, x_n)$ with range of sort $s_0$, its *interpretation* in $\mathcal{M}$ is a function

$$t^{\mathcal{M}} : M^{(s_1)} \times \cdots \times M^{(s_n)} \to M^{(s_0)}$$

(where $x_i$ is of sort $x_i$ for each $i = 1, \ldots n$) defined by induction on the complexity of terms. (See [16, Definition 5.3].) This definition corresponds precisely to the definition of the interpretation of a term in many-sorted first-order logic.

For each $L(M)$-sentence $\sigma$ with range in $I_\sigma$, we define *the value of* $\sigma$ *in* $\mathcal{M}$. This value is a real number in the interval $I_\sigma$ associated to $\sigma$, and is denoted by $\sigma^{\mathcal{M}}$. The definition is by induction on formulas.

(i) For $L(M)$-terms $t_1, t_2$ without variables (of sort $s$), $(d_s(t_1, t_2))^{\mathcal{M}} = d_s^{\mathcal{M}}(t_1^{\mathcal{M}}, t_2^{\mathcal{M}})$;

(ii) For $L(M)$-terms $t_1, \ldots, t_n$ (of sorts $s_1, \ldots, s_n$, respectively) without variables and $n$-ary predicate symbol $P$ of sort $(s_1, \ldots, s_n)$ and with range in $I_P$, $(P(t_1, \ldots, t_n))^{\mathcal{M}} = P^{\mathcal{M}}(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}})$;

(iii) For an $n$-ary connective $u$ and for $L(M)$-sentences $\sigma_1, \ldots, \sigma_n$, we have that

$$(u(\sigma_1, \ldots, \sigma_n))^{\mathcal{M}} = u(\sigma_1^{\mathcal{M}}, \ldots, \sigma_n^{\mathcal{M}});$$

(iv) For any $L(M)$-formula $\varphi(x)$ with range in $I_\varphi$, $(\sup_x \varphi(x))^{\mathcal{M}}$ is the supremum in $I_\varphi$ of the set

$$\{\varphi(a)^{\mathcal{M}} \mid a \in M^{(s)}\};$$

(v) For any $L(M)$-formula $\varphi(x)$ with range in $I_\varphi$, $(\inf_x \varphi(x))^{\mathcal{M}}$ is the infimum in $I_\varphi$ of the set

$$\{\varphi(a)^{\mathcal{M}} \mid a \in M^{(s)}\}.$$

Given an $L(M)$-formula $\varphi(x_1, \ldots, x_n)$ with range in $I_\varphi$, we let $\varphi^{\mathcal{M}}$ be the predicate defined on

$$M^{(s_1)} \times \cdots \times M^{(s_n)}$$

by

$$\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = (\varphi(a_1, \ldots, a_n))^{\mathcal{M}}$$

(where $x_i$ is of sort $s_i$ for $i = 1, \ldots, n$).

If $a_1, \ldots, a_n \in M^{(s_1)} \times \cdots \times M^{(s_n)}$ and $E(x_1, \ldots, x_n)$ is an $L(M)$-statement of the form $\varphi(x_1, \ldots, x_n) = \psi(x_1, \ldots, x_n)$, we say that the statement is *true of* $a_1, \ldots, a_n$ *in* $\mathcal{M}$ if $\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \psi^{\mathcal{M}}(a_1, \ldots, a_n)$. In such a case, we write $\mathcal{M} \vDash E[a_1, \ldots, a_n]$.

An important observation concerning formulas in continuous logic for metric structures is that they always define uniformly continuous functions. This can be proved by induction on the complexity of formulas. (See Theorem 3.5 in [2].)

## 1.3 Basic continuous model theory I

Let $L$ be a signature. An $L$-*theory* is a set of closed $L$-statements. If $T$ is an $L$-theory and $\mathcal{M}$ is an $L$-structure, we say that $\mathcal{M}$ is a *model* of $T$ and write $\mathcal{M} \vDash T$ if $\mathcal{M} \vDash \sigma$ for every closed statement $\sigma \in T$.

If $\mathcal{M}$ is an $L$-structure, then $\mathrm{Th}(M)$ denotes the *theory of* $\mathcal{M}$, i.e. the set of all closed $L$-statements which are true in $\mathcal{M}$. When $T$ is a theory of this kind, $T$ will be said to be *complete*.

We say that the $L$-structures $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent*, and write $\mathcal{M} \equiv \mathcal{N}$, if $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$ for all $L$-sentences $\sigma$.

Let $\mathcal{M}$ and $\mathcal{N}$ be $L$-structures and let $(M^{(s)} \mid s \in S)$ and $(N^{(s)} \mid s \in S)$ be the sorts of $\mathcal{M}$ and $\mathcal{N}$, respectively. We say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ if $M^{(s)} \subseteq N^{(s)}$ for every $s \in S$, $f^{\mathcal{N}}$ extends $f^{\mathcal{M}}$ for every function symbol $f$ of $L$, and if $P^{\mathcal{M}}$ extends $P^{\mathcal{N}}$ for all predicate symbols $P$ of $L$. We write $\mathcal{M} \subseteq \mathcal{N}$ to indicate that $\mathcal{M}$ is a substructure of $\mathcal{N}$.

We say that $\mathcal{M}$ and $\mathcal{N}$ are *isomorphic* if there is a collection of surjective isometries $T^{(s)} : M^{(s)} \to N^{(s)}$, for $s \in S$, which respect the interpretations of function and predicate symbols of $L$, in the following sense:

(i) Whenever $f$ is a function symbol of $L$ of arity $(s_1, \ldots, s_m, s_0)$ and $a_k \in M^{(s_k)}$ for $k = 1, \ldots, m$, we have

$$f^{\mathcal{N}}\left(T^{(s_1)}(a_1), \ldots, T^{(s_m)}(a_m)\right) = T^{(s_0)}\left(f^{\mathcal{M}}(a_1, \ldots, a_m)\right);$$

(ii) Whenever $P$ is a function symbol of $L$ of arity $(s_1, \ldots, s_n)$ and $a_k \in M^{(s_k)}$ for $k = 1, \ldots, n$, we have

$$P^{\mathcal{M}}(a_1, \ldots, a_k) = P^{\mathcal{N}}(T^{(s_1)}(a_1), \ldots, T^{(s_n)}(a_n)).$$

(Sometimes we say that $\mathcal{M}$ and $\mathcal{N}$ are *isometrically isomorphic* to emphasize that isomorphisms must preserve distances.)

An *embedding* of $\mathcal{M}$ into $\mathcal{N}$ is an isomorphism between $\mathcal{M}$ and a substructure of $\mathcal{N}$.

If $\mathcal{M} \subseteq \mathcal{N}$ we say that $\mathcal{M}$ is an *elementary substructure of* $\mathcal{N}$, and write $\mathcal{M} \preceq \mathcal{N}$, if whenever $\varphi(x_1, \ldots, x_n)$ is an $L$-formula and $a_1, \ldots, a_n$ are elements of $M$, we have $\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \varphi^{\mathcal{N}}(a_1, \ldots, a_n)$. In this case, we also say that $\mathcal{N}$ is an *elementary extension of* $\mathcal{M}$.

An *elementary embedding* of $\mathcal{M}$ into $\mathcal{N}$ consists of a family of maps $(T^{(s)} \mid s \in S)$ with $T^{(s)} : M^{(s)} \to N^{(s)}$ for $s \in S$ such that the following condition holds: whenever $\varphi(x_1, \ldots, x_n)$ is an $L$-formula and $a_1, \ldots, a_n$ are elements of $\mathcal{M}$, where $x_k$ is a variable of sort $s_k$ and $a_k \in M^{(s_k)}$ for $k = 1, \ldots, n$, we have

$$\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \varphi^{\mathcal{N}}(T^{(s_1)}(a_1), \ldots, T^{(s_n)}(a_n)).$$

*Remark.* Note that an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$ is simply an isomorphism between $\mathcal{M}$ and an elementary substructure of $\mathcal{N}$.

### Types

Let $L$ be a signature, and let $\mathcal{M}$ be a metric $L$-structure, and let

$$A = (A^{(s)} \mid s \in S),$$

where $A^{(s)} \subseteq M^{(s)}$. Let $e_1 \in M^{(s_1)}, \ldots, e_n \in M^{(s_n)}$ for some sorts $s_1, \ldots, s_n$. Fix distinct variables $x_1, \ldots, x_n$ of $L$ of sorts $s_1, \ldots, s_n$, respectively. The *type of* $(e_1, \ldots, e_n)$ *over* $A$ *in* $\mathcal{M}$, denoted $\mathrm{tp}_{\mathcal{M}}(e_1, \ldots, e_n/A)$, is defined to be the set of all $L(A)$-statements $E(x_1, \ldots, x_n)$ such that $(\mathcal{M}, a)_{a \in A} \models E[e_1, \ldots, e_n]$. Such a type is said to have *arity* $(s_1, \ldots, s_n)$. (By $(\mathcal{M}, a)_{a \in A}$ we mean the expansion of $\mathcal{M}$ obtained by naming all elements in $A$.)

When it can be done without confusion, we omit the superscript $\mathcal{M}$ from this notation. Furthermore, if $A$ is the empty set, we may also omit it from the notation.

Let $T_A$ be a complete theory in the signature $L(A)$ resulting from adding to $L$ a constant symbol for each element of $A$. Any model of $T_A$ is isomorphic to a structure of the form $(\mathcal{M}, a)_{a \in A}$, where $\mathcal{M}$ is a model of $T$. The collection of all types of arity $(s_1, \ldots, s_n)$ over $A$ is denoted by $S_{s_1, \ldots, s_n}(T_A)$, or simply $S_{s_1, \ldots, s_n}(A)$ in case the context makes the theory $T_A$ clear.

For each finite tuple $(s_1, \ldots, s_n)$ of sorts, we define a metric on the set $S_{s_1, \ldots, s_n}(T_A)$ by

$$d(p, q) = \inf \left\{ \max_j d^{\mathcal{M}_A}(b_j, c_j) \mid \mathcal{M}_A \models p[b_1, \ldots, b_n], \mathcal{M}_A \models q[c_1, \ldots, c_n] \right\},$$

where $\mathcal{M}_A$ is allowed to vary among all models of $T_A$. Then

$$(S_{s_1, \ldots, s_n}(T_A), d)$$

is a complete metric space. (See [2, Proposition 8.9].)

## 1.4 Ultraproducts

### Ultralimits

Let $X$ be a topological space, and let $(x_i \mid i \in I)$ be an indexed family of elements of $X$. If $\mathcal{U}$ is an ultrafilter on $I$ and $x \in X$, we write

$$\lim_{i, \mathcal{U}} x_i = x$$

and say that $x$ is the *$\mathcal{U}$-limit* of $(x_i)_{i\in I}$ if for every open neighborhood $O$ of $x$, the set $\{i \in I \mid x_i \in O\}$ is in the ultrafilter $\mathcal{U}$. A basic fact from general topology is that a topological space $X$ is Hausdorff compact if and only if for every indexed family $(x_i \mid i \in I)$ in $X$ and every ultrafilter $\mathcal{U}$ on $I$, the $\mathcal{U}$-limit of $(x_i \mid i \in I)$ exists and is unique.

## Ultraproducts of bounded metric spaces

Let $((M_i, d_i) \mid i \in I)$ be a family of bounded metric spaces with diameter $\leq K$ for some fixed constant $K$. Let $\mathcal{U}$ be an ultrafilter on $I$. Then we can define a function $d : \prod_{i\in I} M_i \times \prod_{i\in I} M_i \to [0, K]$ by

$$d(x, y) = \lim_{i,\mathcal{U}} d_i(x_i, y_i).$$

Then $d$ is a pseudometric on the cartesian product of the $M_i$. We now proceed to take the quotient metric space induced by this pseudometric: for $x, y \in \prod_{i\in I} M_i$, define $x \sim_{\mathcal{U}} y$ to mean that $d(x, y) = 0$. Then $\sim_{\mathcal{U}}$ is an equivalence relation. We define the ultraproduct of $((M_i, d_i) \mid i \in I)$ by writing

$$\left(\prod_{i\in I} M_i\right)_{\mathcal{U}} = \left(\prod_{i\in I} M_i\right) \Big/ \sim_{\mathcal{U}} .$$

Then $d$ induces a metric on $\left(\prod_{i\in I} M_i\right)_{\mathcal{U}}$, which we will denote by $d$. The bounded metric space $\left(\prod_{i\in I} M_i\right)_{\mathcal{U}}$ (equipped with the metric $d$) is called the *$\mathcal{U}$-ultraproduct of* the family $((M_i, d_i) \mid i \in I)$. The equivalence class of $(x_i \mid i \in I) \in \left(\prod_{i\in I} M_i\right)_{\mathcal{U}}$ is denoted by $(x_i \mid i \in I)_{\mathcal{U}}$.

If $(M_i, d_i) = (M, d)$ for all $i \in I$, then the resulting $\mathcal{U}$-ultraproduct is called the *$\mathcal{U}$-ultrapower of* $M$ and is denoted by $(M)_{\mathcal{U}}$. In this situation, the map $T : M \to (M)_{\mathcal{U}}$ defined by $T(x) = ((x_i \mid i \in I)_{\mathcal{U}}$, where $x_i = x$ for all $i \in I$, is an isometric embedding and is called the *diagonal embedding of* $M$ *into* $(M)_{\mathcal{U}}$.

## Ultraproducts of functions

Let $((M_i, d_i) \mid i \in I)$ and $((M'_i, d'_i) \mid i \in I)$ be two families of bounded metric spaces, all of diameter $\leq K$ for a fixed constant $K$. Let $(f_i \mid i \in I)$ be a family of $n$-ary functions with $f_i : M_i^n \to M'_i$ uniformly continuous for all $i \in I$. Suppose further that there is single bounded real valued function $\Delta$ that serves as the modulus of uniform continuity for all the functions $f_i$. Let $\mathcal{U}$ be an ultrafilter on $I$. Then we can define an *ultraproduct function*

$$\left(\prod_{i\in I} f_i\right) : \left(\left(\prod_{i\in I} M_i\right)_{\mathcal{U}}\right)^n \to \left(\prod_{i\in I} M'_i\right)$$

in the following manner. If $(x_i^k \mid i \in I) \in \prod_{i\in I} M_i$ for $k = 1, \ldots, n$, define

$$\left(\prod_{i\in I} f_i\right)\big((x_i^1 \mid i \in I)_{\mathcal{U}}, \ldots, (x_i^n \mid i \in I)_{\mathcal{U}}\big) = \big(f_i(x_i^1, \ldots, x_i^n) \mid i \in I\big)_{\mathcal{U}}.$$

This defines a uniformly continuous function that also has $\Delta$ as its modulus of uniform continuity. (See the discussion of ultraproducts of functions in section 5 of [2].)

### Ultraproducts of $L$-structures

Let $(\mathcal{M}_i \mid i \in I)$ be a family of $L$-structures and let $\mathcal{U}$ be an ultrafilter on $I$. For a given sort $s \in S$, there is an absolute bound $N^{(s)}$ on the diameter of the bounded metric spaces of sort $s$ in the family, and so we may form the $\mathcal{U}$-ultraproduct of the family. For each predicate symbol $P$ in $L$, the functions $P^{\mathcal{M}_i}$ all have the same modulus of uniform continuity $\Delta[P]$ and all take their values in the same interval $I_P$. Therefore, the $\mathcal{U}$-ultraproduct of this family of functions is well-defined. By a similar reasoning, given a function symbol $f$ in $L$, we can define the $\mathcal{U}$-ultraproduct of the family $(f^{\mathcal{M}_i} \mid i \in I)$.

We define the *$\mathcal{U}$-ultraproduct of the family $(\mathcal{M}_i \mid i \in I)$ of $L$-structures* to be the $L$-structure $\mathcal{M}$ specified as follows.

For $s \in S$, the underlying bounded metric space of sort $s$ in $\mathcal{M}$ is given by the ultraproduct

$$M^{(s)} = \left(\prod_{i\in I} M_i^{(s)}\right)_{\mathcal{U}}.$$

For each predicate or function symbol $F$ of $L$, the interpretation of $F$ in $\mathcal{M}$ is given by the ultraproduct of functions

$$F^{\mathcal{M}} = \left(\prod_{i\in I} F^{\mathcal{M}_i}\right)_{\mathcal{U}}.$$

If all the $L$-structures $\mathcal{M}_i$ are equal to the same structure $\mathcal{M}_0$, then $\mathcal{M}$ is called the *$\mathcal{U}$-ultrapower of $\mathcal{M}_0$* and is denoted by $(\mathcal{M}_0)_{\mathcal{U}}$.

## 1.5 Basic continuous model theory II

### Axiomatizability of classes of structures

A class $\mathcal{A}$ of $L$-structures is said to be *axiomatizable* in the signature $L$ if there exists a set $\Sigma$ of $L$-statements such that $\mathcal{A}$ is the class of all models of $\Sigma$. The following important proposition allows one to prove the axiomatizability of a class of structures in an indirect fashion. (See [16, Proposition 13.6] and [2, Proposition 5.13].)

**Proposition 1.5.1.** *A class $\mathcal{A}$ of $L$-structures closed under $L$-isomorphisms is axiomatizable if and only if it is closed under ultraproducts and ultraroots.*

*Remark.* By an *ultraroot* of an $L$-structure $\mathcal{M}$, we mean an $L$-structure $\mathcal{N}$ such that there is an index $I$ and an ultrafilter $\mathcal{U}$ on $I$ for which $(\mathcal{N})_{\mathcal{U}}$ is $L$-isomorphic to $\mathcal{M}$.

### Downward Löwenheim-Skolem Theorem

The *density character* of a topological space is the smallest cardinality of a dense subset of the space. For simplicity of exposition, we state the following theorem in the case in which the signature $L$ contains only one sort, but an analogous statement holds true in the many-sorted setting.

**Theorem 1.5.2** (Downward Löwenheim-Skolem)**.** *Let $\kappa$ be an infinite cardinal number and assume* $\mathrm{card}(L) \leq \kappa$*. Let $\mathcal{M}$ be a metric $L$-structure. Suppose $A$ is a subset of the underlying metric space of $\mathcal{M}$ with density character $\leq \kappa$. Then there exists a closed substructure $\mathcal{N}$ of $\mathcal{M}$ such that:*

(i) $\mathcal{N} \preceq \mathcal{M}$;

(ii) $A \subseteq N \subseteq M$;

(iii) *the density character of $N$ is $\leq \kappa$.*

The above theorem appears as Proposition 7.3 in [2]. It is also essentially Proposition 9.13 in [16].

### Saturation

Let $\Gamma(x_1, \ldots, x_n)$ be a set of $L$-statements and let $\mathcal{M}$ be an $L$-structure. We say that $\Gamma(x_1, \ldots, x_n)$ is *satisfiable in* $\mathcal{M}$ if there exist elements $a_1, \ldots, a_n$ of $\mathcal{M}$ such that $\mathcal{M} \models \Gamma[a_1, \ldots, a_n]$. Given such a set $\Gamma(x_1, \ldots, x_n)$, we define $\Gamma^+$ to be the set of all $L$-statements $|\varphi - \psi| \leq \epsilon$ where $\varphi = \psi$ is in $\Gamma$ and $\epsilon > 0$.

Let $\mathcal{M}$ be an $L$-structure and let $\kappa$ be an infinite cardinal. We say that $\mathcal{M}$ is *$\kappa$-saturated* if whenever $A \subseteq M$ has cardinality $< \kappa$ and $\Gamma(x_1, \ldots, x_n)$ is a set of $L(A)$-statements, if every finite subset of $\Gamma^+$ is satisfiable in $(\mathcal{M}, a)_{a \in A}$, then the entire set $\Gamma$ is satisfiable in $(\mathcal{M}, a)_{a \in A}$.

The following proposition guarantees that the kind of structures with which we will be concerned in this thesis all have $\kappa$-saturated elementary extensions for every infinite cardinal $\kappa$.

**Proposition 1.5.3.** *Let $\mathcal{M}$ be an $L$-structure. For every infinite cardinal $\kappa$, $\mathcal{M}$ has a $\kappa$-saturated elementary extension.*

This is Proposition 7.10 in [2]. The proof for the case in which the signature $L$ is countable can be found in [16], where the result appears under the heading Proposition 9.18.

## Quantifier Elimination

Fix a signature $L$ and an $L$-theory $T$. An $L$-formula $\varphi(x_1, \ldots, x_n)$ is *approximable in $T$ by quantifier-free formulas* if for every $\epsilon > 0$ there is a quantifier-free $L$-formula $\psi(x_1, \ldots, x_n)$ such that for all $\mathcal{M} \vDash T$ and all $a_1, \ldots, a_n \in M$, we have

$$|\varphi^{\mathcal{M}}(a_1, \ldots, a_n) - \psi^{\mathcal{M}}(a_1, \ldots, a_n)| \leq \epsilon.$$

The $L$-theory $T$ is said to *admit quantifier elimination* if every $L$-formula is approximable in $T$ by quantifier-free $L$-formulas.

The following criterion for quantifier elimination will be used further along in this thesis:

**Proposition 1.5.4.** *Let $T$ be an $L$-theory. Then the following conditions are equivalent:*

(i) *$T$ admits quantifier elimination;*

(ii) *If $\mathcal{M}$ and $\mathcal{N}$ are models of $T$, then every embedding of a substructure of $\mathcal{M}$ into $\mathcal{N}$ can be extended to an embedding of $\mathcal{M}$ into an elementary extension of $\mathcal{N}$.*

*Moreover, if $\kappa$ is an infinite cardinal and* $\text{card}(L) \leq \kappa$*, then in the second condition it is sufficient to consider models $\mathcal{M}$ of density character $\leq \kappa$. (Proposition* 12.6 *in [2].)*

For a proof of this fact the reader is referred to [16, Proposition 13.17].

## Stability

Let $T$ be a complete theory and let $\lambda$ be an infinite cardinal. We say that $T$ is $\lambda$-*stable with respect to the d-metric* if for any $\mathcal{M} \vDash T$, for any $A \subseteq \mathcal{M}$ of cardinality $\leq \lambda$, for every sort $s \in S$, $S_s(T_A)$ has density character $\leq \lambda$ with respect to the $d$ metric. We say that $T$ is *stable with respect to the $d$ metric* if $T$ is $\lambda$-stable with respect to the $d$ metric for some infinite cardinal $\lambda$. (For two different treatments of stability in the context of continuous logic, the reader is referred to [2, section 14] and [3, sections 7 and 8].)

# Chapter 2

# Musielak-Orlicz spaces

## 2.1 Orlicz lattices

This section contains a summary of known facts about Orlicz lattices. A good general reference is [26]. As defined in [13], a *convex modular* on a vector lattice $E$ is a map $\Theta : E \to [0,\infty)$ satisfying:

$$\Theta(f) = 0 \Longleftrightarrow f = 0; \tag{M1}$$

$$|f| \le |g| \Longrightarrow \Theta(f) \le \Theta(g); \tag{M2}$$

$$\Theta(\alpha f + (1-\alpha)g) \le \alpha\Theta(f) + (1-\alpha)\Theta(g); \tag{CM}$$

$$|f| \wedge |g| = 0 \Longrightarrow \Theta(f+g) = \Theta(f) + \Theta(g) \tag{DA}$$

(for all $f, g \in E$ and $\alpha \in [0,1]$).

The following lemma can be traced back to [8, page 87].

**Lemma 2.1.1.** *Let $E$ be a vector lattice and $\Theta$ be a convex modular on $E$. Then for any $f, g \in E$ and any $a \in (0,1]$,*

$$\Theta(f) \le \Theta(g) + \frac{a}{2}\left(\Theta(2g) + \Theta\left(\frac{2}{a}(f-g)\right)\right). \tag{M3}$$

*Consequently, for any $f, g \in E$ and any $a \in (0,1]$,*

$$|\Theta(f) - \Theta(g)| \le \frac{a}{2}\left(\max\left(\Theta(2f), \Theta(2g)\right) + \Theta\left(\frac{2}{a}(f-g)\right)\right).$$

*Proof.* Consider the following chain of inequalities:

$$\begin{aligned}
\Theta(f) &= \Theta\left((1-a)g + \frac{a}{2}\left(2g + \left(\frac{2}{a}(f-g)\right)\right)\right) \\
&\le (1-a)\Theta(g) + a\,\Theta\left(\frac{1}{2}(2g) + \frac{1}{2}\left(\frac{2}{a}(f-g)\right)\right) \\
&\le (1-a)\Theta(g) + \frac{a}{2}\left(\Theta(2g) + \Theta\left(\frac{2}{a}(f-g)\right)\right).
\end{aligned}$$

Since $a \in (0,1]$ is arbitrary, we get the desired conclusion. $\square$

**Proposition 2.1.2.** *Let $E$ be a vector lattice and let $\Theta$ be a convex modular on $E$. Then the map*

$$t \mapsto \Theta(tf) : \mathbb{R} \to [0, \infty)$$

*is continuous for any $f \in E$.*

*Proof.* We may assume $\Theta(f) \neq 0$; otherwise $f = 0$ and so $\Theta(tf) = 0$ for all $t$. Let $\epsilon \in (0,1]$. Let $0 < \delta < \min\left(\frac{\epsilon}{\Theta(f)}, 1\right)$ and $0 < t < \delta$. Then we have that

$$\Theta(tf) \leq \Theta(\delta f) \leq \delta \Theta(f) < \epsilon,$$

and so it is the case indeed that $t \mapsto \Theta(tf)$ is continuous at $t = 0$.
Now, let us consider arbitrary $t_0 \in \mathbb{R}$. Let $\epsilon \in (0,1]$, $t \in \mathbb{R}$. By Lemma 2.1.1, we have that, for any $a \in (0,1]$,

$$|\Theta(tf) - \Theta(t_0 f)| \leq \frac{a}{2}\left(\max\left(\Theta(2t_0 f), \Theta(2tf)\right) + \Theta\left(\frac{2}{a}(t - t_0) f\right)\right).$$

Choose $a > 0$ such that

$$\frac{a}{2}\left(\Theta\left((2|t_0| + 1)f\right) + 1\right) < \epsilon,$$

and $\delta \in (0,1]$ such that

$$\Theta\left(\frac{2}{a}\delta f\right) \leq 1.$$

Then $|t - t_0| < \delta$ implies that $|\Theta(tf) - \Theta(t_0 f)| < \epsilon$. This shows that $t \mapsto \Theta(tf)$ is continuous. $\square$

Let $E$ be a vector lattice and let $\Theta$ be a convex modular on $E$. We define $\|\cdot\|_\Theta : E \to [0, \infty]$ by

$$\|f\|_\Theta = \inf\{t \in (0, \infty) \mid \Theta(f/t) \leq 1\} \tag{N1}$$

for all $f \in E$. This is referred to as the *Luxemburg* norm in the literature. In what follows, we will show that it is indeed a norm. (See, for example, [23, page 7].)

**Lemma 2.1.3.** *Let $E$ be a vector lattice and let $\Theta$ be a convex modular on $E$. For any $f \in E$ and any $a > 0$, $\|f\|_\Theta \leq a \Longleftrightarrow \Theta(f/a) \leq 1$.*

*Proof.* The right to left implication is trivial. For the left to right direction, it suffices by Lemma 2.1.2 to consider the case $\|f\|_\Theta < a$. In that case, there exists $0 < b < a$ such that $\Theta(f/b) \leq 1$. By M2 we have that

$\Theta(f/a) \leq \Theta(f/b)$, and so the desired conclusion is achieved. $\square$

**Proposition 2.1.4.** *Let $E$ be a vector lattice and $\Theta$ be a convex modular on $E$. Then $\|\cdot\|_\Theta$ is a norm on $E$.*

*Proof.* It is obvious that $\|0\|_\Theta = 0$. If $\|f\|_\Theta = 0$, then there exists a decreasing sequence $(t_m)$ such that $t_m \to 0$ as $m \to \infty$, and for all $m \in \mathbb{N}$, $\Theta(f/t_m) \leq 1$. Let $n \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ such that $t_m < 1/n$. So $n < 1/t_m$, and so $|nf| < |f/t_m|$, whence by M2, $\Theta(nf) \leq \Theta(f/t_m) \leq 1$. This shows that for all $n \in \mathbb{N}$, $\Theta(nf) \leq 1$. So

$$\Theta(f) = \Theta\left(\frac{nf}{n}\right) \leq \frac{1}{n}\Theta(nf) \to 0$$

as $n \to \infty$, whence $\Theta(f) = 0$. Hence $f = 0$.

Note that by M2 we have that $\Theta(-f) = \Theta(f)$ for all $f \in E$. Also, for all $\alpha > 0$, we have

$$\begin{aligned}
\|\alpha f\|_\Theta &= \inf\left\{t \in (0,\infty) \mid \Theta\left(\frac{\alpha f}{t}\right) \leq 1\right\} \\
&= \inf\left\{t \in (0,\infty) \mid \Theta\left(\frac{f}{t/\alpha}\right) \leq 1\right\} \\
&= \alpha \inf\left\{t/\alpha \in (0,\infty) \mid \Theta\left(\frac{f}{t/\alpha}\right) \leq 1\right\} \\
&= \alpha\|f\|_\Theta.
\end{aligned}$$

Finally, we need to show the triangle inequality for $\|\cdot\|_\Theta$. Let $f, g \in E$. By Lemma 2.1.3, we know that

$$\Theta\left(\frac{f}{\|f\|_\Theta}\right) \leq 1$$

and

$$\Theta\left(\frac{g}{\|g\|_\Theta}\right) \leq 1.$$

Hence, if we set $\alpha = \|f\|_\Theta$ and $\beta = \|g\|_\Theta$, we have

$$\begin{aligned}
\Theta\left(\frac{f+g}{\|f\|_\Theta + \|g\|_\Theta}\right) &= \Theta\left(\frac{\alpha}{\alpha+\beta}\cdot\frac{f}{\alpha} + \frac{\beta}{\alpha+\beta}\cdot\frac{g}{\beta}\right) \\
&\leq \frac{\alpha}{\alpha+\beta}\Theta\left(\frac{f}{\alpha}\right) + \frac{\beta}{\alpha+\beta}\Theta\left(\frac{g}{\beta}\right) \\
&\leq 1.
\end{aligned}$$

An application of Lemma 2.1.3 finishes the argument. Also, one must note that basic properties of the convex modular preclude the possibility that the set $\{t \in (0,\infty) \mid \Theta(f/t) \leq 1\}$ is empty. Thus $\|\cdot\|_\Theta$ takes real values only. ☐

**Corollary.** *Let $E$ be a vector lattice and $\Theta$ be a convex modular on $E$. Then $(E, \|\cdot\|_\Theta)$ is a normed vector lattice.*

*Proof.* It suffices to show that for any $f, g \in E$, $\|f\|_\Theta \leq \|g\|_\Theta$ whenever $|f| \leq |g|$. But Lemma 2.1.3 yields

$$|f| \leq |g| \Longrightarrow (\forall\, a > 0)\, \Theta(f/a) \leq \Theta(g/a) \Longrightarrow \|f\|_\Theta \leq \|g\|_\Theta,$$

and this finishes the proof. ☐

**Definition.** Let $E$ be a normed vector lattice with norm $\|\cdot\|$. If $\Theta$ is a convex modular on $E$, we say that $(E, \Theta)$ is a *modulared* normed vector lattice if $\Theta$ induces the given norm on $E$, i.e., if

$$\|x\| = \inf\{\epsilon > 0 \mid \Theta(x/\epsilon) \leq 1\}$$

for all $x \in E$. An *Orlicz lattice* is a complete modulared normed vector lattice.

For $k > 0$, a convex modular $\Theta$ on a vector lattice $E$ is said to satisfy the $\Delta_2^k$-*condition* if $\Theta(2f) \leq k\Theta(f)$ for any $f \in E$. A convex modular $\Theta$ is said to satisfy the $\Delta_2$-*condition* if it satisfies the $\Delta_2^k$-condition for some $k \geq 2$.

**Lemma 2.1.5.** *Let $k \geq 2$. For any vector lattice $E$ and convex modular $\Theta$ on $E$ satisfying the $\Delta_2^k$-condition, we have that*

$$\Theta : (E, \|\cdot\|_\Theta) \to [0, \infty)$$

*is uniformly continuous on bounded subsets of $(E, \|\cdot\|_\Theta)$. Moreover, for each $n \geq 1$, the modulus of uniform continuity $\Delta_k^{(n)}$ associated to the restriction of $\Theta$ to the ball $B_n := \{f \in E : \|f\|_\Theta \leq n\}$ depends only on $n$ and $k$, and not on which specific vector lattice $(E, \Theta)$ we may have in mind.*

*Proof.* Let $n \geq 1$, and let $B_n := \{f \in E : \|f\|_\Theta \leq n\}$. Let $\epsilon > 0$ be arbitrary. Let $m(n) = \lfloor \log_2(n) \rfloor + 1$. Then for any $g \in B_r$ we have that

$$\Theta(g) = \Theta(\frac{rg}{n}) \leq k^{m(n)}\Theta(\frac{g}{n}) \leq k^{m(n)}.$$

Choose $a \in (0,1)$ small enough to satisfy $\frac{ak}{2}\left(1 + k^{m(n)}\right) \leq \epsilon$. Let $\Delta_k^{(n)}(\epsilon) = a$. Then if $f, g \in B_n$ and

$\|f-g\|_\Theta < \Delta_k^{(n)}(\epsilon)$, we have that

$$|\Theta(f)-\Theta(g)| \leq \frac{ak^{m(n)+1}}{2} + \frac{ak}{2}\,\Theta\left(\frac{f-g}{a}\right) \leq \epsilon.$$

Thus the function $\Delta_k^{(n)} : (0,1) \to (0,1)$ above defined is a modulus of uniform continuity for $\Theta$ on $B_r$. Note also that $\Delta_k^{(n)}$ depends only on $k$ and $n$, as claimed in the statement of the Lemma. □

For a given natural number $k \geq 2$, we expand the signature of Banach lattices $L$ by adding, for each sort $B_n$ ($n \in \mathbb{N}$), a predicate symbol $\Theta^{(n)}$ of arity $(n)$ (equipped with the modulus of uniform continuity $\Delta_k^{(n)}$ defined in the proof of Lemma 2.1.5), and call the new signature $L_k$. By interpreting $\Theta^{(n)}$ as the restriction of the convex modular to the ball of radius $n$ centered at the origin, we obtain that Orlicz lattices satisfying the $\Delta_2^k$-condition are $L_k$-structures. In the positive bounded setting of [16], we have that the class $\mathcal{OL}_k$ of all normed vector lattices of the form $(E, \|\cdot\|_\Theta)$ where $\Theta$ is a convex modular satisfying the $\Delta_2^k$-condition is a *uniform class.*[1]

The following simple lemma shows that the implication $\Theta(f) = 0 \implies \|f\|_\Theta = 0$ holds in a uniform way, depending only on $k$ such that $\Theta$ satisfies the $\Delta_2^k$ condition.

**Lemma 2.1.6.** *Let $k \geq 2$. Let $\delta_k : (0,1) \to (0,\infty)$ be defined by*

$$\delta_k(\epsilon) = \frac{1}{k^{\lfloor log_2(\frac{1}{\epsilon})\rfloor + 1}}.$$

*Let $(E,\Theta)$ be an Orlicz lattice satisfying the $\Delta_2^k$-condition. For any $\epsilon \in (0,1]$, for any $f \in E$, we have that $\Theta(f) < \delta_k(\epsilon) \implies \|f\|_\Theta \leq \epsilon$.*

*Proof.* Observe that $\Theta(f) < \delta(\epsilon)$ implies

$$\begin{aligned}\Theta(f/\epsilon) &\leq \Theta\left(2^{\lfloor \log_2(\frac{1}{\epsilon})\rfloor+1} f\right) \leq k^{\lfloor \log_2(\frac{1}{\epsilon})\rfloor+1}\Theta(f)\\ &= \frac{\Theta(f)}{\delta(\epsilon)} < 1.\end{aligned}$$

Therefore, by Lemma 2.1.3, $\|f\|_\Theta \leq \epsilon$. □

## 2.2 Axioms for Orlicz lattices

Let $L$ be the signature of Banach lattices. Recall that in section 1.1 we observed that Banach lattices are $L$-structures. We proceed to list axioms for the class NVL of normed vector lattices, a class for which Banach

[1] The reader is referred to [16, pages 38–39] for a definition of uniform classes.

lattices constitute complete models. Axioms (i) to (ix) are concerned with how the many versions of a given type of function symbol (one for each $n \in \mathbb{N}$) are related to each other. The basic idea is that the ones with "smaller" domains are restrictions of the ones with "bigger" domains. This is expressed using the inclusion maps $I_{m,n}$.

*Remark.* Equational axioms of the form $\forall \bar{x} \ldots t = s$ (where $t$ and $s$ are terms and $\bar{x}$ is a tuple containing all variables occurring in $t$ and $s$) can be expressed equivalently in continuous logic by $\sup_x \ldots d(t,s) = 0$. In what follows, we will make frequent use of this fact to specify axioms to be added to the ongoing list. For the sake of brevity, we will also write $|x|_n$ as an abbreviation for the $L$-term $(x \vee_n 0^{(n)}) +_{2n} ((-1)_n x \vee_n 0^{(n)})$.

(i) For each $m, n, p \in \mathbb{N}$ with $m < n < p$ the axiom

$$\sup_x d^{(p)}(I_{m,p}(x), I_{n,p}(I_{m,n}(x))) = 0$$

(where $x$ is a variable of sort $m$) ensuring a basic compatibility of the inclusion maps.

(ii) For each $m, n \in \mathbb{N}$ with $m < n$, the axiom

$$d^{(n)}(0^{(n)}, I_{m,n}(0^{(m)})) = 0.$$

(iii) For each $m, n \in \mathbb{N}$ with $m < n$, the axiom

$$\sup_x \mid d^{(m)}(x, 0^{(m)}) - d^{(n)}(I_{m,n}(x), 0^{(n)}) \mid = 0$$

(where $x$ is a variable of sort $m$).

(iv) For each $m, n \in \mathbb{N}$ with $m < n$, the axiom

$$\sup_x \sup_y |d^{(m)}(x,y) - d^{(n)}(I_{m,n}(x), I_{m,n}(y))| = 0$$

(where $x$ and $y$ are variables of sort $m$).

(v) For each $m, n \in \mathbb{N}$ with $m < n$, the axiom

$$\sup_x d^{(n)}(I_{m,n}(-_m x), +_n(-1)_n(I_{m,n}(x))) = 0$$

(where $x$ is a variable of sort $m$).

(vi) For each $\lambda$ satisfying $k-1 < \lambda \le k$ for some $k \ge 1$, and for each $m, n \in \mathbb{N}$ with $m < n$, the axiom

$$\sup_x d^{(kn)}(I_{km,kn}(\lambda_m x), \lambda_n I_{m,n}(x)) = 0$$

(where $x$ is a variable of sort $m$).

(vii) For each $m, n \in \mathbb{N}$ with $m < n$, the axiom

$$\sup_x \sup_y d^{(2n)}(I_{m,n}(x) +_n I_{m,n}(x), I_{2m,2n}(x +_m y)) = 0$$

(where $x$ and $y$ are variables of sort $m$).

(viii) For each $m, n \in \mathbb{N}$ with $m < n$, the axiom

$$\sup_x \sup_y d^{(2n)}(I_{m,n}(x) \vee_n I_{m,n}(x), I_{2m,2n}(x \vee_m y)) = 0$$

(where $x$ and $y$ are variables of sort $m$).

(ix) For each $m, n \in \mathbb{N}$ with $m < n$, the axiom

$$\sup_x \sup_y d^{(2n)}(I_{m,n}(x) \wedge_n I_{m,n}(x), I_{2m,2n}(x \wedge_m y)) = 0$$

(where $x$ and $y$ are variables of sort $m$).

(x) Axioms for vector spaces: For a complete list of first-order axioms for vector spaces, the reader is referred to [10, page 49]. All first-order axioms for vector spaces are equational axioms,[2] and so, by the remark above, we may add continuous axioms for vector spaces.

(xi) Axioms for vector lattices: First order axioms for vector lattices are once again equational axioms. They consist of the following:

(a) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;

(b) $x \vee (y \vee z) = (x \vee y) \vee z$;

[2] By way of illustration, consider the first order axiom

$$\forall x \forall y \quad x + y = y + x.$$

In continuous logic, this axiom is expressed by an infinite family of axioms

$$\sup_x \sup_y d^{(2n)}(x +_n y, y +_n x) = 0.$$

indexed by $n \in \mathbb{N}$.

(c) $x \wedge y = y \wedge x$;

(d) $x \vee y = y \vee x$;

(e) $x \wedge x = x$;

(f) $x \vee x = x$;

(g) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$;

(h) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$;

(i) $(x \wedge y) \vee y = y$;

(j) $(x \wedge y) + z = ((x \wedge y) + z) \wedge (y + z)$;

(k) $(\lambda(x \wedge y) \wedge \lambda x = \lambda(x \wedge y))$ for each $\lambda > 0$.

Since each one of these axioms is an equational axiom, we may make use of the preceding remark to write down continuous axioms for them.

(xii) Axioms ensuring that $\| \cdot \|^{(n)}$ is the restriction of a seminorm to the sort $B_n$ compatible with the operations $+_n$, $-_n$, and $\lambda_n$ for all $\lambda \in \mathbb{R}$. Note that one cannot express in continuous logic the implication $\|x\| = 0 \implies x = 0$. However, the condition $\|x\| = d(x, 0)$, along with the metalogical requirement that $d$ be a metric, ensure that the seminorm $\| \cdot \|$ is actually a norm.

(xiii) For each $m \in \mathbb{N}$, the axioms

$$\sup_x \sup_y \left| d^{(m)}(x, y) - \|x -_m y\|^{(2m)} \right| = 0$$

(where $x$ and $y$ are variables of sort $m$).

(xiv) For each $n \in \mathbb{N}$, the axioms

$$\sup_x \ \left| \, \| |x|_n \|^{(4n)} - \|x\|^{(n)} \, \right| = 0$$

and

$$\sup_x \sup_y \ \left| \, \min(\| |x|_n \wedge_{4n} |y|_n \|^{(8n)}, \|x\|^{(n)}) - \| |x|_n \wedge_{4n} |y|_n \|^{(8n)} \, \right| = 0$$

(where $x$ and $y$ are variables of sort $n$). The collection of all these axioms (for $n \in \mathbb{N}$) ensures that $\| \cdot \|$ is a lattice norm.

Let $\mathcal{OL}_k$ denote the class of all Orlicz lattices satisfying the $\Delta_2^k$-condition, for fixed $k \geq 2$. Let us now write down axioms for the class $\mathcal{OL}_k$ in the signature $L_k$ defined at the end of section 2.1. In order to simplify our notation, we will make use of the connective $\dot{-}\colon [0,\infty)^2 \to [0,\infty)$ defined by

$$x \dot{-} y = \max(x - y, 0).$$

(i) A collection $\Sigma_{\mathrm{NVL}}$ of axioms for normed vector lattices as described above.

(ii) For each $n \in \mathbb{N}$, the axioms

$$\sup_x \ \min(\delta_k(\epsilon) \dot{-} \Theta^{(n)}(x), \|x\|^{(n)} \dot{-} \epsilon) = 0$$

for all $0 < \epsilon \leq 1$, where $x$ is a variable of sort $n$, and $\delta_k : (0,1) \to (0,\infty)$ is the function defined in the statement of Lemma 2.1.6. The collection of all these axioms (for $n \in \mathbb{N}$), together with the collection

$$\{\Theta^{(n)}(0^{(n)}) = 0 \mid n \in \mathbb{N}\},$$

will be denoted by $\Sigma_{M1}$ to indicate that they refer to condition $M1$ in the definition of Orlicz lattices. (Note that the choice of $\delta_k$ reflects the fact that we are axiomatizing the class of Orlicz lattices whose convex modulars satisfy the $\Delta_2^k$-condition for a specific, fixed value of $k \geq 2$.)

(iii) For each $n \in \mathbb{N}$, the axioms

$$\sup_x \ \left|\Theta^{(4n)}(|x|_n) - \Theta^{(n)}(x)\right| = 0$$

(where $x$ is a variable of sort $n$) and

$$\sup_x \sup_y \ \left(\Theta^{(8n)}(|x|_n \wedge_{4n} |y|_n) \dot{-} \Theta^{(4n)}(|x|_n)\right) = 0$$

(where $x$ and $y$ are variables of sort $n$). We will denote the collection of all these axioms (for $n \in \mathbb{N}$) by $\Sigma_{M2}$ to indicate that they refer to condition $M2$ in the definition of Orlicz lattices.

(iv) For each $n \in \mathbb{N}$, the axioms

$$\sup_x \sup_y \ \left(\Theta^{(2n)}(\alpha_n x +_n \beta_n y) \dot{-} \left(\alpha\Theta^{(n)}(x) + \beta\Theta^{(n)}(y)\right)\right) = 0$$

(where $x$ and $y$ are variables of sort $n$) for all $(\alpha, \beta) \in [0,1]^2$ satisfying $\alpha + \beta = 1$. The collection of all these axioms (for $n \in \mathbb{N}$) will be denoted by $\Sigma_{CM}$ to indicate that they refer to the condition $CM$ in the definition of Orlicz lattices.

(v) For each $n \in \mathbb{N}$, the axioms

$$\sup_x \sup_y \left| \Theta^{(32n)}(F_{x,y} +_{16n} G_{x,y}) - (\Theta^{(16n)}(F_{x,y}) + \Theta^{(16n)}(G_{x,y})) \right| = 0,$$

where $x$ and $y$ are variables of sort $n$, and where for $(x, y) \in (B_n)^2$,

$$F_{x,y} = I_{4n,8n}(|x|_n) +_{8n} (-1)_{8n} (|x|_n \wedge_{4n} |y|_n)$$

and

$$G_{x,y} = I_{4n,8n}(|y|_n) +_{8n} (-1)_{8n} (|x|_n \wedge_{4n} |y|_n) .$$

The collection of all these axioms (for $n \in \mathbb{N}$) will be denoted by $\Sigma_{DA}$ to indicate that they capture the condition that the convex modular be disjointly additive, i.e. condition $DA$ in the definition of Orlicz lattices.

(vi) For each $n \in \mathbb{N}$, for $t > 0$ with $m - 1 < \frac{1}{t} \leq m$ ($m \in \mathbb{N}$), the axiom

$$\sup_x \ \min \left( 1 \dot{-} \Theta^{(mn)} (\lambda_n x) \, , \|x\|^{(n)} \dot{-} t \right) = 0$$

(where $x$ is a variable of sort $n$ and $\lambda := \frac{1}{t}$). The collection of all these axioms (for $n \in \mathbb{N}$) together with the axiom

$$\sup_z (\Theta_1(z) \dot{-} 1) = 0$$

(where $z$ is a variable of sort $B_1$) will be denoted by $\Sigma_{N1}$ to indicate that it encodes the condition that the norm and the convex modular be related by equation $N1$.

(vii) For each $n \in \mathbb{N}$, the axiom

$$\sup_x \ \left( \Theta^{(2n)}(2_n x) \dot{-} k\Theta^{(n)}(x) \right) = 0$$

(where $x$ is a variable of sort $n$). The collection of all these axioms (for $n \in \mathbb{N}$) will be denoted by $\Sigma_{\Delta_2^k}$ to indicate that it ensures the satisfaction of the $\Delta_2^k$-condition for the given $k \geq 2$.

We claim that the collection $\Sigma_{\mathcal{OL}_k}$ defined by

$$\Sigma_{\mathcal{OL}_k} := \Sigma_{\text{NVL}} \cup \Sigma_{M1} \cup \Sigma_{M2} \cup \Sigma_{CM} \cup \Sigma_{DA} \cup \Sigma_{N1} \cup \Sigma_{\Delta^k}$$

axiomatizes the class $\mathcal{OL}_k$.

**Proposition 2.2.1.** *Let* $(E,\Theta) \in \mathcal{OL}_k$*. Then* $(E,\Theta) \models \Sigma_{\mathcal{OL}_k}$*.*

*Proof.* It is clear that $(E,\Theta) \models \Sigma_{\mathrm{NVL}}$. Lemma 2.1.6 shows that $(E,\Theta) \models \Sigma_{M1}$. It is obvious that

$$(E,\Theta) \models \Sigma_{CM} \cup \Sigma_{DA} \cup \Sigma_{\Delta_2^k} \cup \Sigma_{M2}.$$

It is also clear that for any $f \in B_1$, $\Theta(f) \leq 1$. Given $f \in E$, $t > 0$, and $\epsilon > 0$,

$$\begin{aligned} \|f\| > (1+\epsilon)t &\Longrightarrow \Theta\left(f/(1+\epsilon)t\right) > 1 \\ &\Longrightarrow \frac{1}{1+\epsilon}\Theta\left(f/t\right) > 1 \\ &\Longrightarrow \Theta\left(f/t\right) > 1+\epsilon. \end{aligned}$$

Hence $\Sigma_{N1}$ is satisfied. □

**Proposition 2.2.2.** *Let* $(E,\Theta)$ *be an* $L_k$*-structure that is complete for its metric. If* $(E,\Theta) \models \Sigma_{\mathcal{OL}_k}$*, then* $(E,\Theta) \in \mathcal{OL}_k$*.*

*Proof.* It immediately follows from $\Sigma_{M1}$ that $\Theta(0) = 0$. If $\Theta(f) = 0$, then by $\Sigma_{M1}$ together with axioms (xii), $d(f,0) = \|f\| \leq \epsilon$ for all $\epsilon > 0$, and so $f = 0$, since $d$ is a metric.

Let $f, g \in E$ satisfy $|f| \wedge |g| = 0$. Using $\Sigma_{M2}$ and $\Sigma_{DA}$, we have that

$$\begin{aligned} \Theta(f+g) &= \Theta(|f+g|) \\ &= \Theta(|f|+|g|) \\ &= \Theta(|f| - |f|\wedge|g| + |g| - |f|\wedge|g|) \\ &= \Theta(|f| - |f|\wedge|g|) + \Theta(|g| - |f|\wedge|g|) \\ &= \Theta(|f|) + \Theta(|g|) \\ &= \Theta(f) + \Theta(g). \end{aligned}$$

Let $f, g \in E$ and assume that $|f| \leq |g|$. Then

$$\begin{aligned} \Theta(f) = \Theta(|f|) &= \Theta(|f| \wedge |g|) \\ &\leq \Theta(|g|) = \Theta(g). \end{aligned}$$

It is easy to show that $(E,\Theta)$ satisfies condition CM. As a direct consequence of $\Sigma_{\Delta_2^k}$, we have that

$(E, \Theta)$ satisfies the inequality $\Theta(2f) \leq k\Theta(f)$ for all $f \in E$.

It remains to show that $(E, \Theta)$ satisfies condition N1. For any nonzero $f \in E$, we have that $\Theta(f/\|f\|) \leq 1$, which implies that

$$\|f\| \geq \inf\{\alpha > 0 \mid \Theta(f/\alpha) \leq 1\}.$$

If $0 < t < \|f\|$, then there is $\epsilon > 0$ such that $(1+\epsilon)t < \|f\|$, and so by $\Sigma_{N1}$, $\Theta(f/t) \geq 1 + \epsilon$. This implies that

$$\|f\| \leq \inf\{\alpha > 0 \mid \Theta(f/\alpha) \leq 1\}.$$

Thus we have that

$$\|f\| = \inf\{t \in (0, \infty) \mid \Theta(f/t) \leq 1\},$$

and this establishes the desired result. □

As a corollary, we have that for $k \geq 2$ fixed, the uniform class $\mathcal{OL}_k$ is axiomatizable by universal closed statements in the signature of vector lattices with a convex modular. Thus $\mathcal{OL}_k$ is closed under ultraproducts. This had already been proved by Wnuk in [26, section 7], as well as by Haydon, Levy, and Raynaud in [13, Proposition 4.17]. In the course of this section, we have simply provided a more explicit model-theoretic point of view.

## 2.3 Musielak-Orlicz spaces

If $(\Omega, \Sigma, \mu)$ is a measure space, we denote by $L_0(\Omega, \Sigma, \mu)$ the space of equivalence classes of measurable real-valued functions defined on $\Omega$, relative to the equivalence relation of equality $\mu$-almost everywhere.

A measure space is *decomposable* if there exists a partition $(\Omega_\alpha \mid \alpha \in I)$ of $\Omega$ such that:

(i) $\Omega_\alpha \in \Sigma$ and $\mu(\Omega_\alpha) < \infty$, for all $\alpha \in I$.

(ii) $\Sigma = \{A \subseteq \Omega \mid \forall \alpha \in I \ \Omega_\alpha \cap A \in \Sigma\}$.

(iii) For all $A \in \Sigma$, $\mu(A) = \sum_{\alpha \in I} \mu(A \cap \Omega_\alpha)$.

Unless otherwise stated, every measure space appearing in the remainder of this work is decomposable.

A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be an *Orlicz function* if $\varphi(0) = 0$, $\varphi(1) = 1$, and $\varphi$ is convex, and continuous. Given $k \geq 2$, if $\varphi(2t) \leq k\varphi(t)$ for all $t \in [0, \infty)$, we say that $\varphi$ *satisfies the* $\Delta_2^k$*-condition.*

*Remark.* In the literature, the requirement $\varphi(1) = 1$ above is often excluded from the definition of Orlicz function, and Orlicz functions $\varphi$ satisfying such a requirement are said to be *normalized.* In the course of this work we will deal only with normalized Orlicz functions.

Let $(\Omega, \Sigma, \mu)$ be a measure space. A *Musielak-Orlicz function* on $(\Omega, \Sigma, \mu)$ is a function

$$\psi : [0, \infty) \times \Omega \to [0, \infty)$$

such that

**A:** $\psi(t, \cdot) \in L_0(\Omega, \Sigma, \mu)$ for every $t \in [0, \infty)$.

**B:** $\psi(\cdot, \omega)$ is an Orlicz function for all $\omega \in \Omega$.

Given a measure space $(\Omega, \Sigma, \mu)$ and a Musielak-Orlicz function $\psi$ on it, we define $\Psi : L_0(\Omega, \Sigma, \mu) \to [0, \infty]$ by

$$\Psi(f) = \int_\Omega \psi\left(|f(\omega)|, \omega\right)\, d\mu(\omega)$$

and set

$$\|f\|_\psi = \inf\left\{\alpha \in [0, \infty) \mid \Psi\left(\frac{f}{\alpha}\right) \leq 1\right\}.$$

Let

$$L_\psi(\Omega, \Sigma, \mu) = \left\{f \in L_0(\Omega, \Sigma, \mu) \,\middle|\, \|f\|_\psi < \infty\right\}.$$

Then $L_\psi(\Omega, \Sigma, \mu)$ is a *Musielak-Orlicz space.* The functional $\Psi$ is a convex modular on $L_\psi(\Omega, \Sigma, \mu)$. The norm $\|\cdot\|_\psi$ is usually called the *Luxemburg norm* in the literature. In the case in which

$$\{\psi(\cdot, \omega) \mid \omega \in \Omega\} = \{\varphi\}$$

for some fixed Orlicz function $\varphi$, then we call the associated Musielak-Orlicz space an *Orlicz space* and we denote it by

$$L_\varphi(\Omega, \Sigma, \mu).$$

Let $k \geq 2$. We say that a Musielak-Orlicz space $L_\psi(\Omega, \Sigma, \mu)$ satisfies the $\Delta_2^k$*-condition* if for $\psi(\cdot, \omega)$ satisfies the $\Delta_2^k$-condition for a.e. $\omega \in \Omega$. In such a case, $\Psi$ satisfies the $\Delta_2^k$-condition as a convex modular. Recall that a Banach lattice $(X, \|\cdot\|)$ is said to be *order-continuous* if for every downwards directed net $(x_i \mid i \in I)$ of elements in $X$ satisfying $\inf x_i = 0$ it is the case that $\|x_i\| \to 0$.

Let $L_\psi := L_\psi(\Omega, \Sigma, \mu)$ be a Musielak-Orlicz space. Then it contains a dense ideal

$$H_\psi = \{f \in L_0(\Omega, \Sigma, \mu) \mid \Psi(tf) < \infty \text{ for all } t \in (0, \infty)\}.$$

In case $L_\psi$ satisfies the $\Delta_2^k$-condition for some $k \geq 2$, then $H_\psi = L_\psi$ and $L_\psi$ is order-continuous. (See pages 31–32 of [13].)

The following representation theorem appears in [13] as Theorem 3.17. We will only need to apply it in cases in which the $\Delta_2^k$-condition is satisfied for some $k \geq 2$, in which case $H_\psi = L_\psi$.

**Theorem 2.3.1.** *Let $(E, \Theta)$ be an Orlicz lattice. Then there exist a measure space $(\Omega, \Sigma, \mu)$ and a Musielak-Orlicz function $\psi$ on $\Omega$ such that $(E, \Theta)$ is isomodularly isomorphic to $(H_\psi(\Omega, \Sigma, \mu), \Psi)$ as Orlicz lattices.*

## 2.4 Axiomatizable classes of Musielak-Orlicz spaces

Let $X$ be the space of all functions $f : [0, \infty) \to [0, \infty)$, equipped with the topology of pointwise convergence. A basic open neighborhood of an element $f$ of $X$ is of the form

$$O(F, \epsilon) := \{g \in X \mid |g(x) - f(x)| < \epsilon \text{ for all } x \in F\},$$

where $F$ is a finite subset of $[0, \infty)$ and $\epsilon > 0$. If we view $X$ as the product of continuum many copies of $[0, \infty)$, this gives $X$ the product topology. On $X$ we may define an partial order in the following way: for $f, g \in X$, $f \leq g$ if $f(x) \leq g(x)$ for all $x \in [0, \infty)$. We say that a subset $A$ of $X$ is *dominated* if there is $f \in X$ such that $g \leq f$ for all $g \in A$. Such an $f$ is said to be a *bound of* $A$.

**Lemma 2.4.1.** *Let $K \subseteq X$. Then $K$ is compact if and only if it is both closed and dominated in $X$.*

*Proof.* ($\Longrightarrow$) Suppose $K$ is compact (and therefore closed in $X$). Because for any $r \in [0, \infty)$, the projection $\pi_r : X \to \{r\} \times [0, \infty)$ is continuous, there exist $M_r \in [0, \infty)$ such that $\pi_r(K) \subseteq \{r\} \times [0, M_r]$. Thus

$$K \subseteq \prod_{r \in [0, \infty)} \{r\} \times [0, M_r]$$

and so $K$ is dominated.
($\Longleftarrow$) Suppose $K$ is closed and dominated in $X$, and let $f$ be a bound of $K$. Then we may consider $K$ as a subset of the compact subspace

$$\prod_{r \in [0, \infty)} \{r\} \times [0, f(r)]$$

of $X$. Since closed subsets of compact spaces are compact, then $K$ is compact. □

For $k \geq 2$, let $\mathcal{F}_k$ denote the class of all Orlicz functions satisfying the $\Delta_2^k$-condition, and equip $\mathcal{F}_k$ with the topology of pointwise convergence.

**Lemma 2.4.2.** *$\mathcal{F}_k$ is compact.*

*Proof.* Let $\phi^\star : [0,\infty) \to [0,\infty)$ be the function defined by

$$\phi^\star(t) = \begin{cases} t, & \text{if } t \in [0,1]; \\ k^{\lfloor \log_2(t) \rfloor + 1} & \text{otherwise.} \end{cases}$$

Then $\mathcal{F}_k$ has $\phi^\star$ as a bound in $X$. On the other hand, a pointwise limit of Orlicz functions satisfying the $\Delta_2^k$-condition is an Orlicz function satisfying the $\Delta_2^k$-condition, and so $\mathcal{F}_k$ is closed in $X$. □

Given an Orlicz function $\varphi$ and a real number $a > 0$, the function $\varphi_a$ defined by $\varphi_a(t) := \frac{\varphi(at)}{\varphi(a)}$ is a *dilation of* $\varphi$. We say that a set $D$ of Orlicz functions is a *Dacunha-Castelle set* if $D$ is a closed (hence compact) subset of $\mathcal{F}_k$ (for some $k$) which is closed under dilations. Let $\mathcal{MO}$ denote the class of all Musielak-Orlicz spaces. Let $\mathcal{MO}_D$ be the class of all normed vector lattices $X$ such that $X$ is isometrically isomorphic to some $L_\psi(\Omega, \Sigma, \mu)$ satisfying $\psi(\cdot, \omega) \in D$ for $\mu$-almost every $\omega \in \Omega$.

Let $D$ be a Dacunha-Castelle set. Let us define $\mathcal{MO}_D^{\text{mod}}$ to be the class of all modulared Banach lattices $(X, \Theta)$ such that $(X, \Theta)$ is isomodularly isomorphic to some modulared Musielak-Orlicz space $(L_\psi(\Omega, \Sigma, \mu), \Psi)$ satisfying the condition that $\psi(\cdot, \omega) \in D$ for $\mu$-almost every $\omega \in \Omega$.

**Theorem 2.4.3** (Dacunha-Castelle's representation theorem)**.** *Let $D$ be a Dacunha-Castelle set. Then $\mathcal{MO}_D^{\text{mod}}$ is closed under ultraproducts.*

For a proof of the preceding theorem, the reader is referred to Théorème 1 in [9].

*Remark.* Since $\mathcal{MO}_D$ consists of the reducts to $L$ of elements of $\mathcal{MO}_D^{\text{mod}}$, $\mathcal{MO}_D$ itself is closed under ultraproducts.

Let

$$G_D = \{\Psi_f \mid L_\psi(\Omega, \Sigma, \mu) \in \mathcal{MO}_D,\ f \in L_\psi(\Omega, \Sigma, \mu)\},$$

where $\Psi_f$ stands for the function $[0,\infty) \to [0,\infty)$ defined by

$$\Psi_f(t) = \frac{\Psi(tf)}{\Psi(f)},$$

and where $\Psi$ is the convex modular associated with $L_\psi(\Omega, \Sigma, \mu)$. Let $H_D^0$ be the convex hull of $D$, and let $H_D$ be the closure of $H_D^0$.

**Theorem 2.4.4.** $G_D \subseteq H_D$.

*Proof.* First we note that for any $f \in L_\psi(\Omega, \Sigma, \mu)$, $\Psi_f = \Psi_{|f|}$. Let $f \in L_\psi(\Omega, \Sigma, \mu)$. If $f$ is a simple function, then $\Psi_f \in H_D^0 \subseteq H_D$. In the general case, let $(f_n \mid n \in \mathbb{N})$ be a monotone increasing sequence of simple functions such that $f_n \uparrow |f|$ a.e. Then, by the Monotone Convergence Theorem, $\Psi(tf_n) \uparrow \Psi(tf)$ for all $t \in \mathbb{R}$. Thus $\Psi_{f_n} \to \Psi_f$ pointwise, and $\Psi_f \in \overline{H_D^0} = H_D$. □

**Theorem 2.4.5.** *If $D$ is a Dacunha-Castelle set, then $\mathcal{MO}_{H_D}$ is a universally axiomatizable class of normed vector lattices.*

*Proof.* A class of $L$-structures closed under $L$-isomorphism is universally axiomatizable if and only if it is closed under ultraproducts and substructures. (See [16, Proposition 13.1] for a proof in the positive bounded setting.) If

$$A = L_\psi(\Omega, \Sigma, \mu) \in \mathcal{MO}_{H_D}$$

and $A_0$ is a vector sublattice of $A$, then $(A_0, \Psi \restriction A_0)$ is an Orlicz lattice. By the representation theorem for Orlicz lattices, $A_0$ can be realized as a Musielak-Orlicz space $L_{\psi_0}(\Omega_0, \Sigma_0, \mu_0)$ with $\psi_0(\cdot, \omega) \in H_D$ for all $\omega \in \Omega_0$.[3] But this means that $A_0$ can be seen as a Musielak-Orlicz space in $\mathcal{MO}_{H_D}$, and this implies that the class $\mathcal{MO}_{H_D}$ is closed under substructures. Since $D$ is a Dacunha-Castelle set, $H_D$ is also a Dacunha-Castelle set, and so the class $\mathcal{MO}_{H_D}$ is closed under ultraproducts as well. □

## 2.5 Lattice finite representability of $\mathcal{MO}_{H_D}$ in $\mathcal{MO}_D$

Let $L_\psi(\Omega, \Sigma, \mu) \in \mathcal{MO}_{H_D}$, and let $\chi_{A_1}, \ldots, \chi_{A_M} \in L_\psi(\Omega, \Sigma, \mu)$ be mutually disjoint, with

$$\mu_0 = \mu\left(\bigcup_{m=1}^{M} A_m\right) < \infty.$$

If $S$ denotes the sublattice of $L_\psi(\Omega, \Sigma, \mu)$ generated by $\chi_{A_1}, \ldots, \chi_{A_M}$, and $F$ is a finite set of rational numbers, we denote by $S_F$ the finite subset of $S$ defined by

$$S_F = \left\{ \sum_{m=1}^{M} q_m \chi_{A_m} \;\middle|\; (q_1, \ldots, q_M) \in F^M \right\}.$$

**Lemma 2.5.1.** *Consider $S_F$ as defined above, and let $\epsilon > 0$ be arbitrary. Then there exist*

$$L_{\psi_F}^{(\epsilon)}(\Omega_F^{(\epsilon)}, \Sigma_F^{(\epsilon)}, \mu_F^{(\epsilon)}) \in \mathcal{MO}_D$$

[3] Because $\psi_0(\cdot, \omega) \in \overline{\{\psi_f \mid f \in A\}} \subseteq G_{H_D} \subseteq H_D$.

*and a vector lattice isomorphism*

$$T : S_F \to L^{(\epsilon)}_{\psi_F}(\Omega^{(\epsilon)}_F, \Sigma^{(\epsilon)}_F, \mu^{(\epsilon)}_F)$$

*satisfying*

$$|\Psi(f) - \Psi^{(\epsilon)}_F(T(f))| < \epsilon$$

*for all* $f \in S_F$*, where* $\Psi$ *and* $\Psi^{(\epsilon)}_F$ *are the convex modulars associated to* $L_\psi(\Omega, \Sigma, \mu)$ *and* $L^{(\epsilon)}_{\psi_F}(\Omega^{(\epsilon)}_F, \Sigma^{(\epsilon)}_F, \mu^{(\epsilon)}_F)$*, respectively.*

*Proof.* For $F$ a finite subset of $\mathbb{Q}$, $\delta > 0$, and $\varphi \in H_D$, define

$$B_{\delta,F}(\varphi) = \{\phi \in H_D \mid |\varphi(|t|) - \phi(|t|)| < \delta \quad \text{for all } t \in F\}.$$

The collection

$$\mathcal{O} = \{B_{\delta,F}(\varphi) \mid \varphi \in H^0_D\}$$

is an open covering of $H_D$. By the compactness of $H_D$ in the topology of pointwise convergence, $\mathcal{O}$ has a finite subcover. By the measurability of $\psi(|t|, \cdot)$ for $t \in F$, we can find a measurable partition $\{B_1, \ldots, B_N\}$ of $\bigcup_{m=1}^M A_m$ refining $\{A_1, \ldots, A_M\}$, and a set $\{\varphi_1, \ldots, \varphi_N\} \subseteq H^0_D$ such that for all $t \in F$, for all $n \in \{1, \ldots, N\}$, for all $\omega \in B_n$,

$$|\psi(|t|, \omega) - \varphi_n(|t|)| < \delta.$$

For each $n \in \{1, \ldots, N\}$,

$$\varphi_n = \sum_{k=1}^{K(n)} \gamma^{(n)}_k \phi^{(n)}_k$$

where $\phi^{(n)}_k \in \mathcal{D}$ for all $k \in \{1, \ldots, K(n)\}$, for all $n \in \{1 \ldots, N\}$, and where

$$\sum_{k=1}^{K(n)} \gamma^{(n)}_k = 1.$$

Define $\Theta^{(\epsilon)}_F : S_F \to [0, \infty)$ by

$$\Theta^{(\epsilon)}_F\left(\sum_{n=1}^N q_n \chi_{B_n}\right) = \sum_{n=1}^N \varphi_n(|q_n|)\mu(B_n)$$

for any $\sum_{n=1}^N q_n \chi_{B_n} \in S_F$. We have that

$$|\Psi(g) - \Theta^{(\epsilon)}_F(g)| \le \mu_0 \delta < \epsilon$$

for all $g \in S_F$. Let $\Omega_F^{(\epsilon)} = \mathbb{R}$, $\Sigma_F^{(\epsilon)} = \mathcal{L}$, and $\mu_F^{(\epsilon)} = \lambda$. Let

$$\left\{ C_k^{(n)} \mid k \in \{1, \ldots, K(n)\},\, n \in \{1, \ldots, N\} \right\}$$

be a collection of pairwise disjoint intervals in $\mathbb{R}$ such that $\mu\left(C_k^{(n)}\right) = \mu(B_n)$ for every $k \in \{1, \ldots, K(n)\}$, for every $n \in \{1, \ldots, N\}$. Let $C = \bigcup_{k,n} C_k^{(n)}$. For $n \in \{1, \ldots, N\}$ and $k \in \{1, \ldots, K(n)\}$, we set

$$\rho_k^{(n)} = \left(\phi_k^{(n)}\right)^{-1} \left(\frac{1}{\gamma_k^{(n)}}\right).$$

Define the function

$$\psi_F^{(\epsilon)} : [0, \infty) \times \mathbb{R} \to [0, \infty)$$

by setting

$$\psi_F^{(\epsilon)}(t, \omega) = \sum_{n=1}^{N} \sum_{k=1}^{K(n)} \left[\phi_k^{(n)}\right]_{\rho_k^{(n)}} (t) \chi_{C_k^{(n)}}(\omega) + \phi_0(t) \chi_{\mathbb{R} \setminus C}(\omega)$$

for all $(t, \omega) \in [0, \infty) \times \mathbb{R}$, where $\phi_0$ is a fixed element in $D$. Then $\psi_F^{(\epsilon)}$ is a Musielak-Orlicz function on $(\mathbb{R}, \mathcal{L}, \lambda)$, and so

$$L_{\psi_F^{(\epsilon)}}(\Omega_F^{(\epsilon)}, \Sigma_F^{(\epsilon)}, \mu_F^{(\epsilon)})$$

is a Musielak-Orlicz space in $\mathcal{MO}_D$. Define the map

$$T : S_F \to L_{\psi_F^{(\epsilon)}}(\Omega_F^{(\epsilon)}, \Sigma_F^{(\epsilon)}, \mu_F^{(\epsilon)})$$

by

$$T\left(\sum_{n=1}^{N} q_n \chi_{B_n}\right) = \sum_{n=1}^{N} \sum_{k=1}^{K(n)} \left(\frac{q_n}{\rho_k^{(n)}}\right) \chi_{C_k^{(n)}}.$$

Then $T$ is a linear, lattice-preserving map, and it satisfies

$$\begin{aligned}
\Theta_F^{(\epsilon)}\left(\sum_{n=1}^{N} q_n \chi_{B_n}\right) &= \sum_{n=1}^{N} \sum_{k=1}^{K(n)} \gamma_k^{(n)} \phi_k^{(n)}(q_n)\, \mu(B_n) \\
&= \sum_{n=1}^{N} \sum_{k=1}^{K(n)} \left[\phi_k^{(n)}\right]_{\rho_k^{(n)}} \left(\left[\rho_k^{(n)}\right]^{-1} q_n\right) \mu(B_n) \\
&= \sum_{n=1}^{N} \sum_{k=1}^{K(n)} \left[\phi_k^{(n)}\right]_{\rho_k^{(n)}} \left(\left[\rho_k^{(n)}\right]^{-1} q_n\right) \mu\left(C_k^{(n)}\right) \\
&= \Psi_F^{(\epsilon)}\left(T\left[\sum_{n=1}^{N} q_n \chi_{B_n}\right]\right).
\end{aligned}$$

This completes our argument. □

**Theorem 2.5.2.** *Let $L$ be a finite-dimensional sublattice of a Musielak-Orlicz space $L_\psi(\Omega, \Sigma, \mu) \in \mathcal{MO}_{H_D}$. Then there exists*

$$L_D = L_{\widehat{\psi}}(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu}) \in \mathcal{MO}_D$$

*such that $L$ is normed vector lattice isomorphic to a sublattice of $L_D$.*

*Proof.* Let $\chi_{A_1}, \ldots, \chi_{A_M}$ be a basis for $L$. Let $\{F_n \mid n \in \mathbb{N}\}$ be an increasing chain of finite sets of rational numbers with $\bigcup_{n=1}^{\infty} F_n = \mathbb{Q}$. By Lemma 2.5.1, for each $n \in \mathbb{N}$, there are

$$L_{\psi_n}^{(1/n)}(\Omega^{(n)}, \Sigma^{(n)}, \mu^{(n)}) \in \mathcal{MO}_D$$

and a vector lattice isomorphism

$$T_n : L_{F_n} \to L_{\psi_n}^{(1/n)}(\Omega^{(n)}, \Sigma^{(n)}, \mu^{(n)})$$

satisfying

$$|\Psi(f) - \Psi_n\left(T_n(f)\right)| < 1/n.$$

Let $\mathcal{U}$ be a Frechet ultrafilter on $\mathbb{N}$, and consider

$$P = \prod_{n,\mathcal{U}} \left( L_{\psi_n}^{(1/n)}(\Omega^{(n)}, \Sigma^{(n)}, \mu^{(n)}), \Psi_n^{(1/n)} \right).$$

Then $P = (L_{\widehat{\psi}}(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu}), \widehat{\Psi})$, where $L_{\widehat{\psi}}(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu}) \in \mathcal{MO}_D$. □

**Corollary.** *Any $L_\psi(\Omega, \Sigma, \mu)$ in $\mathcal{MO}_{H_D}$ can be embedded as a sublattice of an element of the class $\mathcal{MO}_D$.*

*Proof.* Let $L_\psi(\Omega, \Sigma, \mu) \in \mathcal{MO}_{H_D}$. Let $\mathcal{S}$ be the set of simple functions in $L_\psi(\Omega, \Sigma, \mu)$ with support of finite measure. Let $\{G_i \mid i \in I\}$ be the set of all sublattices $G_i$ of simple functions generated by finite pairwise disjoint sets of characteristic functions in $\mathcal{S}$. By Theorem 2.5.2, each $G_i$ is lattice isometrically isomorphic to a sublattice $L_i$ of some $L_{\psi_i}(\Omega_i, \Sigma_i, \mu_i) \in \mathcal{MO}_D$. We define the ordering $i_1 \leq i_2 \iff G_{i_1} \subseteq G_{i_2}$ on $I$. Fix $i_0 \in I$, and let $R(i_0) = \{i \in I \mid i_0 \leq i\}$. Then $\{R(i) \mid i \in I\}$ is a filter basis of subsets of $I$, and we may find an ultrafilter $\mathcal{U}$ containing it. Define $T : \mathcal{S} \to \prod_{i,\mathcal{U}} G_i$ by

$$T(f) = (f_i \mid i \in I)_{\mathcal{U}}$$

where

$$f_i = \begin{cases} f & \text{if } f \in G_i \\ 0 & \text{if } f \notin G_i. \end{cases}$$

Then $T$ is a positive, lattice-preserving, isometric map, and from the density of $\mathcal{S}$, it can be extended to $L_\psi(\Omega, \Sigma, \mu)$. Thus $L_\psi(\Omega, \Sigma, \mu)$ is isometrically isomorphic to a sublattice of $\prod_{i,\mathcal{U}} G_i$, and so $L_\psi(\Omega, \Sigma, \mu)$ is isometrically isomorphic to a sublattice of

$$\prod_{i,\mathcal{U}} L_{\psi_i}(\Omega_i, \Sigma_i, \mu_i) \in \mathcal{MO}_D.$$

Because $L_\psi(\Omega, \Sigma, \mu)$ is arbitrary, the conclusion follows. □

# Chapter 3

# Nakano spaces

## 3.1 Köthe function spaces

In this section we provide a few basic definitions, all of which can be found in [21]. A *Köthe function space* $X$ on a measure space $(\Omega, \Sigma, \mu)$ is a linear sublattice of $L_0(\Omega, \Sigma, \mu)$ satisfying:

(i) $X$ is *solid*, i.e. if $f \in X$ and $g \in L_0(\Omega, \Sigma, \mu)$, then $|g| \leq |f|$ implies $g \in X$;

(ii) $X$ is equipped with a norm $\|\cdot\|$ for which it is complete, and which is *order-compatible*, i.e. $|g| \leq |f| \Longrightarrow \|g\| \leq \|f\|$.

(iii) $X$ is *order-dense* in $L_0(\Omega, \Sigma, \mu)$, i.e. every $A \in \Sigma$ with $\mu(A) > 0$ contains $B \in \Sigma$ such that $\mu(B) > 0$ and $\chi_B \in X$.

A Banach lattice $X$ is said to be *order continuous* if $\|x_\alpha\| \to 0$ for every downwards directed net $(x_\alpha)$ of positive elements of $X$ for which $\inf_\alpha x_\alpha = 0$. Every order continuous Banach lattice can be represented as a Köthe function space over a decomposable measure space. If $X$ is an order-continuous Köthe function space over a decomposable measure space $(\Omega, \Sigma, \mu)$, its *Köthe dual*

$$X' := \{f' \in L_0(\Omega, \Sigma, \mu) \mid f'g \in L_1(\Omega, \Sigma, \mu) \text{ for every } g \in X\},$$

equipped with the norm $\|\cdot\|'$ defined by

$$\|f'\| = \sup\{\int f'f \, d\mu \mid f \in X, \|f\| \leq 1\}$$

is also a Köthe function space over the same measure space.

## 3.2 Positive contractive projections

A Banach lattice $X$ is said to be *strictly monotone* if whenever $x, y \in X$, $x \neq y$ satisfy $0 \leq x \leq y$, it is the case that $\|x\| < \|y\|$.

**Proposition 3.2.1.** *Let $X$ be a strictly monotone Banach lattice $X$, and let $P : X \to X$ be a positive contractive projection on $X$. Then $P(X)$ is a closed sublattice of $X$.*

*Proof.* Let $x \in P(X)$. We have that $0 \leq |x| = |Px| \leq P|x|$. Because $X$ is strictly monotone, if $|x| \neq P(|x|)$, then $\||x|\| < \|P(|x|)\|$. But since $P$ is a contraction, $\|P(|x|)\| \leq \||x|\|$, and so $|x| = P(|x|)$. Hence $|x| \in P(X)$, and $P(X)$ is indeed a sublattice of $X$. The continuity of $P$ forces $P(X)$ to be closed. □

Let $X$ be a Köthe function space over a measure space $(\Omega, \Sigma, \mu)$. Let $f_0 \in L_0(\Omega, \Sigma, \mu)_+$, and let $\Sigma_0$ be a sub-$\sigma$-algebra of $\Sigma$. Following [17], we define

$$X_{f_0}(\Sigma_0) = \{h \in L_0(\Omega, \Sigma_0, \mu) \mid \text{supp}(h) \subseteq \text{supp}(f_0) \ \& \ f_0 \cdot h \in X\}.$$

**Proposition 3.2.2.** *Let $X$ be a Köthe function space over a measure space $(\Omega, \Sigma, \mu)$. Let $f_0 \in L_0(\Omega, \Sigma, \mu)_+$, and let $\Sigma_0$ be a sub-$\sigma$-algebra of $\Sigma$. Then $f_0 \cdot X_{f_0}(\Sigma_0)$ is a closed sublattice of $X$.*

*Proof.* It is routine to verify that $f_0 \cdot X_{f_0}(\Sigma_0)$ is a linear subspace of $X$. In order to show that it is a sublattice, it suffices to show that for any $h \in X_{f_0}(\Sigma_0)$, $\max(h, 0) \in X_{f_0}(\Sigma_0)$. Let $h \in X_{f_0}(\Sigma_0)$. Then $\max(h, 0)$ is $\Sigma_0$-measurable, $\text{supp}(\max(h, 0)) \subseteq \text{supp}(h) \subseteq \text{supp}(f_0)$, and $|\max(h, 0) \cdot f_0| \leq |h \cdot f_0|$. Thus $\max(h, 0) \in X_{f_0}(\Sigma_0)$, and $f_0 X_{f_0}(\Sigma_0)$ is indeed a sublattice. Let now $(f_n \mid n \geq 1)$ be a sequence of elements in $f_0 X_{f_0}(\Sigma_0)$ satisfying $\lim_{n\to\infty} \|f_n - f\| \to 0$, for some $f \in X$. Then we have that $\text{supp}(f) \subseteq \bigcup_{n=1}^{\infty} \text{supp}(f_n) \subseteq \text{supp}(f_0)$ and $f_n \to f$ in measure. So $\min(1, |f_n - f|) \to 0$ in $L_1$, and hence there exists a subsequence $(f_{n_k})$ of $(f_n)$ with $f_{n_k} \to f$ a.e. Thus $\frac{f_{n_k}}{f_0} \to \frac{f}{f_0}$ a.e. and since $\frac{f_{n_k}}{f_0}$ are $\Sigma_0$-measurable, so is $\frac{f}{f_0}$. Finally, $\frac{f}{f_0} \in X_{f_0}(\Sigma_0)$. This shows that $f_0 \cdot X_{f_0}(\Sigma_0)$ is a closed sublattice of $X$. □

Let $X$ be a Köthe function space over a measure space $(\Omega, \Sigma, \mu)$, and let $Y$ be a closed sublattice of $X$. We denote by $\Sigma_Y^0$ the set consisting of all the supports of elements of $Y$. By $\Sigma_Y$ we mean the $\sigma$-algebra generated by $\Sigma_Y^0$.

**Proposition 3.2.3.** *Let $X$ be an order-continuous Köthe function space over a complete, decomposable measure space $(\Omega, \Sigma, \mu)$. Then for every closed sublattice $Y$ of $X$ there exist a sub-$\sigma$-algebra $\Sigma_Y$ of $\Sigma$ and a positive $\Sigma$-measurable function $f_0$ such that $Y = f_0 \cdot X_{f_0}(\Sigma_Y)$.*

*Proof.* Let $\Sigma_Y^0$ be the set of supports of elements of $Y$. Observe that for any $f, g \in Y$ and for $(f_n \mid n \in \mathbb{N})$ in $Y$:

(i) $\text{supp}(f) \bigcup \text{supp}(g) = \text{supp}(\max(|f|, |g|))$;

(ii) $\text{supp}(f) \setminus \text{supp}(g) = \text{supp}((|f| - |f| \wedge |g|)_+)$;

(iii) $\bigcup_{n\in\mathbb{N}} \operatorname{supp}(f_n) = \operatorname{supp}\left(\sum_n 2^{-n}\|f_n\|^{-1}|f_n|\right)$.

Hence $\Sigma_Y^0$ is a sub-$\sigma$-ring of $\Sigma$. Let $\Sigma_Y$ be the $\sigma$-algebra generated by $\Sigma_Y^0$.

Let $(f_\alpha)_{\alpha\in\mathcal{A}}$ be a maximal family of pairwise disjoint positive elements of $Y$, and let

$$f_0 := \sup f_\alpha \in L_0(\Omega, \Sigma, \mu).$$

We claim that $Y = f_0 \cdot X_{f_0}(\Sigma_Y)$.

($\subseteq$) Let $f \in Y$ and $\mathcal{A}_f := \{\alpha \mid \mu\left(\operatorname{supp}(f) \cap \operatorname{supp}(f_\alpha)\right) > 0\}$. By the order-continuity of $X$, $f$ has $\sigma$-finite support, and so $\mathcal{A}_f$ is countable.

*Claim* 1. $\operatorname{supp}(f) \subseteq \operatorname{supp}(f_0)$.

*Proof.* If $f \in Y$, then $\operatorname{supp}(f) \setminus \bigcup_{\alpha\in\mathcal{A}_f} \operatorname{supp}(f_\alpha)$ belongs to $\Sigma_Y$, and so it is the support of an element $g \in Y$ which is disjoint from all the $f_\alpha$, for $\alpha \in \mathcal{A}_f$; by the maximality of $(f_\alpha)$, this implies that $g = 0$. □

*Claim* 2. For every $f \in Y_+$ and $\lambda \in [0, \infty)$, we have $(f - \lambda f_0)_+ \in Y$.

*Proof.* For every finite subset $F$ of $\mathcal{A}$, $(f - \lambda \sum_{\alpha\in F} f_\alpha)_+ \in Y$. The downwards-directed net

$$\left(\left(f - \lambda \sum_{\alpha\in F} f_\alpha\right)_+\right)_F$$

(where $F$ ranges over the collection of finite subsets of $\mathcal{A}$) is order convergent to $(f - \lambda f_0)_+$. □

For every $\lambda \geq 0$, we have that $\operatorname{supp}(\frac{f}{f_0} - \lambda 1)_+ = \operatorname{supp}(f - \lambda f_0)_+ \in \Sigma_Y^0$. For every $\lambda < 0$, $\operatorname{supp}\left(\frac{f}{f_0} - \lambda\right)_+ = \Omega \in \Sigma_Y$. Hence $f/f_0$ is $\Sigma_Y$-measurable and $f \in f_0 \cdot X_{f_0}(\Sigma_Y)$. By linearity, this remains true for arbitrary $f \in Y$.

($\supseteq$) Suppose $f \in f_0 \cdot X_{f_0}(\Sigma_Y)$. Then $f = f_0 \cdot h$ for some $\Sigma_Y$-measurable $h$ with $\operatorname{supp}(h) \subseteq \operatorname{supp}(f_0)$. By linearity, we may assume without loss of generality that $f \geq 0$. Then $h \geq 0$ can be realized as the supremum of a net $(h_i)$ of $\Sigma_Y$-measurable simple functions. By the order-continuity of $X$, to show that $f \in Y$, it is sufficient to show that every $f_0 \cdot h_i \in Y$. By linearity, we may reduce to proving that $f_0\chi_A \in Y$ for all $\alpha \in \mathcal{A}$ and $A \in \Sigma_Y$. This reduces further, by order-continuity, to showing that $f_\alpha\chi_A \in Y$ for all $\alpha \in \mathcal{A}$ and for $A \in \Sigma_Y$. Note that we may further assume that $A \subseteq \operatorname{supp}(f_\alpha)$. By the definition of $\Sigma_Y$, either $A$ or $A^c$ belongs to $\Sigma_Y^0$. In case $A^c \in \Sigma_Y^0$, then $A = \operatorname{supp}(f_\alpha) \setminus A^c \in \Sigma_Y^0$. Hence $A \in \Sigma_Y^0$, and so $A = \operatorname{supp}(g)$ for some $g \in Y_+$. This shows that $f_\alpha\chi_A = \sup_{n\geq 1} f_\alpha \wedge ng$. By the order-continuity of $X$, $f_\alpha\chi_A \in Y$. □

Let $X$ be an order-continuous Köthe function space over the measure space $(\Omega, \Sigma, \mu)$ with a strictly monotone norm, and let $Y$ be the range of a positive contractive projection $P$ on $X$. Let $\Sigma_P$ denote the closure of $\Sigma_Y$ under taking suprema and infima of arbitrary families up to $\mu$-null sets. Let $S_Y$ denote the smallest element of $\Sigma$ containing all the supports of elements in $Y$. (In particular, $S_Y \in \Sigma_P$.)

**Lemma 3.2.4.** *$\Sigma_P$ consists of suprema of arbitrary families in $\Sigma_Y^0 \cup \{S_Y^c\}$, where $S_Y^c = \Omega \setminus S_Y$.*

*Proof.* ($\supseteq$) Suppose $A = \sup\{A_j \mid j \in J\}$ for some family $\{A_j \mid j \in J\} \subseteq \Sigma_Y^0 \cup \{S_Y^c\}$. If

$$J_0 := \{j \in J \mid A_j = S_Y^c\} = \emptyset,$$

we have nothing to prove, so let us assume that $J_0 \neq \emptyset$. In that case, we may write

$$A = \sup\{A_j \mid j \notin J_0\} \vee S_Y^c.$$

The first term of that disjunction is obviously in $\Sigma_Y$ (and hence in $\Sigma_P$), and the second term of that disjunction is in $\Sigma_P$ because its complement is in $\Sigma_P$.

($\subseteq$) Let $A \in \Sigma_P$. Then $A = \sup\{A_j \mid j \in J\}$ for some family $\{A_j \mid j \in J\} \subseteq \Sigma_Y$. Let

$$J_1 := \{j \in J \mid A_j \in \Sigma_Y^0\}$$

and

$$J_2 := \{j \in J \mid A_j^c \in \Sigma_Y^0\}.$$

Observe that for $j \in J_2$, $A_j = (A_j \cap S_Y) \cup S_Y^c$ (and for $j \in J_1$, $A_j = A_j \cap S_P$). Since $S_Y = \sup\{A \mid A \in \Sigma_Y^0\}$, we can readily write $A$ as the supremum of a subfamily of $\Sigma_Y^0 \cup \{S_Y^c\}$. ☐

**Lemma 3.2.5.** *A set $A \in \Sigma$ belongs to $\Sigma_P$ if and only if its intersection with every set $B \in \Sigma_Y^0$ belongs to $\Sigma_Y^0$ and either $A \cap S_P^c =_\mu S_P^c$ or else $A \cap S_Y^c =_\mu \emptyset$.*

*Proof.* ($\Longleftarrow$) Let $A \in \Sigma$ be such that its intersection with every set $B \in \Sigma_Y^0$ belongs to $\Sigma_Y^0$. Suppose further that $A \cap S_Y^c =_\mu \emptyset$. Then

$$A =_\mu \sup\{A \cap B \mid B \in \Sigma_Y^0\}$$

and so $A \in \Sigma_P$. Suppose otherwise that $A \cap S_Y^c =_\mu S_Y^c$. In this case, $A^c \cap S_Y^c =_\mu \emptyset$, and so $A^c \in \Sigma_P$, which entails that $A \in \Sigma_P$.

($\Longrightarrow$) Suppose $A \in \Sigma_P$. Then there is a family $(A_j \mid j \in J) \subseteq \Sigma_Y^0 \cup \{S_Y^c\}$ such that $A = \sup\{A_j \mid j \in J\}$.

Let $B \in \Sigma_Y^0$. Then $A \cap B = \sup\{A_j \cap B \mid j \in J\}$. Recall that underlying our discussion is the assumption that $X$ is order-continuous, which implies that the support of any one of its elements is $\sigma$-finite. Writing $\Sigma_B^0$ for the $\sigma$-ring $\{B \cap C \mid C \in \Sigma_P^0\}$, we get that for every $B \in \Sigma_Y^0$, $\Sigma_B$ is closed under arbitrary suprema, thus closed under suprema of arbitrary families (because $B$ is $\sigma$-finite). In particular, this implies that $A \cap B \in \Sigma_B \subseteq \Sigma_Y^0$. Furthermore, if the family $(A_j \mid h \in J)$ contains $S_Y^c$ then $A \cap S_Y^c = S_Y^c$, and if not, $A \cap S_Y^c = \emptyset$. □

**Corollary.** *Let $f$ be a $\Sigma$-measurable function. Then $f$ is $\Sigma_P$-measurable if and only if its restriction to $S_Y^c$ is constant and its restriction to any element of $\Sigma_Y^0$ is $\Sigma_Y$-measurable.*

*Proof.* ($\Longrightarrow$) Let $f$ be $\Sigma_P$-measurable, and let $B \in \Sigma_Y^0$. For any interval $[a, b)$, we know that $A := f^{-1}([a,b)) \in \Sigma_P$. Then $f \restriction_B^{-1} ([a,b]) = A \cap B$ belongs to $\Sigma_Y^0$ by Lemma 3.2.5 and $f \restriction_{S_Y^c} ([a,b]) = A \cap S_Y^c \in \{\emptyset, S_Y^c\}$. Thus $f \restriction_B$ is $\Sigma_Y$-measurable and $f \restriction_{S_Y^c}$ is constant.
($\Longleftarrow$) Let $f$ be a $\Sigma$-measurable function such that its restriction to $S_Y^c$ is constant and its restriction to any given element of $\Sigma_Y^0$ is $\Sigma_Y$-measurable. For any interval $[a, b)$, we know that $f^{-1}([a,b)) \in \Sigma$. Additionally, we know that $f^{-1}([a,b)) \cap B \in \Sigma_Y$ for all $B \in \Sigma_Y^0$, and that $f^{-1}([a,b)) \cap S_Y^c$ is constant. As an immediate consequence of Lemma 3.2.5, $f^{-1}([a,b)) \in \Sigma_P$. Because $[a, b)$ was taken arbitrarily, we obtain that $f$ is $\Sigma_P$-measurable. □

The following corollary is immediate.

**Corollary.** *If $f$ is a $\Sigma$-measurable function with $\sigma$-finite support, then $f$ is $\Sigma_P$-measurable if and only if it is $\Sigma_Y$-measurable.*

Let $X$ be a Banach space. By $S_X$ we denote the unit sphere of $X$, i.e. $S_X = \{x \in X \mid \|x\| = 1\}$. We define a *duality map* from $S_X$ into subsets of the Banach space dual $X^*$ by the condition that $f \in J(x)$ if and only if $\|f\|_{X^*} = 1$ and $\langle f, x\rangle = 1$. If $J(x)$ contains exactly one functional then element $x$ is called *smooth* in $X$. If every element $x \in S_X$ is smooth in $X$ then $X$ is called *smooth*.

We arrive finally to the main proposition of this section.

**Proposition 3.2.6.** *Let $X$ be a smooth reflexive Köthe function space over the measure space $(\Omega, \Sigma, \mu)$ with a strictly monotone norm. Let $Y = P(X)$, where $P$ is a positive contractive projection. Suppose that the Köthe dual $X'$ of $X$ has a strictly monotone norm. Then there exists a positive $\Sigma$-measurable function $f_0'$ such that $P^*(X') = f_0' \cdot X_{f_0}(\Sigma_P)$.*

*Proof.* Because $X$ is reflexive, it does not contain $c_0$ and hence it is order continuous. (See Propositions 1.$a$.5, 1.$a$.7 and 1.$a$.8 in [21].) Therefore, its Banach lattice dual $X^*$ is itself a Köthe function space over the measure space $(\Omega, \Sigma, \mu)$. If $P^*$ denotes the adjoint of $P$, then $P^*$ is a positive contraction, whence $P^*(X^*)$ is a closed sublattice of $X^*$ by Proposition 3.2.1. Thus there exists a positive $\Sigma$-measurable function $f_0'$ such that $P^*(X^*) = g_0 \cdot X_{f_0'}(\Sigma_{P^*(X^*)})$. The smoothness of $X$ yields a duality map $J : X \setminus \{0\} \to S_{X^*}$. By a result of Calvert (see [7]), $J(P(X)) \subseteq P^*(X^*)$. Since $X^*$ is also smooth, $J \restriction_{S_X}$ is a bijection the unit spheres of $X$ and $X^*$ with inverse $J^* \restriction_{S_{X^*}}$, where $J^* : X^* \setminus \{0\} \to X^{**} = X$ is the duality map of $X^*$. For every nonzero $h \in X$, we have that $\text{supp}(Jh) = \text{supp}(h)$, whence $\Sigma_{P^*(X^*)} = \Sigma_{P(X)}$. Now, $X$ is order-continuous, and so its elements have $\sigma$-finite supports. By Lemma 3.2, $X^*_{f_0'}(\Sigma_{P^*}) = X^*_{f_0'}(\Sigma_P)$. The Köthe function dual $X'$ of $X$ can be identified with $X^*$, and so the same conclusion applies to $X'$. □

## 3.3 Nakano spaces

Let $(\Omega, \Sigma, \mu)$ be a measure space, and let $p \in L_0(\Omega, \Sigma, \mu)$ satisfy

$$p_* := \operatorname{ess\,inf} p(\omega) \geq 1$$

and

$$p^* = \operatorname{ess\,sup} p(\omega) < \infty.$$

Define $\Theta_{p(\cdot)} : L_0(\Omega, \Sigma, \mu) \to [0, \infty]$ by setting

$$\Theta_{p(\cdot)}(f) = \int_\Omega |f(\omega)|^{p(\omega)} \, d\mu(\omega).$$

Let $\| \cdot \|_{p(\cdot)} : L_0(\Omega, \Sigma, \mu) \to [0, \infty]$ be defined by

$$\|f\|_{p(\cdot)} = \inf\{\epsilon > 0 \mid \Theta_{p(\cdot)}(f/\epsilon) \leq 1\}.$$

Let

$$L_{p(\cdot)}(\Omega, \Sigma, \mu) = \left\{ f \in L_0(\Omega, \Sigma, \mu) \;\middle|\; \|f\|_{p(\cdot)} < \infty \right\}.$$

We say that $L_{p(\cdot)}(\Omega, \Sigma, \mu)$ thus defined is a *Nakano space*. It is easy to see that a Nakano space is a Musielak-Orlicz space with Musielak-Orlicz function $\psi$ on $(\Omega, \Sigma, \mu)$ defined by

$$\psi(t, \omega) = t^{p(\omega)},$$

equipped with the Luxemburg norm $\| \cdot \|_{p(\cdot)}$.

The functional $\Theta_{p(\cdot)}$ defined above is a convex modular which satisfies the $\Delta_2^{2^{p^*}}$-condition. Indeed,

$$\Theta_{p(\cdot)}(2f) = \int_\Omega 2^{p(\omega)} |f(\omega)|^{p(\omega)} \, d\mu(\omega) \leq 2^{p^*} \Theta_{p(\cdot)}(f)$$

for all $f \in L_{p(\cdot)}(\Omega, \Sigma, \mu)$.

If the Nakano space $L_{p(\cdot)}(\Omega, \Sigma, \mu)$ comes equipped with the convex modular $\Theta_{p(\cdot)}$, we say that

$$(L_{p(\cdot)}(\Omega, \Sigma, \mu), \Theta_{p(\cdot)})$$

is a *modulared Nakano space.*

Let $X = L_{p(\cdot)}(\Omega, \Sigma, \mu)$ be a Nakano space. For $x \in X$ and $\lambda > 1$, we have

$$\Theta_{p(\cdot)}(x) \leq \lambda \Theta_{p(\cdot)}(x) \leq \lambda^{p_*} \Theta_{p(\cdot)}(x) \leq \Theta_{p(\cdot)}(\lambda x) \leq \lambda^{p^*} \Theta_{p(\cdot)}(x),$$

and if $0 < \lambda < 1$, we have

$$\lambda^{p^*} \Theta_{p(\cdot)}(x) \leq \Theta_{p(\cdot)}(\lambda x) \leq \lambda^{p_*} \Theta_{p(\cdot)}(x) \leq \lambda \Theta_{p(\cdot)}(x) \leq \Theta_{p(\cdot)}(x).$$

Consequently, for every fixed $x \in X \setminus \{0\}$, $t \mapsto \Theta_{p(\cdot)}(tx)$ is strictly increasing, continuous, convex, and even. This implies that the relationship between the convex modular $\Theta_{p(\cdot)}$ and $\|\cdot\|_{p(\cdot)}$ is tight in the following sense:

**Proposition 3.3.1.** *Let $X = L_{p(\cdot)}(\Omega, \Sigma, \mu)$ be a Nakano space. Let $x \in X \setminus \{0\}$. Then we have:*

(i) $\|x\|_{p(\cdot)} = a$ *if and only if* $\Theta_{p(\cdot)}\left(\frac{x}{a}\right) = 1$.

(ii) $\|x\|_{p(\cdot)} < 1 \iff \Theta_{p(\cdot)}(x) < 1$. *(See Theorems* 1.2 *and* 1.3 *in [11].)*

**Corollary.** *Let $X = L_{p(\cdot)}(\Omega, \Sigma, \mu)$ be a Nakano space. The norm $\|\cdot\|_{p(\cdot)}$ is strictly monotone.*

*Proof.* Let $x, y \in X$ with $|x| < |y|$. Then there exists $\epsilon > 0$ such that $A_\epsilon := \{\omega \in \Omega \mid |y(\omega)| \geq (1+\epsilon)|x(\omega)|\}$

satisfies $\mu(A_\epsilon) > 0$. We have that

$$\begin{aligned}\Theta_{p(\cdot)}(x) &= \int_\Omega |x(\omega)|^{p(\omega)}\, d\mu(\omega) \\ &= \int_{A_\epsilon} |x(\omega)|^{p(\omega)}\, d\mu(\omega) + \int_{\Omega\setminus A_\epsilon} |x(\omega)|^{p(\omega)}\, d\mu(\omega) \\ &< \int_{A_\epsilon} ((1+\epsilon)|x(\omega)|)^{p(\omega)}\, d\mu(\omega) + \int_{\Omega\setminus A_\epsilon} |x(\omega)|^{p(\omega)}\, d\mu(\omega) \\ &\leq \int_{A_\epsilon} |y(\omega)|^{p(\omega)}\, d\mu(\omega) + \int_{\Omega\setminus A_\epsilon} |y(\omega)|^{p(\omega)}\, d\mu(\omega) = \Theta_{p(\cdot)}(y).\end{aligned}$$

If we choose $y$ to have $\|y\|_{p(\cdot)} = 1$, then by Proposition 3.3.1 we have that $\|x\|_{p(\cdot)} < 1$, and this proves that $X$ is strictly monotone. □

*Remark.* Since Nakano spaces are Musielak-Orlicz spaces satisfying the $\Delta_2^k$-condition for some $k \geq 2$, then Nakano spaces are automatically order continuous. (See [13, pages 31–32].)

## 3.4 Ultraproducts of Nakano spaces

For $p(\cdot) \in L_0(\Omega, \Sigma, \mu)$ given, we define its *essential range* to be the set

$$\mathcal{R}_{p(\cdot)} = \left\{q \in [1,\infty) \;\middle|\; \forall \epsilon > 0\ \mu\left(\left\{\omega \in \Omega \;\middle|\; |p(\omega) - q| < \epsilon\right\}\right) > 0\right\}.$$

If $X := L_{p(\cdot)}(\Omega, \Sigma, \mu)$ is a Nakano space, we will say that $X$ *has essential range* $\mathcal{R}_{p(\cdot)}$. Similarly, we will say that a modulared Nakano space $(X, \Theta) := (L_{p(\cdot)}(\Omega, \Sigma, \mu), \Theta_{p(\cdot)})$ *has essential range* $\mathcal{R}_{p(\cdot)}$.

*Remark.* Note that $\mathcal{R}_{p(\cdot)}$ is a closed subset of $[1,\infty)$. Indeed, suppose that $q_n \in \mathcal{R}_{p(\cdot)}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} q_n = q$ for some $q \in \mathbb{R}$. Given $\epsilon > 0$, pick $n_0$ sufficiently large so that $|q_{n_0} - q| < \epsilon/3$. Then the set

$$A := \{\omega \in \Omega \mid |p(\omega) - q| < \epsilon\}$$

contains the set

$$B := \{\omega \in \Omega \mid p(\omega) - q_{n_0}| < \epsilon/3\}.$$

Since $q_{n_0} \in \mathcal{R}_{p(\cdot)}$, we have that $\mu(B) > 0$, and so $\mu(A) > 0$.

Let $X := L_{p(\cdot)}(\Omega, \Sigma, \mu)$ be a Nakano space with essential range contained in $[1, N]$. Then $(X, \Theta_{p(\cdot)})$ is an $L_k$-structure for all $k \geq 2^N$. Indeed, for each $n \geq 1$, the restriction to the ball centered at the origin of radius $n$ of the convex modular $\Theta_{p(\cdot)}$ satisfies the modulus of uniform continuity $\Delta_k^{(n)}$ associated to that value of $k$. (For a definition of $\Delta_k^{(n)}$, the reader is referred to Lemma 2.1.5.)

In what follows, let $K$ be a compact subset of $[1, \infty)$.

Define $\mathcal{N}_{\subseteq K}^{\mathrm{mod}}$ to be the closure under isomodular isomorphism of the following class of modulared Nakano spaces:

$$\{(L_{p(\cdot)}(\Omega, \Sigma, \mu), \Theta_{p(\cdot)}) \mid \mathcal{R}_{p(\cdot)} \subseteq K\}.$$

It is easy to check that

$$\left\{\varphi \;\middle|\; \exists p \in K \;\forall t \in \mathbb{R}\; \varphi(t) = t^p\right\}$$

is a Dacunha-Castelle set, and so, as a consequence of Dacunha-Castelle's representation theorem (Theorem 2.4.3), the class $\mathcal{N}_{\subseteq K}^{\mathrm{mod}}$ is closed under ultraproducts.

Define $\mathcal{N}_{K}^{\mathrm{mod}}$ to be the closure under isomodular isomorphism of the following class of modulared Nakano spaces:

$$\{(L_{p(\cdot)}(\Omega, \Sigma, \mu), \Theta_{p(\cdot)}) \mid \mathcal{R}_{p(\cdot)} = K\}.$$

By Theorem 2.4.3, any ultraproduct of elements of this class is a member of $\mathcal{N}_{\subseteq K}^{\mathrm{mod}}$.

Define $\mathcal{N}_{\subseteq K}$ to be the closure under Banach lattice isometry of the following class of Nakano spaces:

$$\left\{L_{p(\cdot)}(\Omega, \Sigma, \mu) \;\middle|\; \mathcal{R}_{p(\cdot)} \subseteq K\right\}.$$

Since $\mathcal{N}_{\subseteq K}$ consists of the reducts to $L$ of elements of $\mathcal{N}_{\subseteq K}^{\mathrm{mod}}$, $\mathcal{N}_{\subseteq K}$ itself is closed under ultraproducts.

Define the class $\mathcal{N}_K$ to be the closure under Banach lattice isometry of the following class of Nakano spaces:

$$\left\{L_{p(\cdot)}(\Omega, \Sigma, \mu) \;\middle|\; \mathcal{R}_{p(\cdot)} = K\right\}.$$

Then any ultraproduct of elements of this class is a member of $\mathcal{N}_{\subseteq K}$.

The aims of this section are to show that both $\mathcal{N}_{K}^{\mathrm{mod}}$ and $\mathcal{N}_K$ are closed under ultraproducts; that $(X, \Theta) \in \mathcal{N}_{K_1}^{\mathrm{mod}} \bigcap \mathcal{N}_{K_2}^{\mathrm{mod}}$ implies $K_1 = K_2$; and that whenever $X$ is not 1-dimensional, $X \in \mathcal{N}_{K_1} \bigcap \mathcal{N}_{K_2}$ implies $K_1 = K_2$. The latter two facts justify saying that $K$ is *the* essential range of all members of the classes $\mathcal{N}_{K}^{\mathrm{mod}}$ and $\mathcal{N}_K$.

Let $X$ be a Banach lattice. A subset $S$ of $X$ is called *solid* if $x \in S$, $y \in X$ and $|y| \leq |x|$ imply that $y \in S$. A solid vector subspace $I$ of $X$ is called an *ideal* of $X$. An ideal $B$ of $X$ is said to be a *band* if $A \subseteq B$ and $\sup A \in X$ imply that $\sup A \in B$.

**Lemma 3.4.1.** *$L_{p(\cdot)}(\Omega, \Sigma, \mu)$ and $L_{q(\cdot)}(\Omega', \Sigma', \mu')$ be two Nakano spaces. Let $K$ be a compact subset of*

$[1,\infty)$ *such that* $L_{p(\cdot)}(\Omega,\Sigma,\mu)$ *and* $L_{q(\cdot)}(\Omega',\Sigma',\mu')$ *are elements of* $\mathcal{N}_{\subseteq K}$. *Let* $A\in\Omega$ *with* $\mu(A)<\infty$, *and let* $B\in\Omega'$ *with* $\mu'(B)<\infty$. *Suppose that*

$$T:(\overline{\langle\chi_A\rangle},\Theta_{p(\cdot)\restriction_A})\to(\overline{\langle\chi_B\rangle},\Theta_{q(\cdot)\restriction_B})$$

*is an isomodular isomorphism between the closed sublattices generated by* $\chi_A$ *and* $\chi_B$ *equipped, respectively, with* $\Theta_{p(\cdot)\restriction_A}$ *and* $\Theta_{q(\cdot)\restriction_B}$. *Let* $\mathfrak{B}(K)$ *denote the family of all Borel subsets of* $K$. *Let* $\pi_A^{p(\cdot)}:\mathfrak{B}(K)\to[0,\infty]$ *be defined by*

$$\pi_A^{p(\cdot)}(S)=\mu\left([p(\cdot)\restriction_A]^{-1}(S)\right),$$

*and let* $\pi_B^{q(\cdot)}:\mathfrak{B}(K)\to[0,\infty)$ *be defined by*

$$\pi_B^{q(\cdot)}(S)=\mu'\left([q(\cdot)\restriction_B]^{-1}(S)\right).$$

*Then* $\pi_A^{p(\cdot)}=\pi_B^{q(\cdot)}$.

*Proof.* For simplicity of notation, we will write $\pi_A$ for $\pi_A^{p(\cdot)}$ and $\pi_B$ for $\pi_B^{q(\cdot)}$. We begin by observing that

$$\begin{aligned}\int_K |t|^p\,d\pi_A(p)&=\int_A |t|^{p(\omega)}\,d\mu(\omega)=\Theta_{p(\cdot)}(t\chi_A)=\Theta_{q(\cdot)}(t\chi_B)\\&=\int_B |t|^{q(\omega)}\,d\mu'(\omega)=\int_K |t|^p\,d\pi_B(p)\end{aligned}$$

for all $t\in\mathbb{R}$. (Note that by $\pi_A$ and $\pi_B$ are bounded measures on $K$.) Hence we have that

$$\int_0^\infty \exp(p\log|t|)\,d\pi_A(p)=\int_0^\infty \exp(p\log|t|)\,d\pi_B(p).$$

Let $F_A:\mathbb{R}\to\mathbb{R}$ be defined by

$$F_A(s)=\int_0^\infty \exp(ps)\,d\pi_A(p),$$

and let $F_B:\mathbb{R}\to\mathbb{R}$ be defined by

$$F_B(s)=\int_0^\infty \exp(ps)\,d\pi_B(p),$$

It is routine to check that for all $k\in\mathbb{N}$, the $k$-th derivative of $F_A$ is

$$F_A^{(k)}(s)=\int_0^\infty p^k\exp(ps)\,d\pi_A(p),$$

and likewise that the $k$-th derivative of $F_B$ is

$$F_B^{(k)}(s) = \int_0^\infty p^k \exp(ps)\, d\pi_B(p).$$

Hence for all $k \in \mathbb{N}$, we have that

$$F_A^{(k)}(0) = \int_0^\infty p^k\, d\pi_A(p) = F_B^{(k)}(0).$$

By the Stone-Weierstrass Theorem, it follows that

$$\int_K f(p)\, d\pi_A(p) = \int_K f(p)\, d\pi_B(p)$$

for every continuous function $f : K \to \mathbb{R}$. In other words,

$$\mathbb{E}[f(p(\cdot))] = \mathbb{E}[f(q(\cdot))]$$

for every continuous function $f : K \to \mathbb{R}$, where $\mathbb{E}[X]$ denotes the classical probabilistic *expectation* of the random variable $X$. By a well-known elementary fact from probability theory, the latter equality implies that $\pi_A = \pi_B$. (See, for example, [18, Theorem 5.3].) $\square$

As a consequence of Lemma 3.4.1, we have the following:

**Corollary.** *If $(L_{p(\cdot)}(\Omega, \Sigma, \mu), \Theta_{p(\cdot)})$ and $(L_{p'(\cdot)}(\Omega', \Sigma', \mu'), \Theta_{p'(\cdot)})$ are isomodularly isomorphic, then $\mathcal{R}_{p(\cdot)} = \mathcal{R}_{p'(\cdot)}$.*

The following proposition allows us to conclude that the class $\mathcal{N}_K^{\mathrm{mod}}$ is closed under ultraproducts.

**Proposition 3.4.2.** *Let $K$ be a fixed compact subset of $[1, \infty)$. Let $I$ be an index set, and suppose that for all $i \in I$, $(X_i, \Theta_i) := (L_{p_i(\cdot)}(\Omega_i, \Sigma_i, \mu_i), \Theta_{p_i(\cdot)})$ satisfies $\mathcal{R}_{p_i(\cdot)} = K$. Let $\mathcal{U}$ be any ultrafilter on $I$. Let*

$$(X, \Theta) := \prod_{i \in I} (X_i, \Theta_i) \Big/ \mathcal{U}.$$

*Then $(X, \Theta)$ is isomodularly isomorphic a to modulared Nakano space*

$$(L_{p(\cdot)}(\Omega, \Sigma, \mu), \Theta_{p(\cdot)})$$

*satisfying $K = \mathcal{R}_{p(\cdot)}$.*

*Proof.* For $p \in K$, define $B_p$ to be the set

$$\left\{(x_i \mid i \in I)_{\mathcal{U}} \in X \mid \forall \epsilon > 0 \ \{i \in I \mid \operatorname{supp}(x_i) \subseteq p_i(\cdot)^{-1}(p-\epsilon, p+\epsilon)\} \in \mathcal{U}\right\}.$$

*Claim* 1. If $x, y \in B_p$, then $x + y \in B_p$.

*Proof.* Let $x = (x_i \mid i \in I)_{\mathcal{U}}$ and $y = (y_i \mid i \in I)_{\mathcal{U}}$. Then

$$x + y = (x_i + y_i \mid i \in I)_{\mathcal{U}}.$$

Fix $\epsilon > 0$. Then

$$\begin{aligned}&\{i \in I \mid \operatorname{supp}(x_i + y_i) \subseteq p_i(\cdot)^{-1}(p-\epsilon, p+\epsilon)\}\\ &\supseteq \{i \in I \mid \operatorname{supp}(x_i) \subseteq p_i(\cdot)^{-1}(p-\epsilon, p+\epsilon)\}\\ &\cap \{i \in I \mid \operatorname{supp}(y_i) \subseteq p_i(\cdot)^{-1}(p-\epsilon, p+\epsilon)\} \in \mathcal{U},\end{aligned}$$

and so the claim has been established. □

It is clear that if $x \in B_p$, then for any $\alpha \in \mathbb{R}$, $\alpha x \in B_p$, and so $B_p$ is a vector subspace of $X$. Let now $x = (x_i \mid i \in I)_{\mathcal{U}} \in B_p$ and $y = (y_i \mid i \in I)_{\mathcal{U}} \in X$, and suppose that $|y| \leq |x|$. For $i \in I$, define $x'_i = |x_i| \wedge |y_i|$.

*Claim* 2. $(|x_i| \mid i \in I)_{\mathcal{U}} = (x'_i \mid i \in I)_{\mathcal{U}}$

*Proof.* Let $\epsilon > 0$. Then

$$\begin{aligned}&\{i \in I \mid |x_i| - x'_i < \epsilon\}\\ &\supseteq \{i \in I \mid |x_i| - |y_i| < \epsilon\} \in \mathcal{U}.\end{aligned}$$

This establishes our claim. □

Thus $|x| = (x'_i \mid i \in I)_{\mathcal{U}}$. We need to show that $|x| \in B_p$. Let $\epsilon > 0$ be arbitrary. Then

$$\begin{aligned}&\{i \in I \mid \operatorname{supp}(x'_i) \in p_i(\cdot)^{-1}(p-\epsilon, p+\epsilon)\}\\ &\supseteq \{i \in I \mid \operatorname{supp}(y_i) \in p_i(\cdot)^{-1}(p-\epsilon, p+\epsilon)\},\end{aligned}$$

whence $|x| \in B_p$. To see that $x \in B_p$, we can write $x = x^+ - x_-$, apply the above reasoning to $x^+$ and to $x_-$, and obtain that both $x_+$ and $x_-$ belong to $B_p$. Because $B_p$ is a vector subspace, we immediately get that $x \in B_p$.

We need to establish the following

*Claim* 3. Let $(x^{(j)} \mid j \in J\} \subseteq B_p$ satisfying $x := \sup\{x^{(j)} \mid j \in J\} \in X$. Then $x \in B_p$.

*Proof.* Without loss of generality, we may assume that $x^{(j)} \geq 0$ for all $j \in J$. We will change, in an ad hoc way, the representation of $x^{(j)}$ as an ultralimit for each $\epsilon > 0$ given. This is perfectly justified so long as the representations chosen are all equivalent under the equivalence relation of the ultraproduct construction. Let $\epsilon > 0$. For each $j \in J$, let $x^{(j)} = (x_i^{(j)} \mid i \in I)_{\mathcal{U}}$ be a representation of $x^{(j)}$ satisfying: for all $i \in I$, $\operatorname{supp}(x_i^{(j)}) \in p_i(\cdot)^{-1}(p-\epsilon, p+\epsilon)$. Note that selecting such a representation is indeed not a problem, since $x^{(j)} \in B_p$. By definition of $x$,

$$x \leq (\sup\{x_i^{(j)} \mid j \in J\} \mid i \in I)_{\mathcal{U}} := \xi.$$

Then, if for $i \in I$ we let $\xi_i = \sup\{x_i^{(j)} \mid j \in J\}$, we have that

$$\{i \in I \mid \operatorname{supp}(\xi_i) \subseteq p_i(\cdot)^{-1}(p-\epsilon, p+\epsilon)\} = I \in \mathcal{U}.$$

Hence $\xi \in B_p$ and so $x \in B_p$. □

This shows that $B_p$ is indeed a band in $X$, as desired.

By Dacunha-Castelle's representation theorem (Theorem 2.4.3), we may write

$$(X, \Theta) = (L_{p(\cdot)}(\Omega, \Sigma, \mu), \Theta_{p(\cdot)})$$

for some (decomposable) measure space $(\Omega, \Sigma, \mu)$ and some $p(\cdot) \in L_0(\Omega, \Sigma, \mu)$ with $\mathcal{R}_{p(\cdot)} \subseteq K$. Since $B_p$ is a band in $X$, we can write

$$B_p = L_{p(\cdot)\restriction_{\Omega_p}}(\Omega_p, \Sigma_p, \mu_p)$$

for some $\Omega_p \subseteq \Omega$, $\Sigma_p = \{\Omega_P \cap A \mid A \in \Sigma\}$, and $\mu_p = \mu \restriction_{\Sigma_p}$.

We wish to prove that $\Theta_{p(\cdot)} \restriction_{B_p} = \Theta_p$, where $\Theta_p : B_p \to [0, \infty)$ is defined by

$$\Theta_p(x) = \int_{\Omega_p} |x(\omega)|^p \, d\mu(\omega).$$

Let $B \in \Sigma_p$ satisfy $\Theta_{p(\cdot)}(\chi_B) < \infty$. Then we can find a representation $\chi_B = (b_i \mid i \in I)_{\mathcal{U}}$ of $\chi_B$ such that $b_i \in L^+_{p_i}(\Omega_i, \Sigma_i, \mu_i)$ and $\Theta_{p_i(\cdot)}(b_i) = \Theta(\chi_B)$. Let $B_i := \operatorname{supp}(b_i) \subseteq p_i(\cdot)^{-1}(p-\epsilon, p+\epsilon)$ for all $i \in I$, and set

$\Theta_p^{(i)} : L_{p_i(\cdot)}(B_i, \Sigma_i, \mu_i) \to [0, \infty]$ defined by

$$\Theta_p^{(i)}(x) = \int_{B_i} \left( \frac{|x(\omega)|}{b_i(\omega)} \right)^p b_i(\omega)^{p_i(\omega)}\, d\mu_i(\omega).$$

If $A \in \Sigma_p$, $A \subseteq B$, then we can write $\chi_{A_i} = (\chi_{A_i} b_i)$ for some $A_i \in \Sigma_i$ with $A_i \subseteq B_i$, and we have to prove:

*Claim* 4. $\lim_{i,\mathcal{U}} \int_{A_i} |t|^{p_i(\omega)} b_i(\omega)^{p_i(\omega)}\, d\mu(\omega) = \lim_{i,\mathcal{U}} \int_{A_i} |t|^p b_i(\omega)^{p_i(\omega)}\, d\mu(\omega)$ for all $t \in \mathbb{R}$.

*Proof.*

$$\begin{aligned}
&\lim_{i,\mathcal{U}} \left| \int_{A_i} |t|^{p_i(\omega)} b_i(\omega)^{p_i(\omega)}\, d\mu(\omega) - \int_{A_i} |t|^p b_i(\omega)^{p_i(\omega)}\, d\mu(\omega) \right| \\
&\le \lim_{i,\mathcal{U}} \int_{A_i} \left| |t|^{p_i(\omega)} - |t|^p \right| b_i(\omega)^{p_i(\omega)}\, d\mu(\omega) \\
&\le \lim_{i,\mathcal{U}} \int_{A_i} |t|^p (|t|^\epsilon - 1) b_i(\omega)^{p_i(\omega)}\, d\mu(\omega) \\
&= |t|^p \left(|t|^\epsilon - 1\right) \lim_{i,\mathcal{U}} \Theta_p^{(i)}(\chi_{A_i}) \\
&= |t|^p \left(|t|^\epsilon - 1\right) \mu(A) := \rho(\epsilon).
\end{aligned}$$

It is clear that $\rho(\epsilon) \to 0$ as $\epsilon \to 1$. This establishes our claim. □

Since for all $A \in \Sigma_P$ with $\mu(A) < \infty$ we have that

$$\int_A |t|^{p(\omega)}\, d\mu(\omega) = \int_A |t|^p\, d\mu(\omega)$$

for all $t \in \mathbb{R}$, by Lemma 3.4.1 we obtain that $\pi_{p(\cdot)} \restriction_A = \pi_p \restriction_A = \delta_p$ (where $\delta_p$ denotes the Dirac measure at point $\{p\}$) for all $A \in \Sigma_p$ with $\mu(A) < \infty$.

*Claim* 5. The set $\{\omega \in \Omega_p \mid p(\omega) \neq p\}$ is $\mu$-null.

*Proof.* We may assume, without loss of generality, that $(\Omega_p, \Sigma_p, \mu_p)$ is a decomposable measure space. (In fact, $(\Omega, \Sigma, \mu)$ can be taken to be decomposable itself.) Therefore we can let

$$(\Omega_p, \Sigma_p, \mu_p) = \bigoplus_{\eta \in \mathcal{E}} (\Omega_p^{(\eta)}, \Sigma_p^{(\eta)}, \mu_p^{(\eta)}),$$

be a decomposition of $(\Omega_p, \Sigma_p, \mu_p)$. Since $\mu_p(\Omega_p^{(\eta)}) < \infty$ for all $\eta \in \mathcal{E}$, we have that $\pi_{p(\cdot)} \restriction_{\Omega_p^{(\eta)}} = \pi_p \restriction_{\Omega_p^\eta}$ for all $\eta \in \mathcal{E}$. Let $\mathbf{N} := \{\omega \in \Omega_p \mid p(\omega) \neq p\}$, and for each $\eta \in \mathcal{E}$, let $\mathbf{N}^{(\eta)} := \{\omega \in \Omega_p^{(\eta)} \mid p(\omega) \neq p\}$. Then we have that

$$\mu_p(\mathbf{N}) = \mu_p\left(\bigcup_{\eta\in\mathcal{E}} \mathbf{N}^{(\eta)}\right) = \sum_{\eta\in\mathcal{E}} \mu_p^{(\eta)}\left(\mathbf{N}^{(\eta)}\right) = 0.$$

This concludes the proof of the claim. □

This shows that $p(\omega) = p$ for a.e. $\omega \in \Omega_p$, and so $p \in \mathcal{R}_{p(\cdot)}$. Since $p$ is an arbitrary element of $K$, we infer that $K \supseteq \mathcal{R}_{p(\cdot)}$. This implies that $K = \mathcal{R}_{p(\cdot)}$. □

**Corollary.** *The classes $\mathcal{N}_K^{\mathrm{mod}}$ and $\mathcal{N}_K^{\mathrm{mod}}$ are closed under ultraproducts.*

*Proof.* By Proposition 3.4.2, the class $\mathcal{N}_K^{\mathrm{mod}}$ is closed under ultraproducts. The class $\mathcal{N}_K$ consists of the $L$-reducts of all members of $\mathcal{N}_K^{\mathrm{mod}}$. (Recall that the latter are $L_k$-structures for all $k \geq 2^{\sup K}$.) Hence any ultraproduct of members of the class $\mathcal{N}_K$ is the $L$-reduct of a corresponding ultraproduct of members of $\mathcal{N}_K^{\mathrm{mod}}$. This implies that the class $\mathcal{N}_K$ is closed under ultraproducts. □

Let $X$ be a Banach lattice and let $1 \leq p < \infty$. $X$ is said to be *p-convex* if there exists a constant $M < \infty$ so that

$$\left\| \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \right\| \leq M \left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}$$

for every choice of vectors $\{x_i\}_{i=1}^n$ in $X$.

On the other hand, $X$ is said to be *p-concave* if there exists a constant $M < \infty$ so that

$$\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p} \leq M \left\| \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \right\|$$

for every choice of vectors $\{x_i\}_{i=1}^n$ in $X$.

**Proposition 3.4.3.** *Let $X := L_{p(\cdot)}(\Omega, \Sigma, \mu)$ be a Nakano space with $p_* = \operatorname{ess\,inf} p(\cdot) \geq 1$. Then $X$ is $p_*$-convex (with constant 1). Similarly, $X$ is $p^*$-concave (with constant 1), where $p^* = \operatorname{ess\,sup} p(\cdot)$.*

*Remark.* Note that $\phi : L_{p(\cdot)}(\Omega, \Sigma, \mu) \to L_{\frac{p(\cdot)}{p_*}}(\Omega, \Sigma, \mu)$ defined by $\phi(x) = |x|^{p_*}\operatorname{sign}(x)$ satisfies

$$\Theta_{\frac{p(\cdot)}{p_*}}(\phi(x)) = \Theta_{p(\cdot)}(x)$$

for any $x \in L_{p(\cdot)}(\Omega, \Sigma, \mu)$. Observe also that $\Theta_{\frac{p(\cdot)}{p_*}}$ is itself a convex modular.

*Proof.* We wish to show that

$$\left\| \left(\sum_{i=1}^n |x_i|^{p_*}\right)^{1/p_*} \right\| \leq \left(\sum_{i=1}^n \|x_i\|^{p_*}\right)^{1/p_*}$$

for any tuple $(x_i)_{i=1}^n \in X^n$. Because $\Theta_{p(\cdot)}$ satisfies the $\Delta_2$-condition, it is enough to show that there is $M' < \infty$ such that

$$\Theta_{p(\cdot)}\left(\frac{(\sum_{i=1}^n |x_i|^{p_*})^{1/p_*}}{(\sum_{i=1}^n \|x_i\|^{p_*})^{1/p_*}}\right) \leq 1.$$

Let $\alpha = (\sum_{i=1}^n \|x_i\|^{p_*})$, and let $A = \{\omega \in \Omega \mid \sum_{i=1}^n |x_i(\omega)|^p < \alpha$ and $B = \Omega \setminus A$. By the preceding remark, we have that

$$\begin{aligned}
\Theta_{p(\cdot)}\left(\frac{(\sum_{i=1}^n |x_i|^{p_*})^{1/p_*}}{(\sum_{i=1}^n \|x_i\|^{p_*})^{1/p_*}}\right) &= \Theta_{\frac{p(\cdot)}{p_*}}\left(\frac{1}{\alpha}\sum_{i=1}^n |x_i|^{p_*}\right) \\
&= \Theta_{\frac{p(\cdot)}{p_*}}\left(\sum_{i=1}^n \frac{\|x_i\|^{p_*}}{\alpha} \cdot \frac{|x_i|^{p_*}}{\|x_i\|^{p_*}}\right) \\
&\leq \sum_{i=1}^n \frac{\|x_i\|^{p_*}}{\alpha} \Theta_{\frac{p(\cdot)}{p_*}}\left(\frac{|x_i|}{\|x_i\|}\right) \\
&= \sum_{i=1}^n \frac{\|x_i\|^{p_*}}{\alpha} \Theta_{p(\cdot)}\left(\frac{x_i}{\|x_i\|}\right) \leq 1,
\end{aligned}$$

and this finishes the proof. □

The following proposition shows the invariance of the essential range under lattice-preserving isometric isomorphisms of Nakano spaces of dimension $\geq 2$.

**Proposition 3.4.4.** *Let $L_{p_1(\cdot)} := L_{p_1(\cdot)}(\Omega_1, \Sigma_1, \mu_1)$ be a Nakano space of dimension $> 1$. Let $L_{p_2(\cdot)} := L_{p_2(\cdot)}\Omega_2, \Sigma_2, \mu_2)$ be another Nakano space which is lattice-isometrically isomorphic to $L_{p_1(\cdot)}$. Then $\mathcal{R}_{p_2(\cdot)} = \mathcal{R}_{p_1(\cdot)}$.*

*Proof.* Let us show first that if $L_{p_1(\cdot)}$ is lattice isomorphic to $L_{p_2(\cdot)}$ then $p_1^* = \operatorname{ess\,sup} p_1(\cdot)$ and $p_2^* = \operatorname{ess\,sup} p_1(\cdot)$ are equal. Assume for example that $p_1^* < p_2^*$. Since we know that $L_{p_1(\cdot)}$ is $p_1^*$-concave with constant 1, it is sufficient to prove that $L_{p_2(\cdot)}$ is not for reaching a contradiction. Observe that $A_2 = p_2(\cdot)^{-1}([p_2^* - \epsilon, p_2^*])$ verifies $\mu_2(A_2) > 0$. Let $y \in \chi_{A_2} L_{p_2(\cdot)}$ and $x \in L_{p_2(\cdot)}$ be two disjointly supported norm 1 vectors. Then for every $0 < \lambda < 1$:

$$\Theta_2(x + \lambda y) = \Theta_2(x) + \Theta_2(\lambda y) \leq \Theta_2(x) + \lambda^{(p_{2*} - \epsilon)}\Theta_2(y) = 1 + \lambda^{(p_2^* - \epsilon)}$$

Hence $\|x + \lambda y\| \leq 1 + \lambda^{(p_2^* - \epsilon)}$. On the other hand

$$(\|x\|^{p_1^*} + \|\lambda y\|^{p_1^*})^{1/p_1^*} = (1 + \lambda^{p_1^*})^{1/p_1^*} = 1 + \frac{1}{p_1^*}\lambda^{p_1^*} + o(\lambda^{p_1^*})$$

Since $\lambda^{p_1^*} \gg \lambda^{(p_2^* - \epsilon)}$ when $\lambda \to 0$, we see that $L_{p_2(\cdot)}$ is not $p_1^*$-concave with constant 1.

Similarly we show that if $L_{p_1(\cdot)}$ is lattice isomorphic to $L_{p_2(\cdot)}$ then $p_{1*} = \text{ess inf } p_1(\cdot)$ and $p_{2*} = \text{ess inf } p_1(\cdot)$ are equal.

Assume now that $L_{p_1(\cdot)}$ is lattice isomorphic to $L_{p_2(\cdot)}$ but some $p$ belongs to $K_1 = \mathcal{R}_{p_1(\cdot)}$ and not to $K_2 = \mathcal{R}_{p_2(\cdot)}$. Then for some $\epsilon > 0$ we have that $[p-2\epsilon, p+2\epsilon]$ does not intersects $K_2$, so if we set

$$A_1 = p_1(\cdot)^{-1}([p-\epsilon, p+\epsilon]), \quad A_2 = p_2(\cdot)^{-1}([p-2\epsilon, p+2\epsilon])$$

we have

$$\mu_1(A_1) > 0, \ \mu_2(A_2) = 0$$

Set for $i = 1, 2$

$$B_i = p_i(\cdot)^{-1}([1, p-2\epsilon]), \ C_i = p_i(\cdot)^{-1}([p+2\epsilon, +\infty])$$

and

$$Y_i = \chi_{B_i} \cdot L_{p_i(\cdot)}, \ Z_i = \chi_{C_i} \cdot L_{p_i(\cdot)}$$

We have clearly $L_{p_2(\cdot)} = Y_2 \oplus Z_2$, a decomposition into two disjoint bands.

Let $T$ be an isometric lattice isomorphism from $L_{p_2(\cdot)}$ onto $L_{p_1(\cdot)}$. Then $L_{p_1(\cdot)} = T(Y_2) \oplus T(Z_2)$. Moreover $T(Y_2)$ and $T(Z_2)$ are two disjoint bands in $L_{p_1(\cdot)}$, hence are Nakano spaces with respective exponent functions $q(\cdot)$, $\rho(\cdot)$ verifying by the above reasoning:

$$q^* \leq p - 2\epsilon, \ \rho_* \geq p + 2\epsilon$$

Consequently

$$T(Y_2) \subset Y_1, \ T(Z_2) \subset Z_1$$

Thus

$$L_{p_1(\cdot)} = Y_1 \oplus Z_1$$

which is impossible because the band $\chi_{A_1} \cdot L_{p(\cdot)}$ is non zero and disjoint from both $Y_1$ and $Z_1$. $\square$

As a consequence of Proposition 3.4.4, any two Banach lattice-isometric Nakano spaces $L_{p_1(\cdot)}(\Omega_1, \Sigma_1, \mu_1)$ and $L_{p_2(\cdot)}(\Omega_2, \Sigma_2, \mu_2)$ of dimension $\geq 2$ have the same essential range, and so it makes sense to say that $K$ is the common essential range of all members of the class $\mathcal{N}_K$.

## 3.5 A duality map for some Nakano spaces

A functional $F$ on a Banach space $X$ is said to be *Gâteaux-differentiable* at the point $f \in X$ if, for arbitrary $h \in X$, the function $F(f+th)$ is differentiable with respect to $t$ and the derivative of this function, for $t=0$, has the form

$$\frac{d}{dt}F(f+th)\restriction_{t=0} = \langle g, h\rangle,$$

where the element $g$ from the topological dual $X^*$ to the space $X$ does not depend on $h$. The element $g$ will be called the *Gâteaux gradient* of the functional $F$ at the point $f$. The map $\nabla F : X \to X^*$, defined by the formula $\nabla F(f) = g$ on all elements $f$ at which $F$ is Gâteaux differentiable, will also be called the Gâteaux gradient.

**Proposition 3.5.1.** *Suppose the functional $F$ is Gâteaux-differentiable on an open set $U$ of the space $X$ and that its Gâteaux gradient is a continuous map $U \to X^*$. Then the functional $F$ is differentiable in $U$ and its gradient coincides with the Gâteaux gradient.*

*Proof.* Let $x \in U$ and $y \in U$ such that the segment $[x, y] \subseteq U$. By definition,

$$\frac{d}{dt}F(x+ty) = \langle \nabla(x+ty), y\rangle$$

for $0 \le t \le 1$. Integrating this equality, we obtain that

$$F(x+y) - F(x) = \int_0^1 \langle \nabla(x+ty), y\rangle \, dt.$$

Therefore

$$\begin{aligned} |F(x+y) - F(x) - \langle \nabla(x), y\rangle| &= \left| \int_0^1 \langle \nabla(x+ty) - \nabla(x), y\rangle \right| \\ &\le \|y\| \int_0^1 \|\nabla(x+ty) - \nabla(x)\|_X \, dt. \end{aligned}$$

From the continuity of $\nabla$, it follows that

$$\lim_{\|y\|\to 0} \frac{|F(x+y) - F(x) - \langle \nabla(x), y\rangle|}{\|y\|} = 0.$$

This finishes the proof. ☐

Let $K$ be a fixed compact subset of $[1, \infty)$, and let $X := L_{p(\cdot)}(\Omega, \Sigma, \mu)$ be a Nakano space for which the essential range of $p(\cdot)$ is $K$. Let $\psi$ be defined by the equation $\psi(t,\omega) = t^{p(\omega)}$. Then $X$ can be regarded as the Musielak-Orlicz space $L_\psi(\Omega,\Sigma,\mu)$.

*Remarks.* (i) $X$ satisfies the $\Delta_2$-condition.

(ii) $\psi(\cdot,\omega)$ is continuously differentiable for every $\omega \in \Omega$.

Let $f, g \in L_{p(\cdot)}(\Omega,\Sigma,\mu)$. Define

$$\xi(r,s,\omega) = \psi\left(\left|\frac{f(\omega)+rg(\omega)}{s}\right|,\omega\right)$$

and

$$\Delta\xi(r,h,s,\omega) = \frac{\xi(r+h,s,\omega)-\xi(r,s,\omega)}{h}.$$

Observe that

$$\lim_{h\to 0}\Delta\xi(r,h,s,\omega) = \frac{1}{s}\frac{\partial\psi}{\partial t}\left(\left|\frac{f(\omega)+rg(\omega)}{s}\right|,\omega\right)g(\omega)$$

for every $\omega \in \Omega$.

Note that $\int_\Omega \xi(r,s,\omega)\,d\mu(\omega)$ and $\int_\Omega \xi(r\pm 1,s,\omega)\,d\mu(\omega)$ all exist. Then so does

$$\int_\Omega \Delta\xi(r,\pm 1,s,\omega)\,d\mu(\omega) = \int_\Omega \xi(r\pm 1,s,\omega)\,d\mu(\omega) - \int_\Omega \xi(r,s,\omega)\,d\mu(\omega).$$

Note that by the convexity of $\psi(\cdot,\omega)$,

$$\Delta\xi(r,-1,s,\omega) \le \Delta\xi(r,h,s,\omega) \le \Delta\xi(r,1,s,\omega)$$

for $|h|$ sufficiently small. Hence

$$|\Delta\xi(r,h,s,\omega)| \le \max\left[|\Delta\xi(r,-1,s,\omega)|,|\Delta\xi(r,1,s,\omega)|\right].$$

Note that the right hand side of the previous inequality is a function in $L_1$ not depending on $h$, and so we may apply Lebesgue's Dominated Convergence Theorem to obtain

$$\lim_{h\to 0}\int \Delta\xi(r,h,s,\omega)\,d\mu(\omega) = \frac{1}{s}\int_\Omega \frac{\partial\psi}{\partial t}\left(\left|\frac{f(\omega)+rg(\omega)}{s}\right|,\omega\right)g(\omega)\,d\mu(\omega).$$

Let $\Phi$ be defined by

$$\Phi(r,s) = \int_\Omega \psi\left(\left|\frac{f(\omega)+rg(\omega)}{s}\right|,\omega\right)\,d\mu(\omega).$$

The discussion above shows that $\frac{\partial \Phi}{\partial r}(r,s)$ exists. In fact,

$$\frac{\partial \Phi}{\partial r}(r,s) = \frac{1}{s}\int_\Omega \frac{\partial \psi}{\partial t}\left(\left|\frac{f(\omega)+rg(\omega)}{s}\right|, \omega\right) g(\omega)\, d\mu(\omega).$$

By a similar argument we can show that

$$\frac{\partial \Phi}{\partial s}(r,s) = -\frac{1}{s^2}\int_\Omega \frac{\partial \psi}{\partial t}\left(\left|\frac{f(\omega)+rg(\omega)}{s}\right|, \omega\right) (f(\omega)+rg(\omega))\, d\mu(\omega).$$

Moreover, both $\frac{\partial \Phi}{\partial r}(r,s)$ and $\frac{\partial \Phi}{\partial s}(r,s)$ are continuous as functions of $(r,s)$ for fixed $f, g$.

Since $\psi(t,\omega) = t^{p(\cdot)}$, we have that $\int_\Omega \psi\left(\left|\frac{f(\omega)}{k}\right|, \omega\right) d\mu(\omega) = 1$ implies that $\|f\|_{p(\cdot)} = k$.

*Remark.* The norm $\|\cdot\|_{p(\cdot)}$ can be defined with the aid of the equation $\int_\Omega \psi\left(\left|\frac{f(\omega)}{k}\right|, \omega\right) d\mu(\omega) = 1$ whenever $\|f\|_{p(\cdot)} \neq 0$. This follows from the fact that the integral involved is finite for all $k \neq 0$, depends continuously on $k$, and is such that

$$\lim_{k\to 0}\int_\Omega \psi\left(\left|\frac{f(\omega)}{k}\right|, \omega\right) d\mu(\omega) = \infty$$

and

$$\lim_{k\to \infty}\int_\Omega \psi\left(\left|\frac{f(\omega)}{k}\right|, \omega\right) d\mu(\omega) = 0.$$

**Theorem 3.5.2.** *If $p_* > 1$ and $p^* < \infty$, then the Luxemburg norm $\|\cdot\|_{p(\cdot)}$ is a Gâteaux differentiable functional, and the duality map*

$$J : L_{p(\cdot)}(\Omega,\Sigma,\mu) \to \left(L_{p(\cdot)}(\Omega,\Sigma,\mu)\right)^*$$

*is defined by*

$$J(f)(g) = \left\langle \frac{1}{A}\cdot \frac{p(\omega)\ \mathrm{sgn}\,(f(\omega))\, f(\omega)^{p(\omega)-1}}{\|f\|_{p(\cdot)}^{p(\omega)-1}}, g(\omega)\right\rangle,$$

*where the constant $A$ is defined by*

$$A := \int_\Omega \frac{p(\omega)|f(\omega)|^{p(\omega)}}{\|f\|_{p(\cdot)}^{p(\omega)}}\, d\mu(\omega).$$

*Remark.* By the considerations at the beginning of this section, the norm is differentiable on $X \setminus \{0\}$.

*Proof.* Consider the equation

$$\int_\Omega \psi\left(\left|\frac{f(\omega)+rg(\omega)}{s}\right|, \omega\right) d\mu(\omega) = 1.$$

This equation defines $r$ as an implicit function of $s$. Since the partial derivatives $\frac{\partial \Phi(r,s)}{\partial r}$ and $\frac{\partial \Phi(r,s)}{\partial s}$ are continuous and since

$$\frac{\partial \Phi(0,s)}{\partial s} = -\frac{1}{s^2}\int_\Omega \psi'\left(\left|\frac{|f(\omega)|}{s}\right|,\omega\right)|f(\omega)|\,d\mu(\omega) < 0$$

whenever $\|f\| \neq 0$, by the Implicit Function Theorem we obtain that

$$\left.\frac{\partial s}{\partial r}\right|_{r=0} = \left.\frac{-\frac{\partial \Phi}{\partial r}(r,s(r))}{\frac{\partial \Phi}{\partial s}(r,s(r))}\right|_{r=0}$$

$$= -\left.\frac{\frac{1}{s}\int_\Omega \frac{\partial \psi}{\partial t}\left(\left|\frac{f(\omega)+rg(\omega)}{s}\right|,\omega\right)g(\omega)\,d\mu(\omega)}{-\frac{1}{s^2}\int_\Omega \frac{\partial \psi}{\partial t}\left(\left|\frac{f(\omega)+rg(\omega)}{s}\right|,\omega\right)[f(\omega)+rg(\omega)]\,d\mu(\omega)}\right|_{r=0}$$

$$= \frac{\int_\Omega \frac{\partial \psi}{\partial t}\left(\left|\frac{f(\omega)}{s(0)}\right|,\omega\right)g(\omega)\,d\mu(\omega)}{\int_\Omega \frac{\partial \psi}{\partial t}\left(\left|\frac{f(\omega)}{s(0)}\right|,\omega\right)\frac{f(\omega)}{s(0)}\,d\mu(\omega)} = \langle h, g\rangle$$

where

$$h = \frac{\frac{\partial \psi}{\partial t}\left(\left|\frac{f(\omega)}{\|f\|_\psi}\right|,\omega\right)}{\int_\Omega \frac{\partial \psi}{\partial t}\left(\left|\frac{f(\omega)}{\|f\|_\psi}\right|,\omega\right)\frac{f(\omega)}{\|f\|_\psi}\,d\mu(\omega)}.$$

This completes the proof. □

As a quick consequence we have the following

**Corollary.** *For any $f \in S_{L_{p(\cdot)}(\Omega,\Sigma,\mu)}$, where $p_* > 1$ and $p^* < \infty$, we have that*

$$J(f) = \frac{p(\omega)\ \mathrm{sgn}\,(f(\omega))\,f(\omega)^{p(\omega)-1}}{\int_\Omega p(\omega)|f(\omega)|^{p(\omega)}\,d\mu(\omega)}.$$

*Remark.* In Theorem 3.5.2, the condition $p(\omega) > 1$ $\mu$-a.e. (in place of $p_* > 1$) is sufficient.

## 3.6 Ultraroots of Nakano spaces I

The principal aim of this section is to prove that if $\inf K > 1$, then the class $\mathcal{N}_K$ is closed under ultraroots. In fact, we will prove something stronger than that, namely that, whenever it is not 1-dimensional, the range of a positive contractive projection on a Nakano space in $\mathcal{N}_K$ is itself a Nakano space in $\mathcal{N}_K$. Our eventual aim is to prove that the requirement $\inf K > 1$ is not necessary for the closure of $\mathcal{N}_K$ under ultraroots. This will be done in section 3.5.

If $P$ is a positive contractive projection of $L_{p(\cdot)}(\Omega, \Sigma, \mu)$ onto the closed sublattice $Y$, then $J(Y) \subseteq P^*\left((L_{p(\cdot)}(\Omega, \Sigma, \mu))^*\right)$. The following theorem uses the notation developed in section 3.2.

**Theorem 3.6.1.** *Let $K$ be a compact subset of $(1, \infty)$ and let $p(\cdot)$ have $K$ as its essential range. Whenever the range of a positive contractive projection $P$ on the Nakano space $X := L_{p(\cdot)}(\Omega, \Sigma, \mu)$ is not 1-dimensional, then it is of the form $f_0 \cdot X_{f_0}(\Sigma_P)$ for some $\Sigma$-measurable $f_0$, with $p(\cdot) \restriction_{\mathrm{supp}(f_0)}$ being $\Sigma_P$-measurable.*

*Proof.* By the results of the section 3.2, the range of $P$ is a closed sublattice of $X$ of the form $Y := X_{f_0}(\Sigma_P)$, for some $f_0 \in L_0(\Omega, \Sigma, \mu)$. It remains to show that $p(\cdot) \restriction \mathrm{supp}(f_0)$ is $\Sigma_P$-measurable. In order to establish this, it is sufficient to show that the restriction of $p(\cdot)$ to the support of any positive element in $Y$ is $\Sigma_Y$-measurable. In fact, since $Y$ is not 1-dimensional, we may and do reduce to considering elements in $Y$ that are not atoms.

Let $f$ be a positive element of $Y$ that is not an atom.

*Claim* 1. There exist $g, h \in S_Y$ with $\mathrm{supp}(g) = \mathrm{supp}(h) = \mathrm{supp}(f)$ such that $\mu(\{\omega \in \Omega \mid g(\omega) = h(\omega)\}) = 0$.

*Proof.* Write $f = f_1 + f_2$ for $f_1 \wedge f_2 = 0$, and let $g = \frac{f}{\|f\|}$. Let $h_0 = f_1 + 2f_2$ and $h = \frac{h_0}{\|h_0\|}$. Then we have that $\|f\| < \|h_0\| < 2\|f\|$. This yields the desired conclusion. □

So we have $g, h \in Y$ satisfying $\mathrm{supp}(g) = \mathrm{supp}(h)$. Then both $\frac{g}{h}$ and $\frac{J(g)}{J(h)}$ are $\Sigma_Y$-measurable. We have that

$$\left|\frac{J(g)}{J(h)}\right| = \left|\frac{g}{h}\right|^{p(\cdot)\restriction_{\mathrm{supp}(g)}-1} \left|\frac{\displaystyle\int_{\Omega_0} p(\omega)|h(\omega)|^{p(\omega)} d\mu(\omega)}{\displaystyle\int_{\Omega} p(\omega)|g(\omega)|^{p(\omega)}\, d\mu(\omega)}\right|,$$

and so

$$\log\left|\frac{J(g)}{J(h)}\right| = \left(p(\omega) \restriction_{\mathrm{supp}(g)} -1\right) \log\left|\frac{g}{h}\right| + \log\left|\frac{\displaystyle\int_{\Omega_0} p(\omega)|h(\omega)|^{p(\omega)} d\mu(\omega)}{\displaystyle\int_{\Omega} p(\omega)|g(\omega)|^{p(\omega)}\, d\mu(\omega)}\right|.$$

This readily shows that $p(\omega) \cdot \chi_{\mathrm{supp}(f)}$ is $\Sigma_P$-measurable, as everything else occurring in the above equation is $\Sigma_P$-measurable. Because $f$ is arbitrary, $p(\cdot) \restriction_{\mathrm{supp}(f_0)}$ is $\Sigma_P$-measurable. □

**Corollary.** *If $K$ is a compact subset of $(1, \infty)$, then the class $\mathcal{N}_{\subseteq K}$ of Nakano spaces is closed under ultraroots.*

*Proof.* If the ultraroot $Y$ of a Nakano space $X \in \mathcal{N}_{\subseteq K}$ is finite-dimensional, then it is equal to $X$, and so it is trivially an element of $\mathcal{N}_{\subseteq K}$.

It suffices to show that if $X = L_{p(\cdot)}(\Omega, \Sigma, \mu)$, then whenever the range $Y := f_0 \cdot X_{f_0}(\Sigma_P)$ of a positive contractive projection $P$ is not 1-dimensional, $(Y, \Theta_{p(\cdot)} \restriction_Y)$ is isomodularly isomorphic to

$$(L_{p(\cdot)\restriction\Omega_0}(\Omega_0, \Sigma_P, \mu), \Theta_{p(\cdot)\restriction_{\Omega_0}}),$$

where $\Omega_0 = \mathrm{supp}(f_0)$. The reason that the above condition is sufficient is that the map

$$T : (X)_{\mathcal{U}} \to X$$

defined by

$$T\left((x_i \mid i \in I)_{\mathcal{U}}\right) = w^*\text{–}\lim_{i,\mathcal{U}} x_i$$

is a positive contractive projection onto $X$.

By the previous theorem, we know that $p(\cdot) \restriction \Omega_0$ is $\Sigma_P$-measurable. Consider the map

$$T : f_0 \cdot X_{f_0}(\Sigma_P) \to L_{p(\cdot)\restriction\Omega_0}(\Omega_0, \Sigma_P, \mu)$$

defined by

$$T(f_0 \cdot h) = \left(\mathbb{E}^{\Sigma_P}|f_0(\omega)|^{p(\omega)}\right)^{\frac{1}{p(\omega)}} \cdot h.$$

This constitutes an isomodular isometry between the two structures, and the corollary is established. □

*Remark.* The expression $\mathbb{E}^{\Sigma_P}\left(|f_0|^{p(\cdot)}\right)$ above is defined by

$$\mathbb{E}^{\Sigma_P}\left(|f_0|^{p(\cdot)}\right) := \sum_{\alpha \in A} \mathbb{E}^{\Sigma_P}\left(|f_\alpha|^{p(\cdot)}\right),$$

where $f_0 = \sum_{\alpha \in A} f_\alpha$ with $f_\alpha \in X^+$ (for all $\alpha \in A$) is a decomposition of $f_0$ according to some decomposition $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$ of $(\Omega, \Sigma, \mu)$. Since $f_\alpha^{p(\cdot)} \in L_1(\Omega, \Sigma, \mu)$ for all $\alpha$, $\mathbb{E}^{\Sigma_P}\left(|f_0|^{p(\cdot)}\right)$ makes sense.

**Corollary.** *If $K$ is a compact subset of $(1, \infty)$, then the class $\mathcal{N}_K$ of Nakano spaces is closed under ultraroots.*

*Proof.* By the previous corollary, if $X$ is an ultraroot of an element $Y$ in $\mathcal{N}_K$, then $X \in \mathcal{N}_{\subseteq K}$. Let $K'$ be the essential range of some representation of $X$ as a Nakano space. Because $Y$ is an ultrapower of $X$, Theorem 2.4.3 guarantees that $Y$ is Banach-lattice isometric to a Nakano space with essential range $K'' \subseteq K'$. By Proposition 3.4.4, $K'' = K$. Therefore, $K' = K$. □

Let us now consider the situation in which $(Y, \Theta)$ is an Orlicz lattice, and some ultrapower of $(Y, \Theta)$ is isomodularly isometric to an element $(X, \widehat{\Theta})$ in $\mathcal{N}_K^{\text{mod}}$, where $K$ is a fixed compact set contained in $(1, \infty)$.

Via the diagonal map $D : Y \to X$ we obtain an identification of $Y$ as a sub-Orlicz lattice of $X$. By what we have established in the previous result, we can assume $X$ can be regarded as $L_{p(\cdot)\restriction\Omega}(\Omega, \widehat{\Sigma}_X, \widehat{\mu})$ and $Y$ as $L_{p(\cdot)}(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu})$ for some decomposable measure space $(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu})$. For $f = (f \mid i \in I)_{\mathcal{U}} \in X$, we have that

$$\begin{aligned}\Theta(f) &= \lim_{i,\mathcal{U}} \Theta(f) = \widehat{\Theta}((f \mid i \in I)_{\mathcal{U}}) = \widehat{\Theta}(D(f)) \\ &= \int_\Omega |D(f)(\omega)|^{p(\omega)}\, d\widehat{\mu}(\omega).\end{aligned}$$

Now, since $f$ is $\Sigma_X$-measurable, so is $D(f)$. This is transparent from

$$D(f)^{-1}(O) = \{\omega \in \widehat{\Omega} \mid D(f) \in O\} = \{\omega \in \widehat{\Omega} \mid f \in O\} = f^{-1}(O)$$

for any open set $O \subseteq \mathbb{R}$.

Hence we have that

$$\Theta(f) = \int_\Omega |D(f)(\omega)|^{p(\omega)}\, d\widehat{\mu}\restriction_{\Sigma_X}(\omega) = \int_\Omega |f(\omega)|^{p(\omega)}\, d\widehat{\mu}\restriction_{\Sigma_X}(\omega)$$

and we have proved the following:

**Corollary.** *Let $K$ be a compact subset of $(1, \infty)$. The classes*

$$\mathcal{N}_{\subseteq K}^{\text{mod}} := \{(E, \Theta) \mid E \in \mathcal{N}_{\subseteq K} \text{ and } \Theta \text{ is its associated convex modular}\}$$

*and*

$$\mathcal{N}_K^{\text{mod}} := \{(E, \Theta) \mid E \in \mathcal{N}_K \text{ and } \Theta \text{ is its associated convex modular}\}$$

*are closed under ultraroots.*

Let us summarize the results we have obtained thus far. If $K$ is a compact subset of $(1, \infty)$, then the classes $\mathcal{N}_{\subseteq K}$, $\mathcal{N}_K$, and $\mathcal{N}_K^{\text{mod}}$ are closed under isomorphisms, ultraproducts and ultraroots.[1] Therefore, the first two classes are axiomatizable in the signature of Banach lattices, and the last class is axiomatizable in signature $L_k$, for $k \geq 2^{\sup K}$.

We aim to prove that the requirement that $\inf K > 1$ is superfluous, by which we mean that we may relax the requirement that $K$ be a compact subset of $(1, \infty)$ to the requirement that $K$ be a compact subset

[1] Additionally, we could define $\mathcal{N}_{\subseteq K}^{\text{mod}}$, and the same would be true of it.

of $[1,\infty)$, and still obtain the truth of the above conclusion. We already know that, allowing $K$ to be a compact subset of $[1,\infty)$, the classes $\mathcal{N}_{\subseteq K}$, $\mathcal{N}_K$, and $\mathcal{N}_K^{\mathrm{mod}}$ are closed under ultraproducts. In order to show that each of them is closed under ultraroots, we need some tools from the classical theory of Banach lattices, which we will briefly bring to the reader's attention in the next section.

## 3.7 Convexification and concavification

So far, we have shown that the classes $\mathcal{N}_{\subseteq K}$, $\mathcal{N}_K$, and $\mathcal{N}_K^{\mathrm{mod}}$ are closed under ultraroots when $K$ is a compact subset of $(1,\infty)$. In order to remove the constraint that $\inf K > 1$, we need to recall a few tools from general Banach lattice theory. The content of this section appears in [21].

Let $\mathcal{H}_n$ denote the family of all functions $f(t_1,\dots,t_n) : \mathbb{R}^n \to \mathbb{R}$ generated, via addition, multiplication by scalars, finite suprema and infima, by the functions $\varphi_i(t_1,\dots,t_n) = t_i$, for $i = 1,\dots,n$. Then every $f \in \mathcal{H}_n$ is continuous on $\mathbb{R}^n$ and satisfies $f(\lambda t_1,\dots,\lambda t_n) = \lambda f(t_1,\dots,t_n)$ (i.e., $f$ is *homogeneous of degree one*). This implies that for any $f \in \mathcal{H}_n$ there exists a constant $M_f < \infty$ so that $|f(t_1,\dots,t_n)| \le M_f\,(|t_1| \vee \dots \vee |t_n|)$ for all $(t_1,\dots,t_n) \in \mathbb{R}^n$.

Let $f \in \mathcal{H}_n$ and let $(x_i)_{i=1}^n$ be a finite set of elements of a Banach lattice $X$. Then we may give meaning to $f(x_1,\dots,x_n) \in X$ by replacing formally each of the variables $t_i$ with the corresponding vector $x_i$. It turns out that $f(x_1,\dots,x_n)$ is uniquely defined for every $f \in \mathcal{H}_n$, and so the map $\tau : \mathcal{N}_n \to X$ defined by $\tau f(t_1,\dots,f_n) = f(x_1,\dots,x_n)$ is linear and preserves lattice operations. Moreover, $\tau : \mathcal{H}_n \to X$ can be seen as a continuous map in the following way: Let $B_n$ be the subset of $\mathbb{R}^n$ of all $n$-tuples $(t_1,\dots,t_n)$ for which $|t_1| \vee \dots \vee |t_n| = 1$ and consider $\mathcal{H}_n$ as a sublattice of $C(B_n)$, the space of all continuous real-valued functions on $B_n$. The map $\tau$ is then continuous when $\mathcal{H}_n$ is endowed with the norm induced by $C(B_n)$.

Seen as a sublattice of $C(B_n)$, $\mathcal{H}_n$ separates points of $B_n$ and contains the function identically equal to one. By Stone-Weierstrass theorem, it follows that $\mathcal{H}_n$ is dense in $C(B_n)$. The closure $\overline{\mathcal{H}_n}$ of $\mathcal{H}_n$ consists of all the functions $f : \mathbb{R}^n \to \mathbb{R}$ which are continuous and homogeneous of degree one in $\mathbb{R}^n$. Hence there is a unique extension $\tau' : \overline{\mathcal{H}_n} \to X$ of $\tau$ which is linear, continuous, and preserves lattice operations.

The above observations are summarized in the following proposition due to Yudin [27] and Krivine [19]. (See [21, Theorem 1.d.1].)

**Proposition 3.7.1.** *Let $X$ be a Banach lattice and let $(x_i)_{i=1}^n$ be a finite subset of $X$. Then there is a unique map $\tau$ from the lattice $\overline{\mathcal{H}_n}$ of all functions which are continuous and homogeneous of degree one on $\mathbb{R}^n$, into $X$ such that:*

(i) *$\tau\varphi_i = x_i$ for $1 \le i \le n$, where $\varphi_i(t_1,\dots,t_n) = t_i$.*

(ii) $\tau$ *is a lattice homomorphism.*

*The map* $\tau$ *satisfies*

$$\|\tau(f)\| \leq \||x_1| \vee \cdots |x_n|\| \cdot \sup\{|f(t_1, \ldots, t_n)| \mid |t_1| \vee \cdots \vee |t_n| = 1\},$$

*for every* $f \in \overline{\mathcal{H}_n}$.

*Remark.* The above functional calculus allows us to define expressions like

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p},$$

where $p \geq 1$, in a perfectly arbitrary Banach lattice $X$. In what follows, we will invoke expressions such as the one above without making reference to the process that renders them meaningful. The reader must keep in mind Proposition 3.7.1.

Following Lindenstrauss and Tzafriri's presentation in [21], we now turn to describing a general procedure for constructing a $p$-convex Banach lattice starting with an arbitrary Banach lattice $X$. This procedure is an abstract description of the map $x \mapsto |x|^p \operatorname{sign}(x)$ which maps $L_{p(\cdot)}(\Omega, \Sigma, \mu)$ onto $L_{p \cdot p(\cdot)}(\Omega, \Sigma, \mu)$. The functional calculus described in 3.7.1 allows us to ascribe meaning to expressions like $x^s$ for $x \in X$.

Let $(X, +, \cdot, \leq, \Theta, \|\cdot\|)$ be an Orlicz lattice and let $p > 1$. For $x, y \in X$ and for a scalar $\alpha \in \mathbb{R}$, we define

$$x \oplus y = (x^{1/p} + y^{1/p})^p$$

and

$$\alpha \odot x = \alpha^p \cdot x,$$

where $(x^{1/p} + y^{1/p})^p$ is the element in $X$ corresponding, by the procedure described in 3.7.1, to the function

$$\begin{aligned} f(t_1, t_2) = {} & ||t_1|^{1/p}\operatorname{sign}(t_1) \\ & + |t_2|^{1/p}\operatorname{sign}(t_2)|^p \operatorname{sign}\left(|t_1|^{1/p}\operatorname{sign}(t_1) + |t_2|^{1/p}\operatorname{sign}(t_2)\right). \end{aligned} \tag{$\star$}$$

Equipped with the order $\preceq$ defined by $x \preceq y \Longleftrightarrow x \leq y$, $(X, \oplus, \odot, \preceq)$ is a vector lattice. The functional $\mathbf{N}(\cdot)$ defined by $\mathbf{N}(x) = \|x\|^{1/p}$ constitutes a lattice norm that makes $(X, \oplus, \odot, \preceq, \mathbf{N}(\cdot))$ into a $p$-convex Banach lattice. Furthermore, $\Theta$ itself is a convex modular on the structure $(X, \oplus, \odot, \preceq)$ and it is easy to check that $\mathbf{N}(x) = \inf\{\epsilon > 0 \mid \Theta\left(\frac{1}{\epsilon} \cdot x\right) \leq 1\}$. The upshot of this is that $(X, \oplus, \odot, \preceq, \Theta, \mathbf{N}(\cdot))$ is a $p$-convex Orlicz lattice. We will denote this structure by $X^{(p)}$ and call it the *p-convexification* of $X$.

*Remark.* In case $X$ is a Musielak-Orlicz space, $X^{(p)}$ can be identified with the space of all the functions $x$ so that $x^p = |x|^p \mathrm{sign}(x) \in X$ endowed with the convex modular of $X$ and with the norm $\mathbf{N}(x) = \||x|^p\|^{1/p}$. In particular, if $X = L_{p(\cdot)}(\Omega, \Sigma, \mu)$, then $X^{(p)} = L_{p \cdot p(\cdot)}(\Omega, \Sigma, \mu)$.

Let $(X, +, \cdot, \leq, \Theta, \|\cdot\|)$ is an Orlicz lattice which is $r$-convex as a Banach lattice for some $1 < p \leq r$. A similar procedure to the one described above allows us to define the *p-concavification* $X_{(p)}$ of $X$. For $x, y \in X$ and $\alpha \in \mathbb{R}$, simply put

$$x \oplus y = (x^p + y^p)^{1/p}$$

and

$$\alpha \odot x = \alpha^{1/p} \cdot x.$$

Additionally, endow this new structure with the convex modular $\Theta$ and with the new norm $\mathbf{N}'(\cdot)$ defined by

$$\mathbf{N}'(x) = \inf \left\{ \epsilon > 0 \mid \Theta\left(\frac{1}{\epsilon} \odot f\right) \leq 1 \right\}.$$

Finally, equip this modulared normed vector space with the ordering $\preceq$ defined by $x \preceq y \iff x \leq y$, which makes it into an Orlicz lattice. In fact, the structure $(X, \oplus, \odot, \preceq, \Theta, \mathbf{N}'(\cdot))$ is an $r/p$-convex Orlicz lattice with $M^{(r/p)}(X_{(p)}) \leq \left(M^{(p)}(X) M^{(r)}(X)\right)^p$.

Two facts are of importance to our aims:

(i) For any Orlicz lattice $X$ and $p > 1$, $X_{(p)}^{(p)} = X$.

(ii) If $1 \leq p < p^*$, then

$$(L_{p(\cdot)}(\Omega, \Sigma, \mu))_{(p)} = L_{\frac{p(\cdot)}{p}}(\Omega, \Sigma, \mu).$$

## 3.8 Ultraroots of Nakano spaces II

In this section we apply the tools above developed to prove that the classes $\mathcal{N}_{\subseteq K}$, $\mathcal{N}_K$, and $\mathcal{N}_K^{\mathrm{mod}}$ are all closed under ultraroots whenever $K$ is a fixed compact subset of $[1, \infty)$. So far we have obtained that said classes are closed under ultraproducts and ultraroots whenever $\inf K > 1$, and the purpose of this section is to remove that requirement. We start by elucidating the relationship between the procedures of $p$-convexification and $p$-concavification and the ultrapower operation for Banach lattices (and Orlicz lattices).

*Remark.* We have stated as a fact that (for $p > 1$) the $p$-convexification of a given Orlicz lattice

$$(X, +, \cdot, \vee, \wedge, \Theta, \|\cdot\|)$$

is again an Orlicz lattice.[2] This entails that the operations on $X^{(p)}$ corresponding to $+$, $\cdot$, $\vee$, $\wedge$, and the predicates on $X^{(p)}$ corresponding to $\Theta$ and $\|\cdot\|$, satisfy moduli of uniform continuity on bounded sets. In fact, if one restricts attention to the unit ball of the $X^{(p)}$, it is easy to describe precisely what these moduli are. For example, a modulus of uniform continuity $\Delta[\oplus]$ for $\oplus$ (restricted to the unit ball) can be defined by

$$\Delta[\oplus](\epsilon) = \Delta[f]\left(\Delta[+]\left(\frac{\epsilon}{2}\right)\right),$$

where $\Delta[f]$ is the modulus of uniform continuity of the map $f$ defined in ($\star$). Recall that $f$ is continuous and homogeneous of degree 1, which entails that $f$ is indeed uniformly continuous on bounded sets. The remaining moduli of uniform continuity may be calculated in a similar way.

The following fact has a routine proof.

**Proposition 3.8.1.** *Let $k \geq 2$. Let $(X, \Theta)$ be an Orlicz lattice satisfying the $\Delta_2^k$-condition. Let $\mathcal{U}$ be an ultrafilter on some index set $I$, and let $p > 1$. Then*

$$\left(\prod_{i \in I} X \Big/ \mathcal{U}\right)^{(p)} = \prod_{i \in I} X^{(p)} \Big/ \mathcal{U},$$

*where the equality sign means that the identity map between the corresponding underlying sets constitutes an isomodular isometry.*

Let $X$ be a Banach lattice, $I$ an index set, and $\mathcal{U}$ an ultrafilter on $I$. Suppose that the ultrapower $(X \mid i \in I)_{\mathcal{U}}$ is an element in the class $\mathcal{N}_K$, where $K$ is a compact subset of $[1, \infty)$. Let $r > 1$. Then $(X_{(r)} \mid i \in I)_{\mathcal{U}} = ((X \mid i \in I)_{\mathcal{U}})_{(r)} \in \mathcal{N}_{rK}$. Thus we have that $X_{(r)} \in \mathcal{N}_{rK}$, and this implies that $X = (X_{(r)})^{(r)} \in \mathcal{N}_K$. This argument works for $(X, \Theta) \in \mathcal{N}_K^{\mathrm{mod}}$ as well, and so we may conclude the following:

**Proposition 3.8.2.** *Let $K$ be a compact subset of $[1, \infty)$. The classes $\mathcal{N}_{\subseteq K}$, $\mathcal{N}_K$, and $\mathcal{N}_K^{\mathrm{mod}}$ are closed under ultraroots. Since they are also closed under isomorphisms and ultraproducts, each of these classes is axiomatizable in the appropriate signature.*

[2] In the previous section we paid attention to the relation $\leq$ rather than to the functions $\vee$ and $\wedge$, but it should be clear to the reader that the two points of view are equivalent.

## 3.9 Quantifier Elimination

As usual, we fix $K$ to be a compact subset of $[1,\infty)$. We will restrict our attention to a particularly well-behaved class of Nakano spaces, namely $\mathcal{AN}_K^{\mathrm{mod}}$, the class of all atomless modulared Banach lattices $(X,\Theta)$ such that $(X,\Theta)$ is isomodularly isomorphic to a modulared Nakano space with essential range $= K$. Recall that $x$ is an *atom* in a Banach lattice $X$ if for all $y \in X$, $y \wedge (x-y) = 0$ implies that $y = 0$ or $y = x$. (Note that this implies, in particular, that $x \in X_+$.) An *atomless* Banach lattice is a Banach lattice in which there are no nonzero atoms, and every atomless Nakano space can be represented as a Nakano space over an atomless measure space.

*Remark.* Note that for any $k \geq 2^{\sup K}$, the class $\mathcal{AN}_K^{\mathrm{mod}}$ is a class of $L_k$-structures.

**Lemma 3.9.1.** *Let $(X,\Theta)$ be an Orlicz lattice, and let $x \in X_+$ be an atom. Then for every $y \in X$, for every $\epsilon > 0$, there exists $\delta > 0$ such that $\Theta\left(|y| \wedge (x - |y|)\right) < \delta$ implies that $\Theta\left(y\right) \leq \epsilon$ or $\Theta\left(|y| - x\right) \leq \epsilon$.*

*Proof.* Suppose, to the contrary, that there is $y \in X$ and $\epsilon_0 > 0$ such that for all $\delta > 0$, $\Theta\left(|y| \wedge x - |y|\right) < \delta$ but $\Theta\left(y\right) > \epsilon_0$ and $\Theta\left(|y| - x\right) > \epsilon_0$. Then $|y| \wedge (x - |y|) = 0$, but $|y| \neq 0$ and $|y| \neq x$. Thus $x$ is not an atom. □

**Proposition 3.9.2.** *Let $k \geq 2^{\sup K}$. The class $\mathcal{AN}_K^{\mathrm{mod}}$ of $L_k$-structures is closed under ultraproducts and ultraroots. Therefore it is axiomatizable in $L_k$.*

*Proof.* Let $k \geq 2^{\sup K}$. Then $\mathcal{AN}_K^{\mathrm{mod}}$ is a class of $L_k$-structures. Let $S$ be the usual sort index of the signature of $L_k$, and for $s \in S$, let $\phi^{(s)}(x,y)$ be defined by

$$\phi^{(s)}(x,y) = \max\left\{\Theta\left(|y| \wedge (|x| - |y|)\right), \left|\frac{1}{2}\Theta(x) - \Theta\left(x \wedge y\right)\right|\right\}.$$

$$T_0 = \left\{\sup_{x^{(s)}} \inf_{y^{(s)}} \phi^{(s)}(x^{(s)}, y^{(s)}) = 0 \;\Big|\; s \in S\right\}.$$

Let $T = Th\left(\mathcal{N}_K^{\mathrm{mod}}\right) \bigcup T_0$.

*Claim* 1. For any $(X,\Theta) \in \mathcal{AN}_K^{\mathrm{mod}}$, $(X,\Theta) \vDash T$.

*Proof.* It is enough to show that for any $x \in X_+$, there exists $y \in X_+$ satisfying $y \wedge (x-y) = 0$ such that $\left|\frac{1}{2}\Theta(x) - \Theta(x \wedge y)\right| = 0$. Given $x \in X_+$, a change of measure allows us to represent $X$ as $L_{p(\cdot)}(\Omega,\Sigma,\mu)$ so that $\Theta = \Theta_{p(\cdot)}$, and $x = \chi_A$ for $A \in \Sigma$ with $\mu(A) < \infty$. Since in such a case $\Theta(x) = \mu(A)$, then the above condition follows from the existence of $B \in \Sigma$ satisfying that $\chi_B \leq \chi_A$ and $\frac{1}{2}\mu(A) - \mu(A \cap B) = 0$. But the latter is a consequence of the atomlessness of $X$ (see, for example, [12, 215D]), and so $(X,\Theta) \vDash T$. □

It remains to show that if $(X,\Theta) \models \Sigma$, then $(X,\Theta) \in \mathcal{AN}_K^{\text{mod}}$. Suppose, to the contrary, that $(X,\Theta) \in \mathcal{N}_K^{\text{mod}} \setminus \mathcal{AN}_K^{\text{mod}}$. Then there is an atom $x \in X$. By Lemma 3.9.1 and the $\Delta_2$-condition, for all $\epsilon > 0$ there exists $\delta > 0$ such that $\Theta(y \wedge x - y) < \delta$ implies that $\Theta(y) \leq \epsilon$ or $|\Theta(y) - \Theta(x)| < \epsilon$, which contradicts $T$ whenever $\epsilon$ is sufficiently small. □

In what follows, by $\overline{\langle x \rangle}$ we mean the closed Orlicz-sublattice generated by $x$ in the ambient Nakano space to which $x$ belongs. By $\mathfrak{B}(K)$ we will mean the $\sigma$-algebra of Borel subsets of $K$ in the relative topology on $K$ induced by the usual topology on $\mathbb{R}$.

Let

$$(L_{p(\cdot)}(\Omega,\Sigma,\mu),\Theta_{p(\cdot)}) \in \mathcal{AN}_K^{\text{mod}}$$

be a separable Nakano space, and let $(E,\Theta) \in \mathcal{N}_K^{\text{mod}}$. Let $\chi_A \in L_{p(\cdot)}(\Omega,\Sigma,\mu)$ with $\mu(A) < \infty$. Suppose there is a surjective lattice-preserving isomodular isometry

$$T : \overline{\langle \chi_{A_1} \rangle} \to \overline{\langle h \rangle},$$

where $h \in E_+$ and $F = \overline{\langle h \rangle}$ is regarded as being equipped with the restricted convex modular $\Theta \restriction_F$. Then we can represent $(E,\Theta)$ as a Nakano space $L_{\widehat{p}(\cdot)}(\widehat{\Omega},\widehat{\Sigma},\widehat{\mu})$ such that $h = \chi_B$ for some $B \in \widehat{\Sigma}$ with $\widehat{\mu}(B) < \infty$, and thus apply Lemma 3.4.1. So in such a setting we have that $\pi_A = \pi_B$.

**Lemma 3.9.3.** *Let*

$$(L_{p(\cdot)}(\Omega,\Sigma,\mu),\Theta_{p(\cdot)}) \in \mathcal{AN}_K^{\text{mod}}$$

*be a separable Nakano space, and let $(E,\Theta) \in \mathcal{N}_K^{\text{mod}}$ be $\omega_1$-saturated. Let $\chi_A \in L_{p(\cdot)}(\Omega,\Sigma,\mu)$ with $\mu(A) < \infty$. Suppose there is a lattice-preserving isomodular isometry*

$$T : \overline{\langle \chi_A \rangle} \to E,$$

*represent $T(\chi_A)$ as $\chi_B$, and extend this representation to one for $E$ as a Nakano space $L_{\widehat{p}(\cdot)}(\widehat{\Omega},\widehat{\Sigma},\widehat{\mu})$. Let $C \in \Sigma$ with $C \subseteq A$. Then there exists $D \in \widehat{\Sigma}$ with $D \subseteq B$ such that the linear extension $T' : \langle \chi_A, \chi_C \rangle \to E$ of $T$ satisfying $T'(\chi_C) = \chi_D$ is a lattice-preserving isomodular isometry.*

*Proof.* By $\omega_1$-saturation, it suffices to find, for $\epsilon > 0$ and $t_1,\ldots,t_M \in \mathbb{Q}^{\geq 0}$, a component $\chi_{D^{\epsilon}_{t_1,\ldots,t_M}}$ of $\chi_B$ such that

$$\left|\Theta_{p(\cdot)}\left(t_m\chi_C\right) - \Theta_{\widehat{p}(\cdot)}\left(t_m\chi_{D^{\epsilon}_{t_1,\ldots,t_M}}\right)\right| < \epsilon.$$

Let $t_1,\ldots,t_M \in \mathbb{Q}^{\geq 0}$ and suppose $t_1 < \cdots < 1 < \cdots < t_M$. Let $\psi_i \in L_1(C,\Sigma_C,\mu)$ be defined by

$\psi_i(\omega) = |t_i|^{p(\omega)}$, where $\Sigma_C = \{G \in \Sigma \mid G \subseteq C\}$. By taking common refinements where appropriate, we may approximate the functions $p(\cdot)$ by a simple function $S^{\epsilon}_{t_1,\ldots,t_n}$ so that

$$\left| \int_{\Omega} \psi_i \, d\mu(\omega) - \int_{\Omega} \left( t_m^{S^{\epsilon}_{t_1,\ldots,t_M}(\omega)} \chi_{\{\omega \in C | t_m^{S^{\epsilon}_{t_1,\ldots,t_M}(\omega)} \neq 1\}}(\omega) \right) d\mu(\omega) \right| < \epsilon,$$

for all $m = 1, \ldots, M$. The simple function $S^{\epsilon}_{t_1,\ldots,t_M}$ induces a Borel partition $\{I_1, \ldots, I_N\}$ of $K$, where $p_1, \ldots, p_N$ are the $N$ distinct values that the simple function $S^{\epsilon}_{t_1,\ldots,t_M}$ takes, and for each $n = 1, \ldots, N$, $p_n \in I_n$. This partition induces a partition

$$\left\{ [\widehat{p} \restriction_B]^{-1}(I_1), \ldots, [\widehat{p} \restriction_B]^{-1}(I_N) \right\}$$

of $B$ satisfying

$$\mu\left( [\widehat{p}(\cdot) \restriction_B]^{-1}(I_n) \right) = \pi_B(I_n) = \pi_A(I_n) \geq \pi_C(I_n) = \mu\left( [p(\cdot) \restriction_C]^{-1}(I_n) \right)$$

for all $n = 1, \ldots, N$. For each $n$, choose $D_n \subseteq [\widehat{p}(\cdot) \restriction_B]^{-1}(I_n)$ satisfying

$$\mu(D_m) = \pi_C(I_n).$$

Let

$$\chi_D = \sum_{n=1}^{N} \chi_{D_m}.$$

□

Let $\chi_{A_1}, \ldots, \chi_{A_m}$ be mutually disjoint in $L_{p(\cdot)}(\Omega, \Sigma, \mu)$ satisfying $\mu(A_i) < \infty$ for all $i = 1, \ldots, m$. Suppose there is a lattice-preserving isomodular isometry

$$T : \langle \chi_{A_1}, \ldots, \chi_{A_m} \rangle \to E,$$

where $(E, \Theta) \in \mathcal{N}_K^{\mathrm{mod}}$ is $\omega_1$-saturated.

For $i = 1, \ldots m$, we may represent $T(\chi_{A_i})$ as $\chi_{B_i}$, and extend this representation to all of $(E, \Theta)$. Suppose we have $\chi_A \in L_{p(\cdot)}(\Omega, \Sigma, \mu)$ with $\chi_A \notin \langle \chi_{A_1}, \ldots, \chi_{A_m} \rangle$. We wish to extend $T$ to a lattice-preserving isomodular isometry taking $\{\chi_{A_1}, \ldots, \chi_{A_m}, \chi_A\rangle$ into $E$. Denote $\langle \chi_{A_1}, \ldots, \chi_{A_m} \rangle$ by $L$. We need to consider two cases:

$\chi_A \in L^{\perp\perp}$: For $i = 1, \ldots, m$, apply Lemma 3.9.3 to $A_i$, $A \cap A_i$, to obtain $D_i \subseteq B_i$ with $D_i \in \widehat{\Sigma}$ and

$\pi_{A\cap A_i} = \pi_{D_i}$. Let $D = \cup_{i=1}^m D_i$. We extend $T$ by imposing $T(\chi_A) = \chi_D$. For any $t \in \mathbb{Q}^{\geq 0}$,

$$\Theta_{p(\cdot)}(t\chi_A) = \sum_{i=1}^m \Theta_{p(\cdot)}(t\chi_{A\cap A_i}) = \sum_{i=1}^m \Theta_{\widehat{p}(\cdot)}(t\chi_{D_i})$$
$$= \Theta_{\widehat{p}(\cdot)}(t\chi_D).$$

Thus $T$ remains an isomodular isometry.

$\chi_A \in L^\perp$: We need to show that for any finite subset $\{t_1, \ldots, t_n\}$ of $\mathbb{Q}^{\geq 0}$, and for every $\epsilon > 0$, we can find $\chi_B \in \{\chi_{B_1}, \ldots, \chi_{B_m}\}^\perp$ such that for any $i \in \{1, \ldots, n\}$,

$$\Theta_{p(\cdot)}\left(t_i\chi_A\right) = \Theta\left(t_i\chi_B\right).$$

Let $\psi_i \in L_1(A, \Sigma_A, \mu)$ be defined by $\psi_i(\omega) = |t_i|^{p(\omega)}$, where

$$\Sigma_i = \{G \in \Sigma \mid G \subseteq A\}.$$

By taking common refinements where appropriate, we may approximate $p(\cdot)$ by a simple function $S^\epsilon_{t_1,\ldots,t_n}$ so that

$$\left|\int_\Omega \psi_i \, d\mu(\omega) - \int_\Omega \left(t_i^{S^\epsilon_{t_1,\ldots,t_n}(\omega)} \chi_{\{\omega\in A | t_i^{S^\epsilon_{t_1,\ldots,t_n}(\omega)} \neq 1\}}(\omega)\right) \, d\mu(\omega)\right| < \epsilon,$$

for all $i = 1, \ldots, n$. The simple function $S^\epsilon_{t_1,\ldots,t_n}$ induces a Borel partition $\{I_1, \ldots, I_k\}$ on $K$, where $p_1, \ldots, p_k$ are the $k$ distinct values that the simple function $S^\epsilon_{t_1,\ldots,t_n}$ takes, and for all $i = 1, \ldots, k$, $p_i \in I_i$. Now, because $E$ is a non-atomic $\omega_1$-saturated structure, $E$ can be represented in the following way.

$$E = \bigoplus_{p\in K} L_p(\Omega_p, \Sigma_p, \mu_p) \oplus L_{p(\cdot)}(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu}) \oplus R,$$

where $R$ stands for the band orthogonal, in $E$, to the direct sum to its left, $\chi_{B_i} \in L_{p(\cdot)}(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu})$ for $i = 1, \ldots, m$, and each measure space mentioned has infinite measure and is decomposable. The simple function $S^\epsilon_{t_1,\ldots,t_n}$ now tells us where to look, in $E$, for $\chi_B$. If

$$S^\epsilon_{t_1,\ldots,t_n} = \sum_{j=1}^k p_j \chi_{C_j},$$

then we define

$$\chi_B = \sum_{j=1}^k \chi_{D_j},$$

where $D_j \in L_{p_j}(\Omega_{p_j}, \Sigma_{p_j}, \mu_{p_j})$ for $j = 1, \ldots, k$, and $\widehat{\mu}(D_j) = \mu(C_j)$ for $j = 1, \ldots, k$. Then for $i = 1, \ldots, n$,

$$\Theta(t_i \chi_B) = \Theta_{p(\cdot)} \left( t_i^{S^{\epsilon}_{t_1,\ldots,t_n}(\omega)} \chi_{\{\omega \in B | t_i^{S^{\epsilon}_{t_1,\ldots,t_n}(\omega)} \neq 1\}}(\omega) \right).$$

The general case can be reduced to considering the two preceding cases, by the disjoint additivity of the convex modular. This argument proves

**Theorem 3.9.4.** *Fix* $k \geq 2^{\sup K}$. *Then* $\mathcal{AN}_K^{\mathrm{mod}}$ *admits quantifier elimination in the signature* $L_k$ *of Orlicz lattices satisfying the* $\Delta_2^k$*-condition.*

**Corollary.** *Fix* $k \geq 2^{\sup K}$. *Then* $Th(\mathcal{AN}_K^{\mathrm{mod}})$ *is complete in the signature* $L_k$ *of Orlicz lattices satisfying the* $\Delta_2^k$*-condition.*

*Proof.* Note that there is an $L_k$-structure, namely the Orlicz lattice $\mathcal{Z} := \langle 0 \rangle$, which embeds into any element of the class $\mathcal{AN}_K$. Let $\mathcal{A}, \mathcal{B} \vDash Th(\mathcal{AN}_K^{\mathrm{mod}})$, and let $f$ and $g$ be embeddings of $\mathcal{Z}$ into $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\sigma$ be any $L_k$-statement. By quantifier elimination, there is a quantifier free $L_k$-formula $\psi$ such that $\psi$ and $\sigma$ are equivalent in all models of $Th(\mathcal{AN}_K^{\mathrm{mod}})$. We have that

$$\mathcal{A} \vDash \sigma \Longleftrightarrow \mathcal{A} \vDash \psi \Longleftrightarrow \mathcal{Z} \vDash \psi \Longleftrightarrow \mathcal{B} \vDash \psi \Longleftrightarrow \mathcal{B} \vDash \sigma,$$

whence $\mathcal{A} \equiv \mathcal{B}$. □

## 3.10 Stability

In [15], Henson announced the $\omega$-stability of the theory of atomless $L_p$-Banach lattices. Subsequently, Pomper proved that types in atomless $L_p$-Banach lattices correspond to conditional distributions (see [24, Chapter 6]). In [1], Ben Yaacov, Berenstein, and Henson provided a new proof of the $\omega$-stability of the theory of atomless $L_p$-Banach lattices using Pomper's observation. We will investigate the issue of stability of the theory of the class $\mathcal{AN}_K^{\mathrm{mod}}$ for $K$ a compact subset of $[1, \infty)$. (Here it is important to recall that an appropriate signature to have in mind is $L_k$, where $k \geq 2^{\sup K}$; this is so because the elements of the class $\mathcal{AN}_K^{\mathrm{mod}}$ are $L_k$-structures for all $k \geq 2^{\sup K}$ but not for other values of $k$.)

We begin with a quick review of conditioning in function spaces, starting with Radón-Nikodým's classical result. If $\Sigma_0$ is a sub-$\sigma$-algebra of $\Sigma$ and $(\Omega, \Sigma, \mu)$ is a measure space, we say that $\Sigma_0$ is *$\mu$-semifinite* if for all $A \in \Sigma_0$ with $\mu(A) > 0$, there exists $B \in \Sigma_0$ with $B \subseteq A$ and $0 < \mu(B) < \infty$. In what follows, we will write $(\Omega_0, \Sigma_0, \mu_0) \subseteq (\Omega, \Sigma, \mu)$ to indicate that $\Omega_0 \subseteq \Omega$, $\Sigma_0$ is a $\mu$-semifinite sub-$\sigma$-algebra of $\{A \cap \Omega_0 \mid A \in \Sigma\}$ and

$\mu_0$ is the restriction of $\mu$ to $\Sigma_0$. (In case $\Omega_0 = \Omega$, $\Sigma_0$ is simply a sub-$\sigma$-algebra of $\Sigma$ and $\mu_0$ is the restriction of $\mu$ to $\Sigma_0$.) If $(\Omega, \Sigma, \mu)$ is decomposable and $(\Omega_0, \Sigma_0, \mu_0) \subseteq (\Omega, \Sigma, \mu)$, then $(\Omega_0, \Sigma_0, \mu_0)$ is decomposable, too.

**Theorem 3.10.1** (Radón-Nikodým). *Let $(\Omega, \Sigma_0, \mu_0) \subseteq (\Omega, \Sigma, \mu)$. Let $A \in \Sigma$ with $\mu(A) < \infty$. Then there exists $g_A \in L_0(\Omega, \Sigma_0, \mu)$ such that*

$$\int_B g_A \, d\mu = \mu\left(A \cap B\right)$$

*for every set $B \in \Sigma_0$.*

We denote the function $g_A$ by $C_{\Sigma_0}\mathcal{P}(A)$ and call it the *conditional probability of $A$ with respect to $\Sigma_0$*.

**Definition.** Let $(\Omega, \Sigma, \mu)$ be a measure space, and consider $f_1, \ldots, f_n \in L_0(\Omega, \Sigma, \mu)$. We define the *joint distribution of $f_1, \ldots, f_n$* to be the extended real-valued function

$$\operatorname{dist}(f_1, \ldots, f_n) : \mathfrak{B}(\mathbb{R}^n) \to \overline{\mathbb{R}}$$

defined by

$$\operatorname{dist}(f_1, \ldots, f_n)(B) = \mu\left(\{\omega \in \Omega : (f_1(\omega), \ldots, f_n(\omega)) \in B\}\right)$$

for all $B \in \mathfrak{B}(\mathbb{R}^n)$.

Given $f_1, \ldots, f_n \in L_0(\Omega, \Sigma, \mu)$ and $B \in \Sigma$, denote the set

$$\{\omega \in \Omega : (f_1(\omega), \ldots, f_n(\omega)) \in B\}$$

by

$$\{\bar{f} \in B\}.$$

**Definition.** Let $(\Omega, \Sigma_0, \mu) \subseteq (\Omega, \Sigma, \mu)$. For $f_1, \ldots, f_n \in L_0(\Omega, \Sigma, \mu)$ we define the *joint conditional distribution of $f_1, \ldots, f_n$ with respect to $\Sigma_0$* to be the function

$$\operatorname{dist}((f_1, \ldots, f_n)/\Sigma_0) : \mathfrak{B}(\mathbb{R}^n) \to \overline{\mathbb{R}}$$

defined by

$$B \mapsto C_{\Sigma_0}\mathcal{P}(\{\bar{f} \in B\})$$

for all $B \in \mathfrak{B}(\mathbb{R}^n)$.

We will use a convenient criterion to check whether two tuples $\bar{f} = (f_1, \ldots, f_n)$ and $\bar{h} = (h_1, \ldots, h_n)$ in $L_0(\Omega, \Sigma, \mu)$ have the same joint conditional distribution with respect to a sub-$\sigma$-algebra $\Sigma_0$.

**Proposition 3.10.2.** *Let $(\Omega, \Sigma, \mu)$ be a finite measure space. Let $\bar{f}$ and $\bar{h}$ be $n$-tuples in $L_0(\Omega, \Sigma, \mu)$. Let $(\Omega, \Sigma_0, \mu_0) \subseteq (\Omega, \Sigma, \mu)$. Then*

$$\operatorname{dist}(\bar{f} \mid \Sigma_0) = \operatorname{dist}(\bar{h} \mid \Sigma_0)$$

*if and only if for all continuous bounded $F : \mathbb{R}^n \to \mathbb{R}$ we have that*

$$\mathbb{E}[F(\bar{f}) \mid \Sigma_0] = \mathbb{E}[F(\bar{h}) \mid \Sigma_0],$$

*where $\mathbb{E}[X \mid \Sigma_0]$ denotes the conditional expectation of $X \in L_1(\Omega, \Sigma, \mu)$ with respect to $\Sigma_0$.*

The key ingredient in our proof of the above proposition is the following lemma, due to Pomper. (See [24, Lemma 6.3.2].)

**Lemma 3.10.3.** *Let $(\Omega, \Sigma, \mu)$ be a measure space. let $B = B_1 \times \cdots \times B_n \subseteq \mathbb{R}^n$, where for all $1 \leq i \leq n$, $B_i = (r_i, \infty)$ for some $r_i \in \mathbb{R}$. For every $k \in \mathbb{N}$ let $F_k : \mathbb{R}^n \to \mathbb{R}$ be defined by*

$$F_k(x_1, \ldots, x_n) = \left( \sum_{i=1}^{n} \left( k(x_i - r_i)^+ \right) \wedge 1 - (n-1) \right)^+ .$$

*For any $n$-tuple $\bar{f}$ in $L_0(\Omega, \Sigma, \mu)$, we have*

$$\lim_{k \to \infty} F_k(f_1, \ldots, f_n) = \chi_{(\bar{f})^{-1}(B)}$$

*and $(F_k(f_1, \ldots, f_n)(\omega) \mid k \in \mathbb{N})$ is a monotonically increasing sequence for all $\omega \in \Omega$.*

*Proof.* Note that

$$\{\omega \in \Omega \mid (f_1(\omega), \ldots, f_n(\omega)) \in B\} = \bigcap_{i=1}^{n} \{\omega \in \Omega \mid f_i(\omega) \in B_i\}.$$

For all $k \in \mathbb{N}$ and $1 \leq i \leq n$, let $s_{k,i}(x_1, \ldots, x_n)$ be the term $k(x_i - r_i)^+ \wedge 1$. Note that for all $\omega \in \Omega$ and for all $(f_1, \ldots, f_n) \in (L_0(\Omega, \Sigma, \mu))^n$ we have

$$s_{k,i}(f_1, \ldots, f_n)(\omega) = 0 \Longleftrightarrow f_i(\omega) \leq r$$

and

$$s_{k,i}(f_1, \ldots, f_n)(\omega) = 1 \Longleftrightarrow f_i(\omega) \geq r_i + \frac{1}{k}.$$

Furthermore, $s_{k,i}(f_1, \ldots, f_n)(\omega) \le s_{l,i}(f_1, \ldots, f_n)(\omega)$ for all $\omega \in \Omega$, all $1 \le i \le n$, and all $k \le l$ in $\mathbb{N}$. Let

$$t_k(x_1, \ldots, x_n) := \left( \sum_{i=1}^{n} s_{k,i}(x_1, \ldots, x_n) - (n-1) \right)^+$$

for all $k \in \mathbb{N}$.

We claim that $\lim_{k\to\infty} t_k(f_1, \ldots, f_n)(\omega) = \chi_{\bar{f}^{-1}(B)}$ for all $\omega \in \Omega$ and for all $\bar{f} \in (L_0(\Omega, \Sigma, \mu))^n$. Fix $\omega \in \Omega$ and $\bar{f} \in (L_0(\Omega, \Sigma, \mu))^n$.

In case $\omega \notin \bar{f}^{-1}(B)$, there exists $1 \le i_0 \le n$ such that $f_{i_0}(\omega) \le r_{i_0}$. Then $s_{k,i_0}(f_1, \ldots, f_n)(\omega) = 0$ for all $k \in \mathbb{N}$, and so $\sum_{i=1}^{n} s_{k,i}(f_1, \ldots, f_n)(\omega) \le n-1$. This implies that $t_k(f_1, \ldots, f_n)(\omega) = 0$ for all $k \in \mathbb{N}$. Therefore $\lim_{k\to\infty} t(f_1, \ldots, f_n)(\omega) = 0 = \chi_{\bar{f}^{-1}(B)}(\omega)$.

In case $\omega \in \bar{f}^{-1}(B)$, then $f_i(\omega) > r_i$ for all $1 \le i \le n$. Choose $k_0 \in \mathbb{N}$ such that $f_i(\omega) \ge r_i + \frac{1}{k_0}$ for all $1 \le i \le n$. Then $s_{k,i}(\omega) = 1$ for all $k \in \mathbb{N}$ with $k \ge k_0$ and all $1 \le i \le n$. Therefore $t_k(f_1, \ldots, f_n)(\omega) = 1$ for all $k \in \mathbb{N}$ with $k \ge k_0$, and therefore $\lim_{k\to\infty} t_k(f_1, \ldots, f_n)(\omega) = 1 = \chi_{\bar{f}^{-1}(B)}(\omega)$.

This establishes the Lemma. □

We are now ready to prove Proposition 3.10.2.

*Proof.* ($\Longleftarrow$) It suffices to show that for any $A \in \Sigma_0$,

$$\int_A C_{\Sigma_0}\mathcal{P}((\bar{f})^{-1}(B))\, d\mu_0(\omega) = \int_A C_{\Sigma_0}\mathcal{P}((\bar{h})^{-1}(B))\, d\mu_0(\omega).$$

Let $A \in \Sigma_0$. By the definition of conditional probability,

$$\int_A C_{\Sigma_0}\mathcal{P}((\bar{f})^{-1}(B))\, d\mu_0(\omega) = \int_A \chi_{(\bar{f})^{-1}(B)}\, d\mu.$$

By Lemma 3.10.3,

$$\begin{aligned} \int_A \chi_{(\bar{f})^{-1}(B)}\, d\mu_0 &= \lim_{k\to\infty} \int_A |F_k(f_1, \ldots, f_n)|\, d\mu \\ &= \lim_{k\to\infty} \int_A |F_k(h_1, \ldots, h_n)|\, d\mu \\ &= \int_A \chi_{(h)^{-1}(B)}\, d\mu_0. \end{aligned}$$

This proves the ($\Longleftarrow$) direction of Proposition 3.10.2. The other direction is obvious. □

If $f \in L_0(\Omega, \Sigma, \mu)$, we write $f^{(k)}$ to denote the uniformly bounded function defined by

$$f^{(k)}(\omega) = \begin{cases} k & \text{if } k \leq f(\omega) \\ f(\omega) & \text{if } -k \leq f(\omega) < k \\ -k & \text{if } -k \geq f(\omega). \end{cases}$$

If $\bar{f} = (f_1, \ldots, f_n)$ is an $n$-tuple of elements in $L_0(\Omega, \Sigma, \mu)$, we write $\bar{f}^{(k)}$ to denote the $n$-tuple

$$\bar{f}^{(k)} = (f_1^{(k)}, \ldots, f_n^{(k)}).$$

**Lemma 3.10.4.** *Let $(\Omega, \Sigma_0, \mu) \subseteq (\Omega, \Sigma, \mu)$ and let $\bar{f}$ be an $n$-tuple in $L_0(\Omega, \Sigma, \mu)$. Then*

$$\lim_{k\to\infty} \operatorname{dist}(\bar{f}^{(k)}/\Sigma_0)(B) = \operatorname{dist}(\bar{f}/\Sigma_0)(B)$$

*for every Borel subset $B$ of $\mathbb{R}^n$.*

*Proof.* Let $B$ be a Borel subset of $\mathbb{R}^n$. Then $\operatorname{dist}(\bar{f}/\Sigma_0)(B) = \mathbb{E}[\chi_{\{\bar{f}\in B\}} \mid \Sigma_0]$. By the uniqueness of the conditional expectation, it is enough to show that

$$\mathbb{E}\left[\lim_{k\to\infty} \mathbb{E}\left[\chi_{\{\bar{f}^{(k)}\in B\}} \mid \Sigma_0\right]\right] = \mathbb{E}\left[\chi_{\{\bar{f}\in B\}}\right].$$

Note that $\chi_{\{\bar{f}^{(k)}\in B\}} \uparrow \chi_{\{\bar{f}\in B\}}$ as $k \to \infty$. By the Monotone Convergence Theorem, we have that

$$\begin{aligned} \mathbb{E}\left[\lim_{k\to\infty} \mathbb{E}\left[\chi_{\{\bar{f}^{(k)}\in B\}} \mid \Sigma_0\right]\right] &= \lim_{k\to\infty} \mathbb{E}\left[\mathbb{E}\left[\chi_{\{\bar{f}^{(k)}\in B\}} \mid \Sigma_0\right]\right] \\ &= \lim_{k\to\infty} \mathbb{E}\left[\chi_{\{\bar{f}^{(k)}\in B\}}\right] = \mathbb{E}\left[\chi_{\{\bar{f}\in B\}}\right]. \end{aligned}$$

This finishes the proof. □

Let $C_b(\mathbb{R}^{n+m})$ be the algebra of continuous bounded functions $\mathbb{R}^{n+m} \to \mathbb{R}$. Let $\mathcal{A}$ be the subalgebra of $C_b(\mathbb{R}^{n+m})$ generated by functions of the form $G(\bar{x}, \bar{y}) = G_1(\bar{x})G_2(\bar{y})$, where $G_1 : \mathbb{R}^n \to \mathbb{R}$ and $G_2 : \mathbb{R}^m \to \mathbb{R}$ are continuous and bounded. By the Stone-Weierstrass Theorem, $\mathcal{A}$ is dense in $C_b(\mathbb{R}^n)$ in the topology of uniform convergence on compacta.

**Proposition 3.10.5.** *Let $(\Omega, \Sigma, \mu)$ be a finite measure space. Let*

$$(\Omega, \Sigma_0, \mu_0) \subseteq (\Omega, \Sigma, \mu).$$

*Let $\bar{f}$ and $\bar{h}$ be $n$-tuples in $L_0(\Omega, \Sigma, \mu)$ and let $g$ be an $m$-tuple in $L_0(\Omega, \Sigma_0, \mu_0)$. Then*

$$\operatorname{dist}(f_1, \ldots, f_n \mid \Sigma_0) = \operatorname{dist}(h_1, \ldots, h_n \mid \Sigma_0)$$

*implies that*

$$\operatorname{dist}(f_1, \ldots, f_n, g_1, \ldots, g_m \mid \Sigma_0) = \operatorname{dist}(h_1, \ldots, h_n, g_1, \ldots, g_m \mid \Sigma_0).$$

*Proof.* By Lemma 3.10.4, it suffices to show the result in case $\bar{f}$, $\bar{g}$, and $\bar{h}$ are in $L_\infty(\Omega, \Sigma, \mu)$. By the density of $\mathcal{A}$ in $C(\mathbb{R}^n)$, it suffices to show that $\mathbb{E}[G_1(\bar{f})G_2(\bar{g}) \mid \Sigma_0] = \mathbb{E}[G(\bar{h})G_2(\bar{g}) \mid \Sigma_0]$ for all $G_1 \in C(\mathbb{R}^n)$, $G_2 \in C(\mathbb{R}^m)$. Since $G_2(\bar{g})$ is $\Sigma_0$-measurable, $\mathbb{E}[G_1(\bar{f})G_2(\bar{g}) \mid \Sigma_0] = G_2(\bar{g})\mathbb{E}[G_1(\bar{f}) \mid \Sigma_0]$. Likewise, $\mathbb{E}[G_1(\bar{h})G_2(\bar{g}) \mid \Sigma_0] = G_2(\bar{g})\mathbb{E}[G_1(\bar{h}) \mid \Sigma_0]$. Since $\operatorname{dist}(\bar{f} \mid \Sigma_0) = \operatorname{dist}(\bar{h} \mid \Sigma_0)$, by Proposition 3.10.2 we have that $\mathbb{E}[G_1(\bar{f}) \mid \Sigma_0] = \mathbb{E}[G_1(\bar{h}) \mid \Sigma_0]$. ☐

The following proposition relates joint conditional distributions to types.

**Proposition 3.10.6.** *Let $K$ be a compact subset of $(1, \infty)$, and let $k \geq 2^{\sup K}$. Let $X \in \mathcal{AN}_K^{\mathrm{mod}}$ and let $\widehat{X}$ be an ultrapower of $X$ in the signature $L_k$. Let $(\Omega, \Sigma, \mu) \subseteq (\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu})$ be decomposable measure spaces and let $p(\cdot) \in L_0(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu})$ be such that $\mathcal{R}_{p(\cdot)} = K$, $X = L_{p(\cdot)\restriction\Omega}(\Omega, \Sigma, \mu)$, and $\widehat{X} = L_{p(\cdot)}(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu})$. Let $\bar{f} = (f_1, \ldots, f_n)$ and $\bar{h} = (h_1, \ldots, h_n)$ be $n$-tuples in the band $X^{\perp\perp}$ generated by $X$ in $\widehat{X}$. Then*

$$\operatorname{dist}(\bar{f} \mid \Sigma) = \operatorname{dist}(\bar{h} \mid \Sigma)$$

*implies*

$$\operatorname{tp}_{L_k}(\bar{f} \mid X) = \operatorname{tp}_{L_k}(\bar{h} \mid X).$$

*Proof.* By quantifier elimination, it suffices to show that $\operatorname{qftp}_{L_k}(\bar{f} \mid X) = \operatorname{qftp}_{L_k}(\bar{h} \mid X)$. This in turn reduces to showing that

$$\Theta_{p(\cdot)}(t(\bar{f}, \bar{g})) = \Theta_{p(\cdot)}(t(\bar{h}, \bar{g}))$$

for any $m$-tuple $\bar{g}$ and any Banach lattice term $t(\bar{x}, \bar{y})$. Because $(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu})$ and $(\Omega, \Sigma, \mu)$ may be taken to be decomposable, we may assume that

$$(\Omega, \widehat{\Sigma} \restriction_\Omega, \widehat{\mu} \restriction_\Omega) = \oplus_{\alpha \in I}(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$$

for some index $I$ such that $\Omega_\alpha \in \Sigma$ with $\mu_\alpha(\Omega_\alpha) < \infty$ for all $\alpha \in I$. In this way, we reduce to the case in which $(\Omega, \widehat{\Sigma} \restriction_\Omega, \widehat{\mu} \restriction_\Omega)$ is a finite measure space. Let $\bar{f} = (f_1, \ldots, f_n)$ and $\bar{h} = (h_1, \ldots, h_n)$ be $n$-tuples in the band $X^{\perp\perp}$ generated by $X$ in $\widehat{X}$. Suppose that

$$\mathrm{dist}(\bar{f} \mid \Sigma) = \mathrm{dist}(\bar{h} \mid \Sigma).$$

Since $X$ is an ultraroot of $\widehat{X}$, $p(\cdot) \restriction_\Omega$ is $\Sigma$-measurable. Let $\bar{g} = (g_1, \ldots, g_m)$ be an $m$-tuple in $X$. By Proposition 3.10.5, we have that

$$\mathrm{dist}(\bar{f}, \bar{g}, p(\cdot) \restriction_\Omega \mid \Sigma) = \mathrm{dist}(\bar{h}, \bar{g}, p(\cdot) \restriction_\Omega \mid \Sigma).$$

Let $\bar{x}$ denote an $n$-ary tuple of variables, and let $\bar{y}$ denote an $m$-ary tuple of variables. Let $t(\bar{x}, \bar{y})$ be an arbitrary Banach lattice term. Let

$$F : \mathbb{R}^n \times \mathbb{R}^m \times K \to \mathbb{R}$$

be defined by

$$F(\bar{x}, \bar{y}, p) = |t(\bar{x}, \bar{y})|^p.$$

Then $F$ is a continuous function. For $M > 0$, let $F_M$ be the function defined by $F_M(\bar{x}, \bar{y}, p) = F(\bar{x}, \bar{y}, p) \wedge M$. Then $F_M$ is a continuous bounded function, and Proposition 3.10.2 yields that

$$\mathbb{E}[F_M(\bar{f}, \bar{g}, p(\cdot)) \mid \Sigma] = \mathbb{E}[F_M(\bar{h}, \bar{g}, p(\cdot)) \mid \Sigma]$$

for all $M > 0$. Hence

$$\mathbb{E}[F_M(\bar{f}, \bar{g}, p(\cdot))] = \mathbb{E}[F_M(\bar{h}, \bar{g}, p(\cdot))]$$

An application of the Monotone Convergence Theorem yields

$$\mathbb{E}[F(\bar{f}, \bar{g}, p(\cdot))] = \mathbb{E}[F(\bar{h}, \bar{g}, p(\cdot))].$$

This means that $\Theta_{p(\cdot)}\left(t(\bar{f}, \bar{g})\right) = \Theta_{p(\cdot)}\left(t(\bar{h}, \bar{g})\right)$, and so we have shown that $\mathrm{qftp}_{L_k}(\bar{f} \mid X) = \mathrm{qftp}_{L_k}(\bar{h} \mid X)$. □

Let $(\Omega_0, \Sigma_0, \mu_0) \subseteq (\Omega, \Sigma, \mu)$. The measure space $(\Omega, \Sigma, \mu)$ is said to be *atomless over* $(\Omega_0, \Sigma_0, \mu_0)$ if for every $A \in \Sigma$ of positive finite measure and for every $f \in L_0(\Omega, \Sigma, \mu)$ with $0 \le f \le C_{\Sigma_0}\mathcal{P}(A)$ there exists a set $B \in \Sigma$ with $B \subseteq A$ for which $f = C_{\Sigma_0}\mathcal{P}(B)$.

*Example.* Let $(\Omega, \Sigma, \mu)$ be any nontrivial measure space. Let $\mathfrak{B}$ denote the $\sigma$-algebra of Borel sets on $[0,1]$ and $\mathfrak{T}$ denote the trivial $\sigma$-algebra $\{\emptyset, [0,1]\}$ of $[0,1]$. If $\nu$ denotes the Lebesgue measure on $[0,1]$, we have that the measure space $(\Omega, \Sigma, \mu) \otimes ([0,1], \mathfrak{B}, \nu)$ is atomless over $(\Omega, \Sigma, \mu) \otimes ([0,1], \mathfrak{T}, \nu)$.

In [5], Berkes and Rosenthal proved the following important fact:

**Theorem 3.10.7.** *Let $(\Omega, \Sigma_0, \mu_0) \subseteq (\Omega, \Sigma, \mu)$ be measure spaces. Suppose that $(\Omega, \Sigma, \mu)$ is atomless over $(\Omega, \Sigma_0, \mu_0)$. Let $(\Omega_1, \Sigma_{,1}, \mu_1)$ be another measure space satisfying $(\Omega, \Sigma_0, \mu_0) \subseteq (\Omega, \Sigma_1, \mu_1)$. Then for any $f \in L_0(\Omega, \Sigma_1, \mu_1)$ there is $h \in L_0(\Omega, \Sigma, \mu)$ such that $\mathrm{dist}(h \mid \Sigma_0) = \mathrm{dist}(f \mid \Sigma_0)$. (This is Theorem 1.5 in [5].)*

This result can be generalized, as has been done in [24, Theorem 6.2.7] or [1, Lemma 2.15], to the $n$-dimensional case.

**Theorem 3.10.8.** *Let $(\Omega, \Sigma_0, \mu_0) \subseteq (\Omega, \Sigma, \mu)$ be measure spaces. Suppose that $(\Omega, \Sigma, \mu)$ is atomless over $(\Omega, \Sigma_0, \mu_0)$. Let $(\Omega, \Sigma_{,1}, \mu_1)$ be another measure space satisfying $(\Omega, \Sigma_0, \mu_0) \subseteq (\Omega, \Sigma_1, \mu_1)$. Then for any $n$-tuple $\bar{f}$ in $L_0(\Omega, \Sigma_1, \mu_1)$ there is an $n$-tuple $\bar{h}$ in $L_0(\Omega, \Sigma, \mu)$ such that $\mathrm{dist}(\bar{h} \mid \Sigma_0) = \mathrm{dist}(\bar{f} \mid \Sigma_0)$.*

We aim to prove that whenever $K$ is a compact subset of $(1, \infty)$ the theory $Th(\mathcal{AN}_K^{\mathrm{mod}})$ is stable with respect to the $d$ metric.

Recall that $Th(\mathcal{AN}_K^{\mathrm{mod}})$ is $\lambda$-stable with respect to the $d$ metric if for every $X \in \mathcal{AN}_K^{\mathrm{mod}}$ and for every substructure $Y \subseteq X$ of cardinality $\leq \lambda$, $S_1(Y)$ has density character $\leq \lambda$ for the $d$-metric topology. By an application of the Downward Löwenheim-Skolem Theorem, it is possible to reduce to the case in which $Y$ is taken to be a submodel of $X$ (i.e., $Y$ can be assumed to be an element of $\mathcal{AN}_K^{\mathrm{mod}}$) rather than merely a substructure.[3] Furthermore, $X$ can be replaced, without loss of generality, by some ultrapower $\widehat{Y}$ of $Y$ which is $\kappa$-saturated for some $\kappa > \lambda$. So we may assume that $X$ is itself such an ultrapower of $Y$.

Proposition 3.10.6 applies in this setting. If $Y = L_{p(\cdot)\restriction\Omega}(\Omega, \Sigma, \mu)$, and $X = L_{p(\cdot)}(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu})$ for some $p(\cdot) \in L_0(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu})$ such that $\mathcal{R}_{p(\cdot)} = K$, we have that $\mathrm{dist}(\bar{f} \mid \Sigma) = \mathrm{dist}(\bar{h} \mid \Sigma)$ implies that $\mathrm{tp}(\bar{f} \mid Y) = \mathrm{tp}(\bar{h} \mid Y)$, whenever $\bar{f}$ and $\bar{h}$ are $n$-tuples in the band $Y^{\perp\perp}$ generated by $Y$ in $X$.

Let $\mathfrak{B}$ denote the $\sigma$-algebra of Borel sets in $[0,1]$ and $\nu$ denote Lebesgue measure on $[0,1]$. Let $\mathbf{p}(\cdot) \in L_0((\Omega, \Sigma, \mu) \otimes ([0,1], \mathfrak{B}, \nu)$ be defined by

$$\mathbf{p}((\omega, r)) = p(\omega).$$

If $\mathfrak{T}$ denotes the trivial $\sigma$-algebra $\{\emptyset, [0,1]\}$ on $[0,1]$, then

$$(\Omega, \Sigma, \mu) \otimes ([0,1], \mathfrak{B}, \nu)$$

[3]The key observation is that if $Y_1 \subseteq Y_2$, then the natural restriction map taking $S_1(Y_2)$ onto $S_1(Y_1)$ is a contraction with respect to the $d$ metric. Hence a dense subset of $S_1(Y_2)$ gets mapped onto a dense subset of $S_1(Y_1)$.

is atomless over

$$(\Omega, \Sigma, \mu) \otimes ([0,1], \mathfrak{T}, \nu).$$

An application of Theorem 3.10.7 yields that every 1-type over $Y$ with a realization in the band $Y^{\perp\perp} \subseteq X$ is realized in

$$Z := L_{\mathbf{p}(\cdot)}((\Omega, \Sigma, \mu) \otimes ([0,1], \mathfrak{B}, \nu)).$$

If the density character of $Y$ is $\lambda$ for some infinite cardinal $\lambda$, then $Z$ has density character $\lambda$ as well.

Let $g \in X$ be orthogonal to $Y$. Let $A$ be a dense set in $Y$ with cardinality $\lambda$. The type $\mathrm{tp}(g/A)$ is determined by the collection

$$\left\{\Theta_{\widehat{p}(\cdot)}(rg^+) \mid r \in \mathbb{Q}^+\right\} \cup \left\{\Theta_{\widehat{p}(\cdot)}(rg^-) \mid r \in \mathbb{Q}^+\right\}.$$

The family of all such collections is equinumerous with the set $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers. Thus the space of types of functions in the band orthogonal to $L_{p(\cdot)}(\Omega, \Sigma, \mu)$ has density character less than or equal to $\mathfrak{c}$, the cardinality of the continuum.

The upshot of these observations is the following:

**Theorem 3.10.9.** *Let $K$ be a compact subset of $(1, \infty)$. Then $Th(\mathcal{AN}_K^{\mathrm{mod}})$ is $\lambda$-stable with respect to the $d$ metric for all $\lambda \geq \mathfrak{c}$.*

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# Curriculum Vitae

Luis Pedro Poitevin was born May 5, 1973 in Freiburg, Germany. He grew up mostly in Guatemala, where he had the unusual privilege of getting a relatively good education. After completing his undergraduate degree at Universidad del Valle de Guatemala, he chose to pursue graduate studies in Mathematics at the University of Illinois at Urbana-Champaign. Two years into the PhD program, he decided to focus on Mathematical Logic, specifically its applications in Analysis. He held several Research Assistantships over the years at the University of Illinois at Urbana-Champaign, and he received the Departmental T.A. Instructional Award in 2004. He is a tenure-track Assistant Professor at Salem State College.