FRAÏSSÉ THEORY FOR METRIC STRUCTURES

BY

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Abstract

In 1954, Roland Fraïssé published a paper that answered the following questions: Given a first-order signature L and a class $\mathfrak{A}$ of finite L-structures that is closed under isomorphism:

1. find necessary and sufficient conditions on $\mathfrak{A}$ that guarantee the existence of a "homogeneous" L-structure $\mathcal{M}$ such that the class of L-structures that are isomorphic to finite L-substructures of $\mathcal{M}$ is $\mathfrak{A}$;

2. find necessary and sufficient conditions on $\mathfrak{A}$ that guarantee the existence of an L-structure $\mathcal{M}$ such that $\mathrm{Th}(\mathcal{M})$ has QE and is ω-categorical, and such that the class of L-structures that are isomorphic to finite L-substructures of $\mathcal{M}$ is $\mathfrak{A}$.

In this thesis we generalize Fraïssé's results to the setting of bounded continuous logic for metric structures. This logic was presented in 2004 by Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, and it may be considered as a generalization of first-order logic.

We also prove a theorem, in the setting of continuous model theory, that is a generalization of a theorem of H. D. Macpherson about the automorphism groups of ω-categorical structures.

To my parents,

Παναγιώτη και Αναστασία

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Chapter 1

Introduction

In 1954, Roland Fraïssé published a paper (see [4]) that has become a classic in Model Theory. In this paper he pointed out that we can think of the class of finite linear orderings as a set of approximations to the ordering of the rationals, and he described a way of building the rationals out of these finite approximations. Fraïssé's construction is important because it works in many other cases too. Starting from a suitable set of finite structures we can build their "limit" and some of the structures built in this way have turned out to be remarkably interesting.

In this document we will use the following notation: We say that a set is countable if its cardinality is equal to the cardinality of a subset of ω. We say that a set is countably infinite if its cardinality is equal to the cardinality of ω.

For simplicity, for the remainder of the introduction L_1 will denote a first-order signature with no function symbols. Some of the theorems that will be stated may be true for more general first-order signatures but this is not going to be relevant to our later analysis.

The series of questions that Fraïssé answered were the following: Given a class $\mathfrak{A}$ of finite L-structures that is closed under isomorphisms:

1. find necessary and sufficient conditions on $\mathfrak{A}$ that guarantee the existence of a "homogeneous" L_1-structure $\mathcal{M}$ such that the class of L_1-structures that are isomorphic to finite L_1-substructures of $\mathcal{M}$ is $\mathfrak{A}$;

2. find necessary and sufficient conditions on $\mathfrak{A}$ that guarantee the existence of an L_1-structure $\mathcal{M}$ such that $\mathrm{Th}(\mathcal{M})$ has QE and is ω-categorical, and such that the class of L_1-structures that are isomorphic to finite L_1-substructures of $\mathcal{M}$ is $\mathfrak{A}$.

In later years several people studied the interplay among $\mathfrak{A}$, $\mathcal{M}$, $\mathrm{Th}(\mathcal{M})$, and $\mathrm{Aut}(\mathcal{M})$.

To state Fraïssé's results precisely we will need some terminology.

Notation 1.1. 1. If $\mathcal{A}$ is an L-structure we set $\mathrm{age}(\mathcal{A})$ to be the class of L_1-structures that are isomorphic to finite L_1-substructures of $\mathcal{M}$.

2. If T is a complete L_1-theory we set age(T) to be the class of L_1-structures that are isomorphic to finite L_1-substructures of models of T.

We will also need the following definition.

Definition 1.2. Let $\mathfrak{A}$ be a class of finite L_1-structures with the following properties:

- $\mathfrak{A}$ is closed under isomorphism.
- (Hereditary property, HP for short) If $\mathcal{M} \in \mathfrak{A}$ and $\mathcal{N}$ is an L_1-substructure of $\mathcal{M}$, then $\mathcal{N} \in \mathfrak{A}$.
- (Joint embedding property, JEP for short) If $\mathcal{M}_1 \in \mathfrak{A}$, $\mathcal{M}_2 \in \mathfrak{A}$, then there exist $\mathcal{M} \in \mathfrak{A}$ and L_1-embeddings $f_1 : \mathcal{M}_1 \to \mathcal{M}$, $f_2 : \mathcal{M}_2 \to \mathcal{M}$.

Then $\mathfrak{A}$ is called an age in L_1. Usually, when L_1 is clear from the context, we may just say that $\mathfrak{A}$ is an age.

Definition 1.3. The cardinality of an age $\mathfrak{A}$ in L_1 is defined to be the number of isomorphism types of elements of $\mathfrak{A}$.

Definition 1.4 (Amalgamation property, AP for short)**.** Let $\mathfrak{A}$ be a class of finite L_1-structures. We say that $\mathfrak{A}$ has the amalgamation property if for every $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2$ in $\mathfrak{A}$ and L_1-embeddings $f_1 : \mathcal{M} \to \mathcal{M}_1$, $f_2 : \mathcal{M} \to \mathcal{M}_2$ there exist $\mathcal{N}$ in $\mathfrak{A}$ and L_1-embeddings $g_1 : \mathcal{M}_1 \to \mathcal{N}$, $g_2 : \mathcal{M}_2 \to \mathcal{N}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Note that in general AP does not imply JEP.

Definition 1.5. Let L_1 be a first-order signature and let $\mathcal{M}$ be an L_1-structure. We say that $\mathcal{M}$ is ultrahomogeneous if every isomorphism between finite substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.

Here are two theorems of Fraïssé that answered the first question.

Theorem 1.6. *Let L_1 be a countable first-order signature and let $\mathcal{M}$ be a countable L_1-structure that is ultrahomogeneous. Let $\mathfrak{A} = age(\mathcal{A})$. Then $\mathfrak{A}$ is a nonempty, countable age with the AP.*

Proof. See [6, Theorem 6.1.7]. □

Theorem 1.7. *Let L_1 be a countable first-order signature and let $\mathfrak{A}$ be a nonempty countable age in L_1 with the AP. Then there is a unique L_1-structure $\mathcal{M}$ such that $\mathcal{M}$ is countable, $age(\mathcal{M}) = \mathfrak{A}$ and $\mathcal{M}$ is ultrahomogeneous.*

Proof. See [6, Theorem 6.1.2]. □

Here are two theorems of Fraïssé that answered the second question.

Theorem 1.8. *Let L_1 be a finite first-order signature and let T be a complete L_1-theory that has QE and is ω-categorical. Then age(T) is an a countably infinite age with the AP.*

Proof. See [6, Corollary 6.4.2]. □

Theorem 1.9. *Let L_1 be a finite first-order signature and let $\mathfrak{A}$ be a countably infinite nonempty age with the AP. Then there exists a unique complete theory T that has QE, is ω-categorical, and satisfies age$(T) = \mathfrak{A}$.*

Proof. See [6, Theorems 6.4.1 and 7.3.6]. □

In this thesis we generalize the above theorems to the continuous setting. Let L denote a bounded continuous signature without function symbols. (See [1], [2].) An age in L is defined in the same way as an age in L_1. We will use the following notation.

Notation 1.10. Let $\mathfrak{A}$ be an age. For every $n \in \omega$ we set $S_n^{\mathfrak{A}}$ to be the set of quantifier-free types in the variables $x_1, \ldots, x_n$ which are realized in structures in $\mathfrak{A}$.

The AP in the continuous setting is defined in exactly in the same way as in the classical first-order setting. Here is a variation of the AP.

Definition 1.11. Let $\mathfrak{A}$ be an age. We say that $\mathfrak{A}$ has the near amalgamation property (near-AP) if for every $\epsilon > 0$, $n, l \in \omega$, $\mathcal{M} \in \mathfrak{A}$, $p \in S_{n+l}^{\mathfrak{A}}$, and $\bar{a} = (a_1, \ldots, a_n) \subseteq M$ such that $\bar{a} \models p \restriction n$, there exists $\mathcal{N}$ in $\mathfrak{A}$ such that $\mathcal{M} \subseteq \mathcal{N}$ and $\bar{b} = (b_1, \ldots, b_n) \subseteq N$ such that $\bar{b} \models p$ and

$$\max_{1 \le i \le n} d^{\mathcal{N}}(a_i, b_i) < \epsilon.$$

We note that the AP implies the near-AP. The near-AP is closely related to the notion of strongly ω-qf-near-homogeneous structures, which is defined next:

Definition 1.12. $\mathcal{M}$ is strongly ω-qf-near-homogeneous if for every $n \in \omega$, n-tuples $\bar{a}, \bar{b} \subseteq M$, such that $\mathrm{qftp}_{\mathcal{M}}(\bar{a}) = \mathrm{qftp}\mathcal{M}(\bar{b})$, there exists an automorphism f of $\mathcal{M}$ such that

$$\max_{1 \le i \le n} d^{\mathcal{M}}(a_i, f(b_i)) < \epsilon.$$

If $\mathfrak{A}$ is an age with the near-AP, then we can define a natural metric on its quantifier-free type spaces.

Definition 1.13. Let $\mathfrak{A}$ be an age. For every $n \in \omega$ and p, $q \in S_n^{\mathfrak{A}}$, we define

$$d_n^{\mathfrak{A}}(p, q) = \inf\{\max_{1 \le i \le n} d^{\mathcal{M}}(a_i, b_i) \mid \mathcal{M} \in \mathfrak{A}, \quad \bar{a}, \bar{b} \subseteq M, \quad \mathcal{M} \models p(\bar{a}), \quad \mathcal{M} \models q(\bar{b})\}.$$

Proposition 1.14. *Let $\mathfrak{A}$ be an age with the near-AP. Then for all $n \in \omega$, $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is a metric space.*

Proof. See 6.11. □

In the classical first-order setting the size of the age is measured by its cardinality. In the continuous setting the size of an age is measured by its density.

Definition 1.15. Let $\mathfrak{A}$ be an age with the near-AP.

1. We say that $\mathfrak{A}$ is $d^{\mathfrak{A}}$-compact if for every $n \in \omega$, the metric space $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is compact.

2. We say that $\mathfrak{A}$ is $d^{\mathfrak{A}}$-separable if for every $n \in \omega$, the metric space $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is separable.

3. We say that $\mathfrak{A}$ is $d^{\mathfrak{A}}$-complete if for every $n \in \omega$, the metric space $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is complete.

Definition 1.16. Let $\mathfrak{A}$ be an age. We say that $\mathfrak{A}$ is totally bounded if for all $\epsilon > 0$ there exists $n \geq 1$ such that for all $\mathcal{M} \in \mathfrak{A}$, $\mathcal{M}$ has an ϵ-net of size $\leq n$.

In this thesis we develop a strong analogy between the classical setting and the continuous setting.

- A $d^{\mathfrak{A}}$-compact age in L is the analogue of an age in L_1 with finite number of structures (up to isomorphism) of cardinality $\leq n$ for all $n \in \omega$.

- A $d^{\mathfrak{A}}$-separable age in L is the analogue of an age in L_1 with countable number of structures (up to isomorphism) of cardinality $\leq n$ for all $n \in \omega$.

- An age in L that is totally bounded is the analogue of an age in L_1 with a finite number of structures (up to isomorphism).

On the other hand, $d^{\mathfrak{A}}$-completeness is a smoothness condition. It is the least kind of regularity that an age should have so that we can study it from a model theoretic point of view.

The analogues of Theorem 1.6 and Theorem 1.7, respectively, are the following:

Theorem 1.17 (See Theorem 6.30)**.** *Let L be a countable bounded continuous signature without function symbols. Let $\mathcal{M}$ be a separable L-structure that is ω-qf-near-homogeneous. Set $\mathfrak{A} = age(\mathcal{M})$. Then $\mathfrak{A}$ is a $d^{\mathfrak{A}}$-separable, $d^{\mathfrak{A}}$-complete age with the near-AP.*

Theorem 1.18 (See Theorem 6.28)**.** *Let L be a countable bounded continuous signature without function symbols. Let $\mathfrak{A}$ be a $d^{\mathfrak{A}}$-separable, $d^{\mathfrak{A}}$-complete age with the near-AP. Then there exists a unique separable L-structure $\mathcal{M}$ that is strongly ω-qf-near-homogeneous and satisfies $age(\mathcal{M}) = \mathfrak{A}$.*

The analogues of Theorem 1.8 and Theorem 1.9, respectively are the following:

Theorem 1.19 (See Theorem 7.5)**.** *Let L be a finite bounded continuous signature without function symbols. Let T be a complete L-theory that has QE and is ω-categorical. Then age(T) is a $d^{\mathfrak{A}}$-compact age that has the AP and is not totally bounded.*

Theorem 1.20 (See Theorem 7.6)**.** *Let $\mathfrak{A}$ be an $d^{\mathfrak{A}}$-compact age that has the AP and is not totally bounded. Then there exists a unique complete L-theory T that has QE, is ω-categorical and satisfies age$(T) = \mathfrak{A}$.*

In the first order setting, if a theory T in a finite signature has QE, then T is necessarily ω-categorical. But this is not the case in continuous model theory. (See 7.13) Then, a natural question arises: Can we characterize QE theories in terms of the properties of their ages in the way we did for ω-categorical theories with QE? It turns out that it is possible, as we describe next.

The following definition describes another variation of the AP.

Definition 1.21. Let L be a finite bounded continuous signature without function symbols and $\mathfrak{A}$ an age in L. We say that $\mathfrak{A}$ has the perturbed amalgamation property (PAP) if for every $n \in \omega$, $\epsilon > 0$, $p(\bar{x}) \in S^{\mathfrak{A}}_{n+1}$, there exists $\delta^{\mathfrak{A}}_n(\epsilon, p) > 0$ such that for every $\mathcal{M} \in \mathfrak{A}$, $\bar{a} = (a_1, \ldots, a_n) \subseteq M$ with $\rho_n(p\restriction n, q) < \delta^{\mathfrak{A}}_n(\epsilon, p)$, where $q = \text{qftp}_{\mathcal{M}}(\bar{a})$, there exist $\mathcal{N} \supseteq \mathcal{M}$ in $\mathfrak{A}$, and a_{n+1} in N such that $\rho_{n+1}(p, r) \leq \epsilon$ where $r = \text{qftp}_{\mathcal{N}}(a_1, \ldots, a_{n+1})$.

The following two theorems explore the interplay between QE theories and properties of their ages.

Theorem 1.22 (See Theorem 5.4)**.** *Let L be a finite bounded continuous signature without function symbols. Let $\mathfrak{A}$ be a ρ-compact age with the PAP. Then there exists a unique complete L-theory T that has QE and satisfies age$(T) = \mathfrak{A}$.*

Theorem 1.23 (See Theorem 5.5)**.** *Let L be a finite bounded continuous signature without function symbols. Let T be a complete L-theory that has QE. Then age(T) is a ρ-compact age with the PAP.*

Now we shift our attention to the automorphism groups of ω-categorical structures. The following theorem provides a link between model theory and the study of permutation groups.

Theorem 1.24. *Let L_1 be a countable first-order signature and $\mathcal{M}$ a countably infinite L_1-structure . Set $T = Th(\mathcal{M})$ and $G = Aut(\mathcal{M})$. The following statements are equivalent:*

1. *T is ω-categorical.*
2. *For every $n \geq 1$, G has only finitely many orbits in its induced action on $\mathcal{M}^n$.*

Proof. See [3, p 30]. □

H. D. Macpherson used Theorem 1.24 to prove the following theorem.

Theorem 1.25. *Let L_1 be a countable first-order signature and $\mathcal{M}$ be a countably infinite L_1-structure such that $Th(\mathcal{M})$ is ω-categorical. Set $G = Aut(\mathcal{M})$. Then there exists a dense subgroup $F \leq G$ which is freely generated by countably many elements, where G is equipped with the pointwise convergence topology.*

Proof. See [8, Theorem 3.1]. □

Theorem 1.24 was generalized in the continuous setting by C. Ward Henson as follows.

Theorem 1.26 (See Theorem 4.25)**.** *Let $\mathcal{M}$ be a noncompact separable L-structure. Set $T = Th(\mathcal{M})$ and $G = Aut(\mathcal{M})$. The following statements are equivalent:*

1. *T is an ω-categorical theory.*
2. *For every $\epsilon > 0$, $n \geq 1$ there exist n-tuples $\bar{a}_1 \dots \bar{a}_l \subset M$, for some $l \in \omega$, such that for every n-tuple $\bar{b} \subseteq M$ there exist $1 \leq j \leq l$ and $F \in G$ such that*

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(F(a_{j,i}), b_i) < \epsilon.$$

In this thesis we use Theorem 1.26 to prove the following generalization of Macpherson's Theorem in the continuous setting.

Theorem 1.27 (See Theorem 8.2)**.** *Let $\mathcal{M}$ be a separable L-structure which is strongly ω-homogeneous, noncompact and such that $Th(\mathcal{M})$ is ω-categorical. Set $G = Aut(\mathcal{M})$. Then there exists a dense subgroup $F \leq G$ which is freely generated by countably many elements, where G is equipped with the pointwise convergence topology.*

Chapter 2

Introduction to model theory

In this chapter we give a quick introduction to continuous model theory. Our exposition follows [1] closely.

2.1 Metric structures and signatures

Let (M, d) be a complete bounded metric space. A predicate on M is a uniformly continuous function from M^n into some bounded interval in $\mathbb{R}$, for some $n \geq 1$. A function or operation on M is a uniformly continuous function from M^n into M, for some $n \geq 1$. In each case n is called the arity of the predicate or the function.

A metric structure $\mathcal{M}$ based on (M, d) consists of a family $(R_i \mid i \in I)$ of predicates on M, a family $(F_j \mid j \in J)$ of functions on M, and a family $(a_k \mid k \in K)$ of elements of M. When we introduce a metric structure we may denote it as

$$\mathcal{M} = (M, R_i, F_j, a_k \mid i \in I, j \in J, k \in K).$$

Any of the index sets I, J, K are allowed to be empty. Indeed, they may all be empty, in which case $\mathcal{M}$ is a pure bounded metric space.

The key restrictions on metric structures are: the metric space is complete and bounded, each predicate takes its values in a bounded interval of reals, and the functions and predicates are uniformly continuous. All of these restrictions play a role in making the theory work smoothly.

To each metric structure we associate a signature L as follows. To each predicate R of $\mathcal{M}$ we associate a predicate symbol P and an integer $a(P)$ which is the arity of R; we denote R by $P^{\mathcal{M}}$. To each function F of $\mathcal{M}$ we associate a function symbol f and an integer $a(f)$ which is the arity of F; we denote F by $f^{\mathcal{M}}$. Finally, to each distinguished element a of M we associate a constant symbol c; we denote a by $c^{\mathcal{M}}$. So, a signature L gives sets of predicate, function and constant symbols, and associates to each predicate and function symbol its arity. In that respect, L is identical to a signature of first-order model theory. In addition, a signature for metric structures must satisfy more: for each each predicate symbol P, it must provide a closed bounded interval I_P of real numbers and a modulus of uniform continuity Δ_P. These should satisfy the requirements that $P^{\mathcal{M}}$ takes its values in I_P and that Δ_P is a modulus of uniform continuity for $P^{\mathcal{M}}$. In addition, for each function symbol f, L must provide a modulus of uniform continuity Δ_f for $f^{\mathcal{M}}$. Finally, L must provide a non-negative real number D_L which is a bound on the diameter of the complete metric space (M, d) on which $\mathcal{M}$ is based. We sometimes denote the metric d given by $\mathcal{M}$ as $d^{\mathcal{M}}$; this would be consistent with our notation for the interpretation in $\mathcal{M}$ of the nonlogical symbols of L. However, we also find it convenient often to use the same notation d for the logical symbol representing the metric as well as for its interpretation in $\mathcal{M}$; this is consistent with usual mathematical practice and with the handling of the symbol $=$ in first-order logic.

When these requirements are all met and when the predicate, function and constant symbols of L correspond exactly to the predicates, functions and distinguished elements of which $\mathcal{M}$ consists, then we say that $\mathcal{M}$ is an L-structure.

Basic concepts such as embedding and isomorphism have natural definitions for metric structures:

Definition 2.1. Let L be a signature for metric structures and suppose that $\mathcal{M}$ and $\mathcal{N}$ are L-structures.

An embedding from $\mathcal{M}$ into $\mathcal{N}$ is a metric space isometry

$$T : (M, d^{\mathcal{M}}) \to (N, d^{\mathcal{N}})$$

that commutes with the interpretations of the predicates, function and constant symbols of L in the following sense:

Whenever P is an n-ary predicate symbol of L and $a_1, \dots, a_n \in M$ we have

$$P^{\mathcal{N}}(T(a_1), \dots, T(a_n)) = P^{\mathcal{M}}(a_1, \dots, a_n);$$

whenever f is an n-ary function symbol of L and $a_1, \dots, a_n \in M$, we have

$$f^{\mathcal{N}}(T(a_1), \dots, T(a_n)) = T(f^{\mathcal{M}}(a_1, \dots, a_n));$$

and whenever c is a constant symbol of L we have

$$c^{\mathcal{N}} = T(c^{\mathcal{M}}).$$

An isomorphism is a surjective embedding. We say that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic, and write $\mathcal{M} \cong \mathcal{N}$, if there exists an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. An automorphism of $\mathcal{M}$ is an isomorphism between $\mathcal{M}$ and itself.

We say $\mathcal{M}$ is a substructure of $\mathcal{N}$ (and we write $\mathcal{M} \subseteq \mathcal{N}$) if $M \subseteq N$ and the inclusion map from M into N is an embedding of $\mathcal{M}$ into $\mathcal{N}$.

2.2 Formulas and their interpretations

Fix a signature L for metric structures, as described above. We assume for simplicity of notation that $D_L = 1$ and that $I_P = [0, 1]$ for every predicate symbol P.

Symbols of L

Among the symbols of L are the predicate, function, and constant symbols; these will be referred to as the nonlogical symbols of L and the remaining ones will be called the logical symbols of L. Among the logical symbols is a symbol d for the metric on the underlying metric space of an L structure; this is treated formally as equivalent to a predicate symbol of arity 2. The logical symbols also include an infinite set V_L of variables; usually we take V_L to be countable, but there are situations in which it is useful to permit a larger number of variables. The remaining logical symbols consist of a symbol for each continuous function $u : [0, 1]^n \to [0, 1]$ of finitely many variables $n \geq 1$ (these play the roles of connectives) and the symbols sup and inf, which play the role of quantifiers in this logic.

The cardinality of L, denoted $\mathrm{card}(L)$, is the smallest infinite cardinal number $\geq$ the number of nonlogical symbols of L.

Terms of L

Terms are formed inductively, exactly as in the first-order logic. Each variable and constant symbol is an L-term. If f is an n-ary function symbol and $t_1, \ldots, t_n$ are L-terms, then $f(t_1, \ldots, t_n)$ is an L-term. All L-terms are constructed in this way.

Atomic formulas of L

The atomic formulas of L are the expressions of the form $P(t_1, \ldots, t_n)$ in which P is an n-ary predicate symbol of L and $t_1, \ldots, t_n$ are L-terms, as well as $d(t_1, t_2)$ in which t_1 and t_2 are L-terms.

Note that the logical symbol d for the metric is treated formally as a binary predicate symbol, exactly analogous to how the equality symbol $=$ is treated in first order logic.

Formulas of L

Formulas are also constructed inductively and the basic structure of the induction is similar to the corresponding definition in first-order logic. Continuous functions play the role of connectives and sup and inf are used formally in the way that quantifiers are used in the first-order logic. The precise definition is as follows:

Definition 2.2. The class of L-formulas is the smallest class of expressions satisfying the following requirements:

1. Atomic formulas of L are L-formulas.

2. If $u : [0, 1]^n \to [0, 1]$ is continuous and $\varphi_1, \ldots, \varphi_n$ are L-formulas, then $u(\varphi_1, \ldots, \varphi_n)$ is an L-formula.

3. If φ is an L-structure and x is a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are L-formulas.

An L formula is quantifier free if it is generated inductively from atomic formulas without using the last clause, namely neither $\sup_x$ nor $\inf_x$ are used.

Many syntactic notions from first-order logic can be carried over word for word into this setting. We will assume that this has been done by the reader for many such concepts, including subformula and syntactic substitution of a term for a variable, or formula for a subformula, and so forth.

Free and bound occurrences of variables in L-formulas are defined in a way similar to how this is done in first-order logic. Namely, an occurrence of the variable x is bound if lies within a subformula of the form $\sup_x \varphi$ or $\inf_x \varphi$, and otherwise it is free.

An L-sentence is an L-formula that has no free variables.

When t is a term and the variables occurring in it are among the the variables $x_1, \ldots, x_n$ (which we always take to be distinct in this context), we indicate this by writing t as $t(x_1, \ldots, x_n)$.

Similarly, we write an L-formula as $\varphi(x_1, \ldots, x_n)$ to indicate that its free variables are among $x_1, \ldots, x_n$.

Prestructures

It is common in mathematics to construct a metric space as the quotient of a pseudometric space or as the completion of such a quotient, and the same is true of metric structures. For that reason we need to consider what we call prestructures and to develop the semantics of continuous logic for them.

As above, we take L to be a fixed signature for metric structures. Let (M_0, d_0) be a pseudometric space, satisfying that its diameter is $\leq D_L$. (That is, $d_0(x, y) \leq D_L$ for all $x, y \in M_0$.) An L-prestructure $\mathcal{M}_0$ based on (M_0, d_0) is a structure consisting of the following data:

1. for each predicate symbol P of L (of arity n) a function $P^{\mathcal{M}_0}$ from M_0^n into I_P that has Δ_P as a modulus of uniform continuity;

2. for each function symbol f of L (of arity n) a function $f^{\mathcal{M}_0}$ from M_0^n into M_0 that has Δ_f as a modulus of uniform continuity; and

3. for each constant symbol c of L an element $c^{\mathcal{M}_0}$ of M_0.

Definition 2.3. Let $\mathcal{M}_0$ be an L-prestructure. We say that $\mathcal{M}_0$ is a metric L-prestructure if $\mathcal{M}_0$ is based on a metric space (M_0, d_0).

Given an L-prestructure $\mathcal{M}_0$, we may form its quotient prestructure as follows. Let (M, d) be the quotient metric space induced by (M_0, d) with quotient map $\pi : \mathcal{M}_0 \to \mathcal{M}$. Then

1. for each predicate symbol P of L (of arity n) define $P^{\mathcal{M}}$ from M^n into I_P by setting $P^{\mathcal{M}}(\pi(x_1), \ldots, \pi(x_n)) = P^{\mathcal{M}_0}(x_1, \ldots, x_n)$ for each $x_1, \ldots, x_n \in M_0$;

2. for each function symbol f of L (of arity n) define $f^{\mathcal{M}}$ form M^n into M by setting $f^{\mathcal{M}}(\pi(x_1), \ldots, \pi(x_n)) = \pi(f^{\mathcal{M}_0}(x_1, \ldots, x_n))$ for each $x_1, \ldots, x_n \in M_0$;

3. for each constant symbol c of L define $c^{\mathcal{M}} = \pi(c^{\mathcal{M}_0})$.

It is obvious that (M, d) has the same diameter as (M_0, d_0). Also, as noted in [1, page 11], for each predicate symbol P and each function symbol f of L, the predicate $P^{\mathcal{M}}$ is well defined and has Δ_P as a modulus of uniform continuity and the function $f^{\mathcal{M}}$ is well defined and has Δ_f as a modulus of uniform continuity. In other words, this defines an L-prestructure (which we denote by $\mathcal{M}$) based on the (possibly not complete) metric space (M, d).

Finally we may define an L-structure $\mathcal{N}$ by taking a completion of $\mathcal{M}$. This is based on a complete metric space (N, d) that is a completion of (M, d) and its additional structure is defined in the following natural way (made possible by the fact that the predicates and functions given by $\mathcal{M}$ are uniformly continuous):

1. for each predicate symbol P of L (of arity n) define $P^{\mathcal{M}}$ from N^n into I_P to be the unique such function that extends $P^{\mathcal{M}}$ and is continuous;

2. for each function symbol f of L (of arity n) define $f^{\mathcal{N}}$ from N^n into N to be the unique such function that extends $f^{\mathcal{M}}$ and is continuous;

3. for each constant symbol c of L define $c^{\mathcal{N}} = c^{\mathcal{M}}$.

It is obvious that (N, d) has the same diameter as (M, d). Also, as noted in [1, page 8], for each predicate symbol P and each function symbol f of L, the predicate $P^{\mathcal{N}}$ has Δ_P as a modulus of uniform continuity, and the function $f^{\mathcal{N}}$ has Δ_f as a modulus of uniform continuity. In other words, $\mathcal{N}$ is an L-structure.

Semantics

Let $\mathcal{M}$ be any L-prestructure, with (M, d) as its underlying pseudometric space, and let A be any subset of M. We extend L to a signature $L(A)$ by adding a new constant symbol $c(a)$ to L for each element $a \in A$. We extend the interpretation given by $\mathcal{M}$ in a canonical way, by taking the interpretation of $c(a)$ to be equal to a itself for each $a \in A$. We call $c(a)$ the name of a in $L(A)$. Indeed, we will often write a instead of $c(a)$ where no confusion can result from doing so.

Given an $L(M)$-term $t(x_1, \ldots, x_n)$ we define, exactly as in first-order logic, the interpretation of t in $\mathcal{M}$, which is a function $t^{\mathcal{M}} : M^n \to M$.

We now come to the key definition in continuous logic for metric structures, in which the semantics of this logic is defined. For each $L(M)$-sentence σ, we define the value of σ in $\mathcal{M}$. This value is a real number in the interval $[0, 1]$ and it is denoted by $\sigma^{\mathcal{M}}$. The definition is by induction on formulas. Note that in the definition all terms mentioned are $L(M)$-terms in which no variables occur.

Definition 2.4.

1. $(d(t_1, t_2))^{\mathcal{M}} = d^{\mathcal{M}}(t_1^{\mathcal{M}}, t_2^{\mathcal{M}})$ for any t_1, t_2;

2. for any n-ary predicate symbol P of L and any $t_1, \ldots, t_n$,

$$(P(t_1, \ldots, t_n))^{\mathcal{M}} = P^{\mathcal{M}}(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}});$$

3. for any $L(M)$-sentences $\sigma_1, \ldots, \sigma_n$ and any continuous function $u : [0, 1]^n \to [0, 1]$,

$$(u(\sigma_1, \ldots, \sigma_n))^{\mathcal{M}} = u(\sigma_1^{\mathcal{M}}, \ldots, \sigma_n^{\mathcal{M}});$$

4. for any $L(M)$-formula $\varphi(x)$,

$$(\sup_x \varphi(x))^{\mathcal{M}}$$

is the supremum in $[0, 1]$ of the set $\{\varphi^{\mathcal{M}}(a) \mid a \in M\}$;

5. For any $L(M)$-formula $\varphi(x)$,

$$(\inf_x \varphi(x))^{\mathcal{M}}$$

is the infimum in $[0, 1]$ of the set $\{\varphi^{\mathcal{M}}(a) \mid a \in M\}$.

Definition 2.5. Given an $L(M)$-formula $\varphi(x_1,\ldots,x_n)$ we let $\varphi^{\mathcal{M}}$ denote the function from M^n to $[0,1]$ defined by

$$\varphi^{\mathcal{M}}(a_1,\ldots a_n) = (\varphi(a_1,\ldots,a_n))^{\mathcal{M}}.$$

Definition 2.6. Let φ be an L-formula.

1. We say that φ is a $\forall$ L-formula if for every $\epsilon > 0$, there exists a formula σ of the form $\sup_{\bar{x}} \psi$, where ψ is a quantifier-free formula, such that for every L-structure $\mathcal{M}$ and $a \in M$, $|\varphi^{\mathcal{M}}(a) - \sigma^{\mathcal{M}}(a)| \leq \epsilon$.

2. We say that φ is a $\forall\exists$ L-formula if for every $\epsilon > 0$, there exists a formula σ of the form $\sup_{\bar{x}} \inf_{\bar{y}} \psi$, where ψ is a quantifier-free formula, such that for every L-structure $\mathcal{M}$ and $a \in M$, $|\varphi^{\mathcal{M}}(a) - \sigma^{\mathcal{M}}(a)| \leq \epsilon$.

A key fact about formulas in continuous logic is that they define uniformly continuous functions. Indeed, the modulus of uniform continuity for a predicate does not depend on $\mathcal{M}$ but only on the data given by the signature L.

Theorem 2.7. *Let $t(x_1,\ldots,x_n)$ be an L-term and $\varphi(x_1,\ldots,x_n)$ an L-formula. Then there exist functions Δ_t and Δ_φ from $(0,1]$ to $(0,1]$ such that for any L-prestructure $\mathcal{M}$, Δ_t is a modulus of uniform continuity for the function $t^{\mathcal{M}} : M^n \to M$ and Δ_φ is a modulus of uniform continuity for the predicate $\varphi^{\mathcal{M}} : M^n \to [0,1]$.*

Proof. See [1, Theorem 3.5]. □

Theorem 2.8. *Let $\mathcal{M}_0$ be an L-prestructure with underlying pseudometric space (M_0, d_0); let $\mathcal{M}$ be its quotient L-prestructure with quotient map $\pi : \mathcal{M}_0 \to \mathcal{M}$ and let $\mathcal{N}$ be the L-structure that results from completing $\mathcal{M}$. Let $t(x_1,\ldots,x_n)$ be any L-term and $\varphi(x_1,\ldots,x_n)$ be any L-formula. Then:*

1. $t^{\mathcal{M}}(\pi(a_1),\ldots,\pi(a_n)) = t^{\mathcal{M}_0}(a_1,\ldots,a_n)$ *for all* $a_1\ldots,a_n \in M_0$;

2. $t^{\mathcal{N}}(b_1,\ldots,b_n) = t^{\mathcal{M}}(b_1,\ldots,b_n)$ *for all* $b_1,\ldots,b_n \in M$;

3. $\varphi^{\mathcal{M}}(\pi(a_1),\ldots,\pi(a_n)) = \varphi^{\mathcal{M}_0}(a_1,\ldots,a_n)$ *for all* $a_1,\ldots,a_n \in M_0$;

4. $\varphi^{\mathcal{N}}(b_1,\ldots,b_n) = \varphi^{\mathcal{M}}(b_1,\ldots,b_n)$ *for all* $b_1,\ldots,b_n \in M$.

Proof. See [1, Theorem 3.7]. □

Conditions of L

An L condition E is a formal expression of the form $\varphi = 0$, where φ is an L-formula. We call E closed if φ is a sentence. If $x_1,\ldots,x_n$ are distinct variables, we write an L-condition as $E(x_1,\ldots,x_n)$ to indicate that it has the form $\varphi(x_1,\ldots,x_n) = 0$ (in other words, that the free variables of E are among $x_1,\ldots,x_n$).

Let E_i be the $L(M)$ condition $\varphi_i(x_1, \ldots, x_n) = 0$, for $i = 1, 2$. We say that E_1, E_2 are logically equivalent if for every L-structure $\mathcal{M}$ and every $a_1, \ldots, a_n$ we have

$$\mathcal{M} \models E_1[a_1, \ldots, a_n] \quad \text{iff} \quad \mathcal{M} \models E_2[a_1, \ldots, a_n].$$

If E is the $L(M)$-condition $\varphi(x_1, \ldots, x_n) = 0$ and $a_1, \ldots, a_n$ are in M, we say E is true of $a_1, \ldots, a_n$ in M and write $\mathcal{M} \models E[a_1, \ldots, a_n]$ if $\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = 0$.

If E is the $L(M)$-condition $\varphi(x_1, \ldots, x_n) = 0$ where φ is a $\forall$ L-formula, then we say that E is a $\forall$ L-condition.

If E is the $L(M)$-condition $\varphi(x_1, \ldots, x_n) = 0$ where φ is a $\forall\exists$ L-formula, then we say that E is a $\forall\exists$ L-condition.

Definition 2.9. We define a binary function $\dot{-} : \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ by

$$x \dot{-} y = \begin{cases} (x - y) & \text{if } x \geq y \\ 0 & \text{otherwise.} \end{cases}$$

An important type of condition which appears frequently in this thesis is

$$\min(\phi \dot{-} \psi, \sigma \dot{-} \epsilon) = 0 \quad \text{which means} \quad \text{if } \psi < \phi \text{ then } \sigma \leq \epsilon.$$

2.3 Model theoretic concepts

Fix a signature L for metric structures. In this section we introduce several of the most fundamental model theoretic concepts and discuss some of their basic properties.

Definition 2.10. A theory in L is a set of closed L-conditions. If T is a theory in L and $\mathcal{M}$ an L-structure, we say that $\mathcal{M}$ is a model of T and write $\mathcal{M} \models T$ if $\mathcal{M} \models E$ for every condition E in T. We write $\mathrm{Mod}_L(T)$ for the collection of all L-structures that are models of T. (If L is clear from the context, then we write simply $\mathrm{Mod}(T)$.)

Let T, Σ be L-theories. We say that T is equivalent to Σ if $\mathrm{Mod}(T) = \mathrm{Mod}(\Sigma)$.

Also, we say that a theory T is satisfiable if there exists an L-structure $\mathcal{M}$ such that $\mathcal{M} \models T$.

If T is an L-theory and E is a closed L-condition, we say that E is a logical consequence of T and write $T \models E$ if $\mathcal{M} \models E$ holds for every model $\mathcal{M}$ of T. We set $\mathrm{Con}(T)$ to be the set of all closed L-conditions that are logical consequences of T.

Let T be an L-theory and Σ a set of closed L-conditions. We say that Σ axiomatizes T if $\mathrm{Con}(\Sigma) = \mathrm{Con}(T)$.

If $\mathcal{M}$ is an L-structure, the theory of $\mathcal{M}$, denoted $\mathrm{Th}(\mathcal{M})$, is the set of closed L-conditions that are true in $\mathcal{M}$. If $\mathrm{Con}(T)$ is a theory of this form, then T will be called complete.

We say that an L-theory T is a $\forall$ L-theory if there is a set of closed $\forall$ L-conditions Σ such that $\mathrm{Con}(T) = \mathrm{Con}(\Sigma)$.

We say that an L-theory T is a $\forall\exists$ L-theory if there exists a set of closed $\forall\exists$ L-conditions Σ such that $\mathrm{Con}(T) = \mathrm{Con}(\Sigma)$.

Definition 2.11. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are L-structures.

1. We say that $\mathcal{M}$, $\mathcal{N}$ are elementary equivalent, and write $\mathcal{M} \equiv \mathcal{N}$, if $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$ for all L-sentences σ. Equivalently, this holds if $Th(\mathcal{M}) = Th(\mathcal{N})$.

2. If $\mathcal{M} \subseteq \mathcal{N}$ we say that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, and write $\mathcal{M} \preceq \mathcal{N}$, if whenever $\varphi(x_1, \ldots, x_n)$ is an L-formula and $a_1, \ldots, a_n$ are elements of M, we have

$$\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \varphi^{\mathcal{N}}(a_1, \ldots, a_n).$$

   In this case, we also say that $\mathcal{N}$ is an elementary extension of $\mathcal{M}$.

3. A function F from a subset of M into N is an elementary map from $\mathcal{M}$ into $\mathcal{N}$ if whenever $\varphi(x_1, \ldots, x_n)$ is an L-formula and $a_1, \ldots, a_n$ are elements of the domain of F, we have

$$\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \varphi^{\mathcal{N}}(F(a_1), \ldots, F(a_n)).$$

4. An elementary embedding of $\mathcal{M}$ into $\mathcal{N}$ is a function from all of M into N that is an elementary map from $\mathcal{M}$ into $\mathcal{N}$.

Remark 2.12. 1. Every elementary map from one metric structure into another is distance preserving.

2. The collection of elementary maps is closed under composition and formation of the inverse.

3. Every isomorphism between metric structures is an elementary embedding.

2.4 Model theoretic theorems

Here we state some of the most important theorems of continuous model theory which are generalizations of the corresponding theorems of first-order model theory.

The following theorem is called the Compactness Theorem for continuous model theory.

Theorem 2.13. *Let T be an L-theory. If every finite subset of T has a model then T has a model.*

Proof. See [1, Theorem 5.8]. $\square$

If Λ is a linearly ordered set, a Λ-chain of L-structures is a family of L-structures $(\mathcal{M}_\lambda \mid \lambda \in \Lambda)$ such that $\mathcal{M}_\lambda \subseteq \mathcal{M}_\eta$ for $\lambda < \eta$. If this holds, we can define the union of $(\mathcal{M}_\lambda \mid \lambda \in \Lambda)$ as an L-prestructure in an obvious way. This union is based on a metric space, but it may not be complete. After taking the completion we get an L-structure that we will refer to as the union of the chain and that we will denote by $\bigcup_{\lambda \in \Lambda} \mathcal{M}_\lambda$.

Definition 2.14. A chain of structures $(\mathcal{M}_\lambda \mid \lambda \in \lambda)$ is called an elementary chain if $\mathcal{M}_\lambda \preceq \mathcal{M}_\eta$ for $\lambda < \eta$.

Proposition 2.15. *If $(\mathcal{M}_\lambda \mid \lambda \in \Lambda)$ is an elementary chain and $\lambda \in \Lambda$, then $\mathcal{M}_\lambda \preceq \bigcup_{\lambda \in \Lambda} \mathcal{M}_\lambda$.*

Proof. See [1, Proposition 7.2]. $\square$

Recall that the density character of the of a topological space is the smallest cardinality of a dense subset of the space.

Proposition 2.16 (Downward Löwenheim-Skolem Theorem)**.** *Let κ be an infinite cardinal and assume that that card$(L) \leq \kappa$. Let $\mathcal{M}$ be an L-structure and suppose $A \subseteq M$ has density$(A) \leq \kappa$. Then there exists a substructure $\mathcal{N}$ of $\mathcal{M}$ such that*

1. $\mathcal{N} \preceq \mathcal{M}$;
2. $A \subseteq N \subseteq M$;
3. density$(N) \leq \kappa$.

Proof. See [1, Proposition 7.3]. $\square$

2.5 Spaces of types

In this section we consider a fixed signature L for metric structures and a fixed L-theory T. Until further notice we assume that T is a complete theory.

Suppose that $\mathcal{M}$ is a model of T and $A \subseteq M$. Denote the $L(A)$-structure $(\mathcal{M}, a)_{a\in A}$ by $\mathcal{M}_A$, and set T_A to be the $L(A)$-theory of $\mathcal{M}_A$. Note that any model of T_A is isomorphic to a structure of the form $(\mathcal{N}, a)_{a\in A}$ where $\mathcal{N}$ is a model of T.

Definition 2.17. Let T_A be as above, β an ordinal, and $(x_i)_{i\in\beta}$ distinct variables.

A set p of $L(A)$-conditions with all free variables among $(x_i)_{i\in\beta}$ is called a β-type over A if there exists a model $(\mathcal{M}, a)_{a\in A}$ of T_A and elements $(e_i)_{i\in\beta}$ of M such that p is the set of all $L(A)$-conditions $E(x_{i_1}, \dots, x_{i_n})$ for which $\mathcal{M}_A \models E[e_{i_1}, \dots, e_{i_n}]$, where $n \in \min(\omega, \beta)$ and $i_1, \dots, i_n \in \beta$.

When this relationship holds, we denote p by $\mathrm{tp}_{\mathcal{M}}((e_i)_{i\in\beta}/A)$ and we say that $(e_i)_{i\in\beta}$ realizes p in $\mathcal{M}$. (The subscript $\mathcal{M}$ will be omitted if doing so causes no confusion; A will be omitted if it is empty.)

The collection of all such β-types over A is denoted by $S_\beta(T_A)$ or simply by $S_\beta(A)$ if the context makes the theory T_A clear.

Now we introduce the logic topology on types. Fix T_A as above. If $\varphi(x_{i_1}, \dots, x_{i_n})$ is an L-formula, for some $i_1, \dots, i_n \in \beta$, and $\epsilon > 0$, we let $[\varphi < \epsilon]$ denote the set

$$\{q \in S_\beta(T_A) \mid \text{for some } 0 \leq \delta < \epsilon \text{ the condition } \varphi \dot{-} \delta = 0 \text{ is in } q\}.$$

Definition 2.18. The logic topology on $S_\beta(T_A)$ is defined as follows. If p is is $S_\beta(T_A)$, the basic open neighborhoods of p are the sets of the form $[\varphi < \epsilon]$ for which the condition $\varphi = 0$ is in p and $\epsilon > 0$.

Definition 2.19. Let β be an ordinal, and let $(x_i)_{i\in\beta}$ be distinct variables.

A set p of quantifier-free $L(A)$-conditions with all free variables among $(x_i)_{i\in\beta}$ is called a quantifier-free β-type over A if there exists a model $(\mathcal{M}, a)_{a\in A}$ of T_A and elements $(e_i)_{i\in\beta}$ of M such that p is the set of all quantifier-free $L(A)$-conditions $E(x_{i_1}, \dots, x_{i_n})$ for which $\mathcal{M}_A \models E[e_{i_1}, \dots, e_{i_n}]$, where $n \in \min(\omega, \beta)$ and $i_1, \dots, i_n \in \beta$.

When this relationship holds, we denote p by $\mathrm{qftp}_{\mathcal{M}}((e_i)_{i\in\beta}/A)$ and we say that $(e_i)_{i\in\beta}$ realizes p in $\mathcal{M}$. (A will be omitted if it is empty.)

Let p be a quantifier-free β-type over A. If $\gamma \leq \beta$, then we write $p \restriction \gamma$ to denote the set formed by the set of all $L(A)$-conditions whose free variables are among $(x_i)_{i\in\gamma}$.

If $p((x_i)_{i\in\beta})$ is a quantifier-free β-type over A and $\phi(x_{i_1}, \dots, x_{i_n})$ an $L(A)$-formula where $n \in \min(\omega, \beta)$ and $i_1, \dots, i_n \in \beta$. We write ϕ^p to denote the unique real number r for which $|\phi - r| = 0$ is in p. Equivalently, ϕ^p is defined to be the value $\phi(\bar{a})$ when $\bar{a}$ is any realization of p in an L-structure.

Proposition 2.20. *For any $n \geq 1$, $S_n(T_A)$ is compact and Hausdorff with respect to the logic topology.*

Proof. See [1, Proposition 8.6]. □

Now we introduce the d-metric on types. Let T_A be as above. For each $n \geq 1$ we define a natural metric on $S_n(T_A)$; it is induced as a quotient of the given metric d on M^n, where $(\mathcal{M}, a)_{a \in A}$ is a suitable model of T_A, so we also denote this metric on types by d.

To define the metric, let $\mathcal{M}_A = (\mathcal{M}, a)_{a \in A}$ be any model of T_A in which each type in $S_n(T_A)$ is realized, for each $n \geq 1$. Let (M, d) be the underlying metric space of $\mathcal{M}$. For $p, q \in S_n(T_A)$ we define $d(p, q)$ to be

$$\inf\{\max_{1 \leq j \leq n} d(b_j, c_j) \mid \mathcal{M}_A \models p[b_1, \ldots, b_n], \quad \mathcal{M}_A \models q[c_1, \ldots, c_n]\}.$$

Note that this expression for $d(p, q)$ does not depend on $\mathcal{M}_A$, since $\mathcal{M}_A$ realizes every type of a $2n$-tuple $(b_1, \ldots, b_n, c_1, \ldots, c_n)$ over A. It follows that d is a pseudometric on $S_n(T_A)$. Note that if $p, q \in S_n(A)$, then by the Compactness Theorem and our assumption about $\mathcal{M}_A$, there exist realizations $(b_1, \ldots, b_n)$ of p and $(c_1, \ldots, c_n)$ of q in $\mathcal{M}_A$ such that $\max_j d(b_j, c_j) = d(p, q)$. In particular, if $d(p, q) = 0$, then $p = q$; so d is indeed a metric on $S_n(T_A)$.

Proposition 2.21. *The d-topology is finer than the logic topology on* $S_n(T_A)$*.*

Proof. See [1, Proposition 8.7]. □

Proposition 2.22. *The metric space* $(S_n(T_A), d)$ *is complete.*

Proof. See [1, Proposition 8.8]. □

Chapter 3

Ages

In this chapter we fix a bounded continuous signature L without function symbols.

In Chapter 3 we give the definitions of an age and a ρ-compact age and prove some basic propositions that we will need later.

3.1 Basic definitions

Notation 3.1. We denote by $\mathbb{M}_L$ the class of all L-structures, and by $\mathfrak{M}_L$ the class of all finite L-structures. If the signature L is clear from the context we simply write $\mathbb{M}$ and $\mathfrak{M}$ respectively.

Definition 3.2. Let $\mathfrak{A} \subseteq \mathfrak{M}$. Then $\mathfrak{A}$ is called an age in L if it has the following properties:

- $\mathfrak{A}$ is closed under isomorphism.
- (Hereditary property, HP for short) If $\mathcal{M} \in \mathfrak{A}$ and $\mathcal{N}$ is an L-substructure of $\mathcal{M}$, then $\mathcal{N} \in \mathfrak{A}$.
- (Joint embedding property, JEP for short) If $\mathcal{M}_1, \mathcal{M}_2 \in \mathfrak{A}$, then there exist $\mathcal{M} \in \mathfrak{A}$ and L-embeddings $f_1 : \mathcal{M}_1 \to \mathcal{M}$, $f_2 : \mathcal{M}_2 \to \mathcal{M}$.

Usually L will be clear and we will simply say $\mathfrak{A}$ is an age.

Notation 3.3.

1. Let $\mathcal{M}$ an L-structure. We set $\text{age}(\mathcal{M})$ to be the class of all finite L-structures which are isomorphic to a substructure of $\mathcal{M}$.
2. Let T be an L-theory. We set $FS(T)$ to be the class of all finite L-structures which are isomorphic to a substructure of a model of T.
3. Let $\mathfrak{A} \subseteq \mathfrak{M}$ and α be an ordinal number. We set

$$S_\alpha^{\mathfrak{A}} = \{\text{qftp}_{\mathcal{M}}((a_i)_{i\in\alpha}) \mid \mathcal{M} \in \mathbb{M}, \quad \text{age}(\mathcal{M}) \subseteq \mathfrak{A}, \quad (a_i)_{i\in\alpha} \subseteq M\}.$$

4. Let $\mathcal{M}$ be an L-structure and $A \subseteq M$. We set

$$S_\alpha^{\mathfrak{A}}(A) = \{\mathrm{qftp}_{\mathcal{M}}((a_i)_{i\in\alpha}/A) \mid \mathcal{M} \in \mathbb{M}, \quad \mathrm{age}(\mathcal{M}) \subseteq \mathfrak{A}, \quad (a_i)_{i\in\alpha} \subseteq M\}.$$

5. We set $\mathbb{M}^{\mathfrak{A}}$ to be the class of all L-structures $\mathcal{M}$ such that $\mathrm{age}(\mathcal{M}) \subseteq \mathfrak{A}$.

We will also use the following notation.

Notation 3.4. Let $\mathcal{M}$ be an L-structure.

1. The diagram of $\mathcal{M}$, denoted by $\mathrm{Diag}(\mathcal{M})$, is the set of quantifier-free closed $L(M)$-conditions that are true in $\mathcal{M}$.

2. The elementary diagram of $\mathcal{M}$, denoted by $\mathrm{EDiag}(\mathcal{M})$, is the set of closed $L(M)$-conditions that are true in $\mathcal{M}$.

The following proposition indicates that ages occur very often.

Proposition 3.5. *1. If $\mathcal{M}$ is an L-structure, then age$(\mathcal{M})$ is an age.*

2. If T is a complete L-theory, then $FS(T)$ is an age.

Proof. (1): Clearly $\mathrm{age}(\mathcal{M})$ has the HP and the JEP.

(2): Clearly $FS(T)$ has the HP. We show that it has the JEP. Let $\mathcal{M}, \mathcal{N}$ in $FS(T)$. Without losing generality we may assume that $M \cap N = \emptyset$. It is enough to show that the $L(M \cup N)$-theory

$$\mathrm{Diag}(\mathcal{M}) \cup \mathrm{Diag}(\mathcal{N}) \cup T$$

has a model. This is an immediate consequence of the compactness theorem and the fact that T is complete. □

Notation 3.6. If T is a complete L-theory we write $\mathrm{age}(T)$ instead of $FS(T)$.

Definition 3.7. (Amalgamation property, AP for short) Let $\mathfrak{A} \subseteq \mathfrak{M}$. We say that $\mathfrak{A}$ has the amalgamation property if for every $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2 \in \mathfrak{A}$ and embeddings $f_1 : \mathcal{M} \to \mathcal{M}_1$, $f_2 : \mathcal{M} \to \mathcal{M}_2$, there are $\mathcal{N}$ in $\mathfrak{A}$ and embeddings $g_1 : \mathcal{M}_1 \to \mathcal{N}$, $g_2 : \mathcal{M}_2 \to \mathcal{N}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Proposition 3.8. *Let $\mathfrak{A}$ be an age. The following statements are equivalent:*

1. $\mathfrak{A}$ has the AP.

2. *For every* $\mathcal{M} \in \mathfrak{A}$, $n, l \in \omega$, $p \in S^{\mathfrak{A}}_{n+l}$, *and* $\bar{a} = (a_1 \dots, a_n) \subseteq M$ *such that* $\bar{a} \models p \restriction n$, *there exist* $\mathcal{N}$ *in* $\mathfrak{A}$ *such that* $\mathcal{M} \subseteq \mathcal{N}$ *and* $a_{n+1} \dots a_{n+l}$ *in* N *such that* $(a_1, \dots a_{n+l}) \models p$.

3. *For every* $\mathcal{M} \in \mathfrak{A}$, $n \in \omega$, $p \in S^{\mathfrak{A}}_{n+1}$, *and* $\bar{a} = (a_1 \dots, a_n) \subseteq M$ *such that* $\bar{a} \models p \restriction n$, *there exist* $\mathcal{N}$ *in* $\mathfrak{A}$ *such that* $\mathcal{M} \subseteq \mathcal{N}$ *and* a_{n+1} *in* N *such that* $(a_1, \dots a_{n+1}) \models p$.

Proof. Straightforward from the definitions. □

3.2 ρ-compact ages

For this section we assume, in addition, that L is finite.

Definition 3.9. For every $n \in \omega$, we define a metric on $S^{\mathfrak{M}}_n$ in the following way. Let $(\varphi_i(\bar{x}))_{1 \leq i \leq k}$ be an enumeration of all atomic L-formulas in the variables $x_1, \dots, x_n$. If $p(\bar{x})$, $q(\bar{x})$ are in $S^{\mathfrak{M}}_n$ then we define

$$\rho_n(p, q) = \max_{1 \leq i \leq k} |\varphi_i(\bar{x})^p - \varphi_i(\bar{x})^q|.$$

Notation 3.10. Let $p \in S^{\mathfrak{M}}_n$ for some $n \in \omega$, and let $(\varphi_i(\bar{x}))_{1 \leq i \leq k}$ be an enumeration of all atomic L-formulas in the free variables $x_1, \dots, x_n$. We set

$$\tau_p(\bar{x}) = \max_{1 \leq i \leq k} |\varphi_i(\bar{x}) - \varphi_i(\bar{x})^p|.$$

Remark 3.11. Let $p \in S^{\mathfrak{M}}_n$ for some $n \in \omega$. Then for every metric L-prestructure $\mathcal{M}$ and n-tuple $\bar{a} \subseteq M$

$$\rho_n(p, \mathrm{qftp}_{\mathcal{M}}(\bar{a})) = \tau^{\mathcal{M}}_p(\bar{a}).$$

Proposition 3.12. *For every* $n \in \omega$, $(S^{\mathfrak{M}}_n, \rho_n)$ *is a metric space.*

Proof. Let $n \in \omega$, and $(\varphi_i)_{1 \leq i \leq k}$ be an enumeration of all atomic L-formulas in the free variables $x_1, \dots, x_n$. We define a map $\Phi : S^{\mathfrak{M}}_n \to [0, 1]^k$ as follows: for every $p \in S^{\mathfrak{M}}_n$

$$p \longrightarrow (\varphi^p_i \mid 1 \leq i \leq k).$$

Let $([0, 1]^k, d)$ denote $[0, 1]^k$ equipped with the max metric. We note that for every $p, q \in S^{\mathfrak{M}}_n$, $\rho_n(p, q) = d(\Phi(p), \Phi(q))$. We deduce that we can identify $(S^{\mathfrak{M}}_n, \rho_n)$ with a metric subspace of $([0, 1]^k, d)$. Since $([0, 1]^k, d)$ is a metric space, we conclude that $(S^{\mathfrak{M}}_n, \rho_n)$ is a metric space. □

Notation 3.13. Let $\mathfrak{A} \subseteq \mathfrak{M}$. For every $n \in \omega$, we write $(S_n^{\mathfrak{A}}, \rho_n)$ to denote the the restriction of the metric space $(S_n^{\mathfrak{M}}, \rho_n)$ to $S_n^{\mathfrak{A}}$.

Definition 3.14. Let $\mathfrak{A} \subseteq \mathfrak{M}$. We say that $\mathfrak{A}$ is ρ-compact if for all $n \in \omega$, $(S_n^{\mathfrak{A}}, \rho_n)$ is a complete metric space.

Remark 3.15. We note that for all $n \in \omega$, $(S_n^{\mathfrak{A}}, \rho_n)$ is a metric subspace of $(S_n^{\mathfrak{M}}, \rho_n)$ which is always totally bounded. Hence $(S_n^{\mathfrak{A}}, \rho_n)$ is compact iff it is closed in $(S_n^{\mathfrak{M}}, \rho_n)$ iff it is a complete metric space.

Proposition 3.16. *Let T be a complete L-theory. Then age(T) is a ρ-compact age.*

Proof. By Proposition 3.5 we deduce that age(T) is an age. The fact that it is ρ-compact is an immediate consequence of the compactness theorem. □

In the following proposition we show that a ρ-compact age is in some sense axiomatizable.

Proposition 3.17. *Let $\mathfrak{A}$ be a ρ-compact age. Then there exists an L-theory Σ such that for every metric L-prestructure $\mathcal{M}$, $\mathcal{M} \models \Sigma$ iff age($\mathcal{M}$) $\subseteq \mathfrak{A}$.*

Proof. For all $n \in \omega$, we have that $(S_n^{\mathfrak{A}}, \rho_n)$ is totally bounded. Therefore, for all $n \in \omega$, $\epsilon > 0$, there exists an ϵ-net, $\{p_{n,\epsilon,1}, \ldots, p_{n,\epsilon,m_{n,\epsilon}}\} \subseteq S_n^{\mathfrak{A}}$. Set Σ to be the set of all closed L-conditions of the form

$$\sup_{\bar{x}} \min_{1 \leq i \leq m_{n,\epsilon}} (\tau_{p_{n,\epsilon,i}}(\bar{x}) \dot{-} \epsilon) = 0,$$

for all $n \in \omega$, $\epsilon > 0$. For every metric L-prestructure $\mathcal{M}$ with age($\mathcal{M}$) $\subseteq \mathfrak{A}$, $\mathcal{M} \models \Sigma$. This proves ($\Leftarrow$). For the direction ($\Rightarrow$), let $\mathcal{M}$ be a metric L-prestructure and assume that age($\mathcal{M}$) $\not\subseteq \mathfrak{A}$. Then there exists a finite L-substructure $\mathcal{A}$ of $\mathcal{M}$ such that $\mathcal{A} \notin \mathfrak{A}$. Let $\bar{a}$ be an n-tuple which enumerates A, for some $n \in \omega$, and set $p = \text{qftp}_{\mathcal{A}}(\bar{a})$. We will use the following notation.

Notation 3.18. Let (X, d) be a metric space and $Y \subseteq X$ with $Y \neq \emptyset$. We define

$$d(x, Y) = \inf\{d(x, y) \mid y \in Y\}.$$

We continue the proof of the Proposition 3.17. Since $(S_n^{\mathfrak{A}}, \rho_n)$ is a compact subspace of $(S_n^{\mathfrak{M}}, \rho_n)$, we deduce that $S_n^{\mathfrak{A}}$ is a closed subset of $(S_n^{\mathfrak{M}}, \rho_n)$. Since $p \notin S_n^{\mathfrak{A}}$, we have $\rho_n(p, S_n^{\mathfrak{A}}) > 0$. Set $\epsilon = \rho_n(p, S_n^{\mathfrak{A}})/2$. Clearly

$$\min_{1 \leq i \leq m_{n,\epsilon}} \rho_n(p_{n,\epsilon,i}, p) \geq 2\epsilon.$$

Therefore

$$\min_{1\leq i\leq m_{n,\epsilon}} \tau^{\mathcal{M}}_{p_{n,\epsilon,i}}(\bar{a}) \geq 2\epsilon,$$

and so

$$\min_{1\leq i\leq m_{n,\epsilon}} (\tau^{\mathcal{M}}_{p_{n,\epsilon,i}}(\bar{a}) \dot{-} \epsilon) \geq \epsilon.$$

Therefore

$$\sup_{\bar{x}} \min_{1\leq i\leq m_{n,\epsilon}} (\tau_{p_{n,\epsilon,i}}(\bar{x}) \dot{-} \epsilon) > 0$$

is true in $\mathcal{M}$. We conclude that $\mathcal{M}$ does not satisfy Σ. □

Notation 3.19. Let $\mathfrak{A}$ be a ρ-compact age. We denote by $\Sigma^{\mathfrak{A}}$ the L-theory Σ constructed in the proof of Proposition 3.17.

Corollary 3.20. *Let $\mathfrak{A}$ be a ρ-compact age. If $\mathcal{M}$ is a metric L-prestructure with age($\mathcal{M}$) $\subseteq \mathfrak{A}$, then age($\overline{\mathcal{M}}$) $\subseteq \mathfrak{A}$.*

Proof. Since age($\mathcal{M}$) $\subseteq \mathfrak{A}$, Proposition 3.17 implies that $\mathcal{M} \models \Sigma^{\mathfrak{A}}$. From Proposition 2.8, $\overline{\mathcal{M}} \models \Sigma^{\mathfrak{A}}$ and so again from Proposition 3.17, age($\overline{\mathcal{M}}$) $\subseteq \mathfrak{A}$. □

Definition 3.21. Let $\mathfrak{A} \subseteq \mathfrak{M}$. The completion of $\mathfrak{A}$ is the class $\mathfrak{B}$ of all finite L-structures $\mathcal{M}$ which satisfy the following condition: $\mathcal{M}$ is in $\mathfrak{B}$, where $M = \{a_1, \dots a_n\}$ for some $n \geq 1$, if $\mathrm{qftp}_{\mathcal{M}}(a_1, \dots, a_n)$ is in the closure of $S^{\mathfrak{A}}_n$ in $(S^{\mathfrak{M}}_n, \rho_n)$.

Remark 3.22. If $\mathfrak{B}$ is the completion of $\mathfrak{A}$ then $\mathfrak{A} \subseteq \mathfrak{B}$.

Proposition 3.23. *Let $\mathfrak{A}$ be an age. If $\mathfrak{B}$ is the completion of $\mathfrak{A}$, then $\mathfrak{B}$ is a ρ-compact age.*

Proof. The fact that $\mathfrak{B}$ is ρ-compact is immediate. To show that $\mathfrak{B}$ is an age, it is enough to show that it has the HP and JEP.

For the HP: Let $\mathcal{M} \in \mathfrak{B}$ and $\mathcal{N} \subseteq \mathcal{M}$. Let $\bar{a} = (a_1, \dots, a_m)$ be an enumeration of M, for some $m \geq 1$, such that for some $n \in \omega$, $\bar{a} \restriction n$ is an enumeration of N. Set $p = \mathrm{qftp}_{\mathcal{M}}(\bar{a})$.Since $p \in S^{\mathfrak{B}}_m$, there exists a sequence $(p_i \mid i \in \omega)$ in $S^{\mathfrak{A}}_m$ such that $\lim_{i\to\infty} \rho_m(p, p_i) = 0$. Clearly,

$$\lim_{i\to\infty} \rho_n(p\restriction n, p_i\restriction n) = 0 \tag{3.1}$$

For all $i \in \omega$, $p_i\restriction n \in S^{\mathfrak{A}}_n$ since $\mathfrak{A}$ has the HP property. By (3.1), and since $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B}$ is ρ-compact,we conclude that $p\restriction n \in S^{\mathfrak{B}}_n$ which in turns implies that $\mathcal{N} \in \mathfrak{B}$.

For the JEP: Let $\mathcal{A}, \mathcal{B} \in \mathfrak{B}$. Let $\bar{a}, \bar{b}$ be enumerations of A, B respectively. Set $p = \mathrm{qftp}_{\mathcal{A}}(\bar{a}), q = \mathrm{qftp}_{\mathcal{B}}(\bar{b})$. There exist sequences $(p_i \mid i \in \omega)$ in $S^{\mathfrak{A}}_{n_1}$, $(q_i \mid i \in \omega)$ in $S^{\mathfrak{A}}_{n_2}$, for some $n_1, n_2 \in \omega$ such that

$$\lim_{i \to \infty} \rho_{n_1}(p, p_i) = 0 \quad \text{and} \quad \lim_{i \to \infty} \rho_{n_2}(q, q_i) = 0.$$

Since $\mathfrak{A}$ has the JEP, there exists $l \in \omega$ such that for every $i \in \omega$ there exists $r_i \in S^{\mathfrak{A}}_l$ such that if $\bar{c}_i$ is a realization of r_i then there exist $1 \leq j_1, \ldots, j_{n_1} \leq l$ and $1 \leq k_1, \ldots, k_{n_2} \leq l$ such that

$$(c_{i,j_1}, \ldots, c_{i,j_{n_1}}) \models p_i \quad \text{and} \quad (c_{i,k_1}, \ldots, c_{i,k_{n_2}}) \models q_i$$

respectively. Since for all $i \in \omega$, $r_i \in S^{\mathfrak{A}}_l$, $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B}$ is ρ-compact we deduce that the sequence $(r_i \mid i \in \omega)$ has a limit point r in $S^{\mathfrak{B}}_l$. Without losing generality (we may pass to a subsequence of $(r_i \mid i \in \omega)$), we may assume that

$$\lim_{i \to \infty} \rho_l(r, r_i) = 0.$$

By applying the pigeon-hole principle, there is a subsequence $\{r_{i_m} \mid m \in \omega\}$ of $\{r_i \mid i \in \omega\}$, and finite sequences of natural numbers $1 \leq j_1, \ldots, j_{n_1} \leq l$, $1 \leq k_1, \ldots, k_{n_2} \leq l$ such that for every $m \in \omega$

$$(c_{i_m,j_1}, \ldots, c_{i_m,j_{n_1}}) \models p_{i_m} \quad \text{and} \quad (c_{i_m,k_1}, \ldots, c_{i_m,k_{n_2}}) \models q_{i_m}.$$

Since $r \in S^{\mathfrak{B}}_l$, there is a finite structure $\mathcal{M}$ in $\mathfrak{B}$ and $\bar{c} \subseteq M$ such that $\bar{c} \models r$. From the above we deduce that

$$(c_{j_1}, \ldots, c_{j_{n_1}}) \models p \quad \text{and} \quad (c_{k_1}, \ldots, c_{k_{n_2}}) \models q.$$

We conclude that we can embed both $\mathcal{A}$ and $\mathcal{B}$ into $\mathcal{M}$, and therefore $\mathfrak{B}$ has the JEP.

□

Chapter 4

Basic model theory

In this chapter we fix a bounded continuous signature L without function symbols.

In Chapter 4 we present the basic model theoretic tools that we will use later in this document. In Sections 4.1 and 4.5 we follow [1] closely. In Sections 4.2, 4.3, and 4.4 we prove straightforward generalizations of analogous results in the first-order setting, and we follow [6] closely.

4.1 Quantifier elimination

Definition 4.1. Let T be an L-theory.

1. An L-formula $\varphi(x_1, \ldots, x_n)$ is approximable in T by quantifier-free formulas if for every $\epsilon > 0$ there is a quantifier-free L-formula $\psi(x_1, \ldots, x_n)$ such that for all $\mathcal{M} \models T$ and all $a_1, \ldots, a_n$ in M, one has

$$|\varphi^{\mathcal{M}}(a_1, \ldots, a_n) - \psi^{\mathcal{M}}(a_1, \ldots, a_n)| \leq \epsilon.$$

2. We say that T has quantifier elimination if every L-formula is approximable in T by quantifier-free formulas. In this case we also say that T has QE.

Proposition 4.2. *Let T be an L-theory and $\varphi(x_1, \ldots, x_n)$ be an L-formula. The following statements are equivalent.*

1. *φ is approximable in T by quantifier-free formulas.*
2. *Whenever we are given*
   - *models $\mathcal{M}$ and $\mathcal{N}$ of T;*
   - *substructures $\mathcal{M}_0 \subseteq \mathcal{M}$ and $\mathcal{N}_0 \subseteq \mathcal{N}$;*
   - *an isomorphism Φ from $\mathcal{M}_0$ onto $\mathcal{N}_0$;*
   - *elements $a_1, \ldots, a_n$ of $\mathcal{M}_0$;*

we have

$$\varphi^{\mathcal{M}}(a_1, \dots, a_n) = \varphi^{\mathcal{N}}(\Phi(a_1), \dots, \Phi(a_n)).$$

Moreover, for the implication (2) $\Rightarrow$ (1) *it suffices to assume (2) only for the cases in which* $\mathcal{M}_0$*,* $\mathcal{N}_0$ *are finitely generated.*

Proof. See [5, Proposition 13.12]. □

Proposition 4.3. *Let* T *be an* L*-theory. The following statements are equivalent:*

1. T *admits QE.*

2. *For every separable model* $\mathcal{M}$ *of* T*,* $n \in \omega$*,* $p \in S^{FS(T)}_{n+1}$*, and* $\bar{a} \subseteq M$ *with* $\bar{a} \models p \upharpoonright n$*, there exists an* $\aleph_1$*-saturated extension* $\mathcal{N} \succeq \mathcal{M}$ *and* $a_{n+1} \in N$ *such that* $(\bar{a}, a_{n+1}) \models p$*.*

Proof. See [5, Proposition 13.17]. □

Theorem 4.4. *Let* T *be an* L*-theory. If* T *has QE, then* $FS(T)$ *has the AP.*

Proof. Immediate from Proposition 4.3. □

Proposition 4.5. *Let* T *be a satisfiable* L*-theory that has QE. If there exists an* L*-structure* $\mathcal{A}$ *which embeds into every* $\aleph_1$*-saturated model of* T*, then* T *is a complete* L*-theory.*

Proof. We show that for every $\mathcal{M}$, $\mathcal{N} \models T$ we have that $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. This proves that $\mathrm{Con}(T) = \mathrm{Th}(\mathcal{M})$ for some (every) $\mathcal{M} \models T$. This suffices since $\mathrm{Th}(\mathcal{M})$ is a complete L-theory.

Fix $\mathcal{M}$, $\mathcal{N} \models T$. Let $\mathcal{M}_1, \mathcal{N}_1$ be $\aleph_1$-saturated elementary extensions of $\mathcal{M}, \mathcal{N}$ respectively, and $\mathcal{A}$ be the common L-structure which embeds into both $\mathcal{M}_1, \mathcal{N}_1$.

Suppose that the closed L-condition $\sigma = 0$ is in $\mathrm{Th}(\mathcal{M}_1)$. So we have $\sigma^{\mathcal{M}_1} = 0$. Set

$$\psi(x) = \max(d(x, x), \sigma).$$

Let $a \in A$. We have that $\psi^{\mathcal{M}_1}(a) = 0$. Because $T \subseteq \mathrm{Th}(\mathcal{M})$ has QE we have that $\mathcal{A} \preceq \mathcal{M}_1$. So we have $\psi^{\mathcal{A}}(a) = 0$. As before we have that $\mathcal{A} \preceq \mathcal{N}_1$ and so $\psi^{\mathcal{N}_1}(a) = 0$. This implies that $\sigma^{\mathcal{N}_1} = 0$, and so $(\sigma = 0) \in \mathrm{Th}(\mathcal{N}_1)$. So $\mathrm{Th}(\mathcal{M}_1) \subseteq \mathrm{Th}(\mathcal{N}_1)$ and in a similar way we may prove that $\mathrm{Th}(\mathcal{N}_1) \subseteq \mathrm{Th}(\mathcal{M}_1)$. We deduce that $\mathrm{Th}(\mathcal{M}_1) = \mathrm{Th}(\mathcal{N}_1)$. Since $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{M}_1)$ and $\mathrm{Th}(\mathcal{N}) = \mathrm{Th}(\mathcal{N}_1)$, we conclude that $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. □

4.2 Inductive theories

Definition 4.6. 1. Let $\mathcal{A}$, $\mathcal{B}$ be L-structures such that $\mathcal{A} \subseteq \mathcal{B}$. We say that $\mathcal{A}$ is existentially closed (e.c) in $\mathcal{B}$ (and we write $\mathcal{A} \subseteq_{ec} \mathcal{B}$) if for every quantifier-free L-formula $\phi(\bar{x}, \bar{y})$, and finite tuple $\bar{a} \subseteq A$

$$(\inf_{\bar{x}} \phi(\bar{x}, \bar{a}))^{\mathcal{A}} = (\inf_{\bar{x}} \phi(\bar{x}, \bar{a}))^{\mathcal{B}}.$$

2. Let K be a class of L-structures. We say that a structure $\mathcal{A}$ in K is e.c. in K if for every $\mathcal{B}$ in K with $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A}$ is existentially closed in $\mathcal{B}$.

3. If K is the class of all models of an L-theory T, we refer to e.c. structures in K as e.c models of T.

Definition 4.7. A class of L-structures K is called inductive if the following holds:

1. K is closed under isomorphism.

2. K is closed under unions of chains.

Definition 4.8. Let T be an L-theory. We say that T is inductive if the class $K = \text{Mod}(T)$ is inductive.

Inductive classes are useful because of the following theorem.

Theorem 4.9. *Let K be an inductive class of L-structures and $\mathcal{A}$ a structure in K. Then there is an e.c. structure $\mathcal{B}$ in K such that $\mathcal{A} \subseteq \mathcal{B}$.*

Proof. Similar to the first-order case. For example, see [6, Theorem 7.2.1]. □

The following theorem in one direction characterizes syntactically which theories are inductive and in the other direction indicates that there are several examples of inductive theories.

Theorem 4.10. *Let T be an L-theory. The following statements are equivalent:*

1. T is inductive.

2. T is axiomatized by a set of $\forall\exists$ closed L-conditions.

Proof. (1)⇒(2): See [9, Theorem 3.6].

(2)⇒(1): Similar to the first-order case. For example, see [6, Theorem 2.2.4]. □

Notation 4.11. Let T be an L-theory. We denote by $T_\forall$ the set of all the logical consequences E of T, such that E is a closed $\forall$ L-condition.

The following lemma and proposition will be used later.

Lemma 4.12. *Let T be an L-theory. If $\mathcal{A} \models T_\forall$, then there exists $\mathcal{B} \models T$ such that $\mathcal{A} \subseteq \mathcal{B}$.*

Proof. (of the lemma) Let $\mathcal{A} \models T_\forall$. Let $\bar{a}$ be an enumeration of A. Set $p(\bar{y}) = \text{qftp}_{\mathcal{A}}(\bar{a})$ It is enough to show that

$$p(\bar{y}) \cup T$$

has a model. We use compactness. For a contradiction, assume that there exists a finite $D \subseteq A$ and $\sigma(\bar{x}) = 0$ in $\text{qftp}_{\mathcal{A}}(\bar{d})$, where $\bar{d}$ is an enumeration of D, such that

$$\{\sigma(\bar{x}) = 0\} \cup T$$

has no model. Then,

$$T \models \sup_{\bar{x}}(r \dot{-} \sigma(\bar{x})) = 0$$

for some $r > 0$. Since $\mathcal{A} \models T_\forall$ we deduce

$$(\sup_{\bar{x}}(r \dot{-} \sigma(\bar{x})))^{\mathcal{A}} = 0.$$

This implies

$$(\sup_{\bar{x}}(r \dot{-} \sigma(\bar{x})))^{\mathcal{D}} = 0,$$

which is a contradiction. □

Proposition 4.13. *Let T be a $\forall\exists$ L-theory and $\mathcal{A}$ a model of T. The following statements are equivalent:*

1. *$\mathcal{A}$ is an e.c model of T.*

2. *$\mathcal{A}$ is an e.c. model of $T_\forall$.*

Proof. (1)⇒ (2): Let $\mathcal{B} \models T_\forall$ with $\mathcal{A} \subseteq \mathcal{B}$, $\psi(\bar{x}, \bar{y})$ a quantifier-free L-formula, and $\bar{a} \subseteq A$. It is enough to show that

$$(\inf_{\bar{y}} \psi(\bar{a}, \bar{y}))^{\mathcal{A}} = (\inf_{\bar{y}} \psi(\bar{a}, \bar{y}))^{\mathcal{B}}.$$

By Lemma 4.12, there exists $\mathcal{C} \models T$ with $\mathcal{B} \subseteq \mathcal{C}$. Since $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, we deduce that

$$(\inf_{\bar{y}} \psi(\bar{a}, \bar{y}))^{\mathcal{A}} \geq (\inf_{\bar{y}} \psi(\bar{a}, \bar{y}))^{\mathcal{B}} \geq (\inf_{\bar{y}} \psi(\bar{a}, \bar{y}))^{\mathcal{C}}.$$

Because $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{A}$ is an e.c. model of T we have that

$$(\inf_{\bar{y}} \psi(\bar{a}, \bar{y}))^{\mathcal{A}} = (\inf_{\bar{y}} \psi(\bar{a}, \bar{y}))^{\mathcal{C}}.$$

Hence,

$$(\inf_{\bar{y}} \psi(\bar{a}, \bar{y}))^{\mathcal{A}} = (\inf_{\bar{y}} \psi(\bar{a}, \bar{y}))^{\mathcal{B}}.$$

(2)⇒(1): Assume (2). First we show that $\mathcal{A}$ is a model of T. Since T is a $\forall\exists$ L-theory, a typical L-condition in T can be written in the form $\sup_{\bar{x}} \inf_{\bar{y}} \psi(\bar{x}, \bar{y}) = 0$ with $\psi(\bar{x}, \bar{y})$ quantifier-free. Let $\bar{a}$ be a tuple in $\mathcal{A}$. We must show that $(\inf_{\bar{y}} \psi(\bar{a}, \bar{y}))^{\mathcal{A}} = 0$. By Lemma 4.12, since $\mathcal{A} \models T_{\forall}$ there is $\mathcal{C} \models T$ with $\mathcal{A} \subseteq \mathcal{C}$. Then, $(\inf_{\bar{y}} \psi(\bar{a}, \bar{y}))^{\mathcal{C}} = 0$ and so $(\inf_{\bar{y}} \psi(\bar{a}, \bar{y}))^{\mathcal{A}} = 0$ since $\mathcal{A}$ is an e.c. model of $T_{\forall}$. Thus $\mathcal{A}$ is a model of T. Clearly $\mathcal{A}$ is an e.c. model of T, since every model of T extending $\mathcal{A}$ is in fact a model of $T_{\forall}$ too. □

4.3 Model-complete theories

Definition 4.14. An L-theory is said to be model-complete if every embedding between its models is elementary.

Theorem 4.15. *Let T be an L-theory. The following statements are equivalent:*

1. *T is model complete.*
2. *Every model of T is an e.c. model of T.*
3. *If $\mathcal{A}$, $\mathcal{B}$ are models of T and $e : \mathcal{A} \to \mathcal{B}$ is an embedding, then there are an elementary extension $\mathcal{D}$ of $\mathcal{A}$ and an embedding $g : \mathcal{B} \to \mathcal{D}$ such that $g \circ e$ is the identity on $\mathcal{A}$.*

Proof. (1)⇒(2): Let $\mathcal{A}, \mathcal{B} \models T$ with $\mathcal{A} \subseteq \mathcal{B}$. Since T is model complete we deduce that $\mathcal{A} \preceq \mathcal{B}$. Let $\phi(\bar{x}, \bar{y})$ be a quantifier-free L-formula and $\bar{a}$ a finite tuple in A. Since $\mathcal{A} \preceq \mathcal{B}$, we conclude

$$(\inf_{\bar{x}} \phi(\bar{x}, \bar{a}))^{\mathcal{A}} = (\inf_{\bar{x}} \phi(\bar{x}, \bar{a}))^{\mathcal{B}}.$$

(2)⇒(3): Let $\mathcal{A}, \mathcal{B} \models T$ such that $\mathcal{A} \subseteq \mathcal{B}$. It is enough to show that $\mathrm{EDiag}(\mathcal{A}) \cup \mathrm{Diag}(\mathcal{B})$ is satisfiable. This is done by compactness and by using the fact that $\mathcal{A} \subseteq_{ec} \mathcal{B}$.

(3)⇒(1): Let $\mathcal{A}_0, \mathcal{B}_0 \models T$ such that $\mathcal{A}_0 \subseteq \mathcal{B}_0$. Using the hypothesis repeatedly we build a chain $\mathcal{A}_0 \subseteq \mathcal{B}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{B}_1 \dots$, all being models of T, and such that for all $i \in \omega$, $\mathcal{A}_i \preceq \mathcal{A}_{i+1}$ and $\mathcal{B}_i \preceq \mathcal{B}_{i+1}$. Let

$\mathcal{C}$ be the completion of the set-theoretic union of the whole chain. Then $\mathcal{C} = \bigcup \mathcal{A}_i = \bigcup \mathcal{B}_i$. By Proposition 2.15, $\mathcal{A}_0 \preceq \mathcal{C}$ and $\mathcal{B}_0 \preceq \mathcal{C}$. It follows that $\mathcal{A}_0 \preceq \mathcal{B}_0$. □

Proposition 4.16. *Let T be an L-theory. If T is model-complete then T is a $\forall\exists$ theory.*

Proof. By Proposition 2.15, $\mathrm{Mod}(T)$ is closed under unions of elementary chains and because T is model-complete we deduce that $\mathrm{Mod}(T)$ is closed under unions of chains. By Theorem 4.10 we conclude that T is a $\forall\exists$ theory. □

4.4 Model companions

Definition 4.17. Let T be an L-theory.

1. We say that an L-theory T' is a model companion of T if T' is model-complete and $T'_\forall = T_\forall$.

2. We say that T is companionable if it has a model companion.

Theorem 4.18. *Let T be a $\forall\exists$ L-theory.*

1. *If T is companionable, then up to equivalence of theories, its model companion is unique and is the theory of the class of the e.c. models of T.*

2. *T is companionable iff the class of e.c. models of T is axiomatized by an L-theory.*

Proof. (1): Assume T is companionable with a model companion T'. Since T' is model-complete, by Theorem 4.15 the models of T' are precisely the e.c. models of T'. By Proposition 4.13, the models of T' are precisely the e.c. models of $T'_\forall = T_\forall$. Since T is a $\forall\exists$ theory, again by Proposition 4.13, the e.c. models of $T_\forall$ are precisely the e.c. models of T. So T' axiomatizes the theory of the class of e.c models of T. The uniqueness of T' is clear.

(2): The direction($\Rightarrow$) is immediate from (1).

For the other direction assume that the class of e.c. models of T is axiomatized by an L-theory T'.

First we prove that $T_\forall = T'_\forall$. It is enough to show that that every model of T has an extension which is a model of T', and that every model of T' has an extension which is a model of T. The first is true by Corollary 4.9 and the second is obviously true.

Also we note that T' is a $\forall\exists$ theory: By Proposition 2.15, $\mathrm{Mod}(T')$ is closed under unions of elementary chains and because T' is model-complete we deduce that $\mathrm{Mod}(T')$ is closed under unions of chains. By Theorem 4.10 we conclude that T' is a $\forall\exists$ theory.

By hypothesis, $\mathrm{Mod}(T')$ is the class of e.c. models of T. Because T is $\forall\exists$ theory, by Proposition 4.13, $\mathrm{Mod}(T')$ is the class of e.c models of $T_\forall$. Since $T_\forall = T'_\forall$, $\mathrm{Mod}(T')$ is the class of e.c. models of $T'_\forall$. Since T' is a $\forall\exists$ theory, again by Proposition 4.13, we conclude that $\mathrm{Mod}(T')$ is precisely the class of e.c. models of the L-theory T'. By Theorem 4.15, T' is model-complete. □

Proposition 4.19. *1. If T, T' are model-complete theories and $FS(T) = FS(T')$, then T, T' are equivalent.*

2. If T, T' are theories that have QE and $FS(T) = FS(T')$, then T, T' are equivalent.

Proof. We will need the following lemmas:

Lemma 4.20. *Let T be an L-theory. Then the models of $T_\forall$ are precisely the substructures of models of T.*

Proof. Clearly the substructures of models of T are models of $T_\forall$. The other direction of the containment is an immediate corollary of Lemma 4.12. □

Lemma 4.21. *Let T, T' be L-theories. Then $FS(T) = FS(T')$ iff $T_\forall$ is equivalent to $T'_\forall$.*

Proof. (of the lemma) ($\Rightarrow$): Set $\mathfrak{A} = FS(T)$ and $\mathfrak{B} = FS(T')$. It is enough to show that for all L-structures $\mathcal{M}$, $\mathcal{M} \models T_\forall$ iff $\mathcal{M} \models T'_\forall$. Let $\mathcal{M} \models T_\forall$. Because $\mathfrak{A} = \mathfrak{B}$, we have that $\mathrm{Diag}(\mathcal{M}) \cup T'$ is finitely satisfiable. Compactness implies that there exists an L-structure $\mathcal{N}$ such that $\mathcal{N} \models \mathrm{Diag}(\mathcal{M}) \cup T'$. This implies that $\mathcal{M}$ can embedded in a model of T'. By Lemma 4.20 we conclude that that $\mathcal{M} \models T'_\forall$. In the same way we prove $\mathcal{M} \models T'_\forall$ implies $\mathcal{M} \models T_\forall$.

($\Leftarrow$): By Lemma 4.20, the models of $T_\forall$, $T'_\forall$ are precisely the L-substructures of models T, T' respectively, and so $T_\forall = T'_\forall$ implies that $FS(T) = FS(T')$. □

Now we prove the proposition.

(1): Because $FS(T) = FS(T')$, by Lemma 4.21, $T_\forall = T'_\forall$. This implies that both theories are model companions of the the L-theory $T_\forall$. From Theorem 4.18, the theories T, T' are equivalent.

(2): It follows from (1). □

4.5 ω-categoricity

Definition 4.22. Let κ be a cardinal $\geq \mathrm{card}(L)$. We say that T is κ-categorical if T has noncompact models and whenever $\mathcal{M}$ and $\mathcal{N}$ are noncompact models of T having density character equal to κ, one has that $\mathcal{M}$ is isomorphic to $\mathcal{N}$.

Theorem 4.23. *Suppose that the number of nonlogical symbols in L is countable. Let T be a complete L-theory that has noncompact models. The following statements are equivalent:*

1. *T is ω-categorical;*

2. *For each $n \geq 1$, the metric space $(S_n(T), d)$ is compact.*

Proof. See [1, Theorem 12.10]. □

Theorem 4.24. *Suppose that T is ω-categorical and $\mathcal{M}$ is the separable model of T. Then $\mathcal{M}$ is strongly ω-near-homogeneous in the following sense: if $\bar{a}, \bar{b} \subseteq \mathcal{M}$ are n-tuples, then for every $\epsilon > 0$ there is automorphism F of $\mathcal{M}$ such that*

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(F(a_i), b_i) \leq d(tp(\bar{a}), tp(\bar{b})) + \epsilon.$$

Proof. See [1, Corollary 12.11]. □

In Chapter 8 we need the implication (1)⇒(2) in the following theorem. (This theorem is an observation due to C. Ward Henson.)

Theorem 4.25. *Let $\mathcal{M}$ be a noncompact separable L-structure. Set $T = Th(\mathcal{M})$ and $G = Aut(\mathcal{M})$. The following statements are equivalent:*

1. *T is an ω-categorical theory.*

2. *For every $\epsilon > 0$, $n \geq 1$ there exist n-tuples $\bar{a}_1 \dots \bar{a}_l \subset M$, for some $l \in \omega$, such that for every n-tuple $\bar{b} \subseteq M$ there exist $1 \leq j \leq l$ and $F \in G$ such that*

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(F(a_{j,i}), b_i) < \epsilon.$$

Proof. (1)⇒(2): Fix $\epsilon > 0$, $n \geq 1$ and $\bar{b} \subseteq M$. Set $p = \text{qftp}_{\mathcal{M}}(\bar{b})$. Since T is ω-categorical, by Theorem 4.23 $(S_n(T), d)$ is compact. This implies that there exists a finite $\epsilon/2$-net $\{p_1, \dots, p_l\}$, for some $l \in \omega$, of $(S_n(T), d)$. Therefore there exists $1 \leq j \leq l$ such that $d(p, p_j) < \epsilon/2$. Since $\mathcal{M}$ is the unique separable model of T, for every $1 \leq j \leq l$ there exist $\bar{a}_j \subseteq M$ such that $\bar{a}_j \models p_j$. Then by Proposition 4.24 there exists an automorphism F of $\mathcal{M}$ such that

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(F(a_{j,i}), b_i) \leq d(p_j, p) + \epsilon/2 < \epsilon.$$

(2)⇒(1): By Proposition 4.23 it is enough to show that for all $n \geq 1$, $(S_n(T), d)$ is compact. By Proposition 2.22 it is enough to show that for all $n \geq 1$, $(S_n(T), d)$ is totally bounded. Fix $n \geq 1$. Set D to be the set of n-types realized in $\mathcal{M}$. The hypothesis in (2) implies that D is totally bounded. To show that $(S_n(T), d)$ is totally bounded, it is enough to show that D is dense in $(S_n(T), d)$.

Let $p \in S_n(T)$. Since D is dense in $S_n(T)$ in the logic topology and L is countable, we deduce that there exists a sequence S in D that converges to p in the logic topology. Since D is totally bounded for the d-metric, we deduce that there exists a subsequence S' of S such that S' is a Cauchy sequence in $(S_n(T), d)$. Clearly S' converges to p in the d-metric. We conclude that D is dense in $(S_n(T), d)$.

□

Definition 4.26. Let $\mathcal{M}$ be an L-structure. We say that $\mathcal{M}$ is strongly ω-homogeneous if for every finite $A \subseteq M$ if $f : A \to M$ is an elementary map with respect to $\mathcal{M}$, then there exists an automorphism g of $\mathcal{M}$ that extends f.

The following easy fact will be used later.

Proposition 4.27. *Let $\mathcal{M}$ be a separable L-structure such that Th($\mathcal{M}$) is ω-categorical. The following statements are equivalent:*

1. *$\mathcal{M}$ is strongly ω-homogeneous.*
2. *For every finite tuple $\bar{a} \subseteq M$, Th($\mathcal{M}, \bar{a}$) is ω-categorical.*

Proof. (1)⇒(2): Let $\widetilde{\mathcal{N}}$ be an $L(\bar{a})$-structure such that $\widetilde{\mathcal{N}} \models \mathrm{Th}(\mathcal{M}, \bar{a})$. We will show that $\widetilde{\mathcal{N}} \cong (\mathcal{M}, \bar{a})$. Let $\mathcal{N}$ be the reduct of $\widetilde{\mathcal{N}}$ to L. Clearly $\mathcal{N} \models \mathrm{Th}(\mathcal{M})$ and because $\mathrm{Th}(\mathcal{M})$ is ω-categorical we have that $\mathcal{M} \cong \mathcal{N}$. We deduce that $\widetilde{\mathcal{N}} \cong (\mathcal{M}, \bar{b})$ for some $\bar{b} \subseteq M$ with $\mathrm{tp}_{\mathcal{M}}(\bar{a}) = \mathrm{tp}_{\mathcal{M}}(\bar{b})$. Since $\mathcal{M}$ is strongly ω-homogeneous we conclude that $(\mathcal{M}, \bar{a}) \cong (\mathcal{M}, \bar{b})$ and hence $\widetilde{\mathcal{N}} \cong (\mathcal{M}, \bar{a})$.

(2)⇒(1): Let $\bar{a}$, $\bar{b}$ be finite tuple in M such that $tp_{\mathcal{M}}(\bar{a}) = tp_{\mathcal{M}}(\bar{b})$. Because $tp_{\mathcal{M}}(\bar{a}) = tp_{\mathcal{M}}(\bar{b})$ we have that $\mathrm{Th}(\mathcal{M}, \bar{a}) = \mathrm{Th}(\mathcal{M}, \bar{b})$. Because $(\mathcal{M}, \bar{b}) \models \mathrm{Th}(\mathcal{M}, \bar{a})$ and $\mathrm{Th}(\mathcal{M}, \bar{a})$ is ω-categorical we have that $(\mathcal{M}, \bar{a}) \cong (\mathcal{M}, \bar{b})$. We conclude that $\mathcal{M}$ is strongly ω-homogeneous. □

Chapter 5

Fraïssé Theorems for complete theories with QE

In this chapter we fix a bounded continuous signature L whose only nonological symbols are a finite number of constant and predicate symbols.

In Chapter 5 we prove a generalization to the continuous setting of the Fraïssé Theorems for complete theories that have QE.

5.1 A Fraïssé Theorem for complete theories with QE

Definition 5.1. Let $\mathfrak{A}$ be an age. We say that $\mathfrak{A}$ has the perturbed amalgamation property (PAP) if for every $n \in \omega$, $\epsilon > 0$, and $p(\bar{x}) \in S^{\mathfrak{A}}_{n+1}$, there exists $\delta^{\mathfrak{A}}_n(\epsilon, p) > 0$ such that for every $\mathcal{M} \in \mathfrak{A}$, $\bar{a} = (a_1, \ldots, a_n) \subseteq M$ with $\rho_n(p \restriction n, q) < \delta^{\mathfrak{A}}_n(\epsilon, p)$, where $q = \mathrm{qftp}_{\mathcal{M}}(\bar{a})$, there exist $\mathcal{N} \supseteq \mathcal{M}$ in $\mathfrak{A}$, and a_{n+1} in N such that $\rho_{n+1}(p, r) \leq \epsilon$, where $r = \mathrm{qftp}_{\mathcal{N}}(a_1, \ldots, a_{n+1})$.

Definition 5.2. Let $\mathfrak{A}$ be an age. We say that $\mathfrak{A}$ has the uniformly perturbed amalgamation property (UPAP) if for every $n \in \omega$ and $\epsilon > 0$, there exists $\delta^{\mathfrak{A}}_n(\epsilon) > 0$ such that for every $\mathcal{M} \in \mathfrak{A}$ whenever we are given

- $p(\bar{x}) \in S^{\mathfrak{A}}_{n+1}$
- $\bar{a} = (a_1, \ldots, a_n) \subseteq M$ such that $\rho_n(p \restriction n, q) < \delta^{\mathfrak{A}}_n(\epsilon)$ where $q = \mathrm{qftp}_{\mathcal{M}}(\bar{a})$

then there exist $\mathcal{N}$ in $\mathfrak{A}$ such that $\mathcal{M} \subseteq \mathcal{N}$ and a_{n+1} in N such that $\rho_{n+1}(p, r) \leq \epsilon$, where $r = \mathrm{qftp}_{\mathcal{N}}(a_1, \ldots, a_{n+1})$.

Remark 5.3. UPAP is stronger than the PAP, but it turns out that these concepts are equivalent for ρ-compact ages.

Our main goal in this chapter is to prove the following two theorems, which we call Fraïssé Theorems for complete QE theories.

Theorem 5.4. *Let $\mathfrak{A}$ be a ρ-compact age with the PAP. Then there exists a unique (up to equivalence of theories) complete L-theory T that has QE and satisfies age$(T) = \mathfrak{A}$.*

A strong converse of Theorem 5.4 is the following theorem.

Theorem 5.5. *Let T be a complete L-theory that has QE. Then age(T) is a ρ-compact age with the UPAP.*

We prove Theorem 5.4 in 5.17 and Theorem 5.5 in 5.23.

We will need the following propositions.

Proposition 5.6. *Let $\mathfrak{A}$ be a ρ-compact age.*

1. *If $\mathfrak{A}$ has the PAP, then $\mathfrak{A}$ has the AP.*

2. *In particular, if $\mathfrak{A}$ has the UPAP, then $\mathfrak{A}$ has the AP.*

Proof. (1): We will use Proposition 3.8. Fix $\mathcal{A} \in \mathfrak{A}$, $n \in \omega$, $p \in S^{\mathfrak{A}}_{n+1}$, and $\bar{a} = (a_1, \ldots, a_n) \subseteq A$ such that $\bar{a} \models p \restriction n$. Let $\bar{b}$ be an enumeration of $A \setminus \bar{a}$. Assume that $\bar{b}$ is an l-tuple for some $l \in \omega$. Set $q = \mathrm{qftp}_{\mathcal{M}}(\bar{b})$. Since $\mathfrak{A}$ has the PAP, for every $m \geq 1$ there exists $\mathcal{A}_m \in \mathfrak{A}$ such that $\mathcal{A} \subseteq \mathcal{A}_m$ and $c_m \in \mathcal{A}_m$ such that $\rho_n(p, r_m) \leq 1/m$ where $r_m = \mathrm{qftp}_{\mathcal{A}_m}(\bar{a}, c_m)$. Set $d_m = \bar{b}\bar{a}a_m$ and $s_m = \mathrm{qftp}_{\mathcal{A}_m}(\bar{d}_m)$. Since the sequence $(s_m \mid m \geq 1)$ is in $S^{\mathfrak{A}}_{l+n+1}$ and $(S^{\mathfrak{A}}_{l+n+1}, \rho_{l+n+1})$ is compact, we deduce that the sequence mentioned has a limit point s. Let $\bar{d} \models s$. Let $\mathcal{B}$ be the L-structure generated by $\bar{d}$ and $f : \mathcal{A} \to \mathcal{B}$ the embedding defined as follows: For every $1 \leq i \leq l$ we set $f(b_i) = d_i$; for every $1 \leq i \leq n$ we set $f(a_i) = d_{l+i}$. The above embedding shows that without losing generality we may assume that $\mathcal{A} \subseteq \mathcal{B}$. Clearly $(\bar{a}, d_{l+n+1}) \models p$. This suffices.

(2): Follows easily from (1) since UPAP implies PAP. $\square$

Definition 5.7. Let $\mathfrak{A}$ be an age.

1. We say that $\mathfrak{A}$ has the strong JEP if for every family of L-structures $(\mathcal{A}_i)_{i \in I}$ in $\mathbb{M}^{\mathfrak{A}}$ there exist $\mathcal{B} \in \mathbb{M}^{\mathfrak{A}}$ and for every $i \in I$ an embedding $f_i : \mathcal{A}_i \to \mathcal{B}$.

2. We say that $\mathfrak{A}$ has the strong AP if for every family of L-structures $\mathcal{A}, (\mathcal{B}_i)_{i \in I}$ in $\mathbb{M}^{\mathfrak{A}}$ and embeddings $f_i : \mathcal{A} \to \mathcal{B}_i$ for every $i \in I$, there exist an L-structure $\mathcal{C}$ in $\mathbb{M}^{\mathfrak{A}}$ and, for every $i \in I$, an embedding $g_i : \mathcal{B}_i \to \mathcal{C}$ such that for every $i, j \in I$, $g_i \circ f_i = g_j \circ f_j$.

3. We say that $\mathfrak{A}$ has the strong PAP if for every $n \in \omega$, $\epsilon > 0$, and $p(\bar{x}) \in S^{\mathfrak{A}}_{n+1}$, there exist $\Delta^{\mathfrak{A}}_n(\epsilon, p) > 0$ such that for every $\mathcal{M} \in \mathbb{M}^{\mathfrak{A}}$, $\bar{a} = (a_1, \ldots, a_n) \subseteq M$ with $\rho_n(p \restriction n, q) < \Delta^{\mathfrak{A}}_n(\epsilon, p)$ where $q = \mathrm{qftp}_{\mathcal{M}}(\bar{a})$, there exist $\mathcal{N} \supseteq \mathcal{M}$ in $\mathbb{M}^{\mathfrak{A}}$, and a_{n+1} in N such that $\rho_{n+1}(p, r) \leq \epsilon$ where $r = \mathrm{qftp}_{\mathcal{N}}(a_1, \ldots, a_{n+1})$.

Proposition 5.8. *Let $\mathfrak{A}$ be an age. Conditions (1), (2) are equivalent.*

1. *$\mathfrak{A}$ has the strong PAP.*

2. *For every $n \in \omega$, $\epsilon > 0$, and $p \in S^{\mathfrak{A}}_{n+1}$, there exists $\Delta^{\mathfrak{A}}_n(\epsilon, p) > 0$ such that for every $\mathcal{M} \in \mathbb{M}^{\mathfrak{A}}$, $\bar{a} = (a_1, \ldots, a_n) \subseteq M$ such that $\rho_n(p \restriction n, q) < \Delta^{\mathfrak{A}}_n(\epsilon, p)$, where $q = qftp_{\mathcal{M}}(\bar{a})$, there exist $\mathcal{N} \supseteq \mathcal{M}$ in $\mathbb{M}^{\mathfrak{A}}$ and a_{n+1} in N such that*

$$\min(\Delta^{\mathfrak{A}}_n(\epsilon, p) \dot{-} \tau_{p \restriction n}(\bar{a}), \tau_p(\bar{a}, a_{n+1}) \dot{-} \epsilon) = 0.$$

is true in $\mathcal{N}$.

Proof. Clear from the definition of the strong PAP. □

Proposition 5.9. *Let $\mathfrak{A}$ be a ρ-compact age.*

1. *If $\mathfrak{A}$ has the JEP, then $\mathfrak{A}$ has the strong JEP.*
2. *If $\mathfrak{A}$ has the AP, then $\mathfrak{A}$ has the strong AP.*
3. *If $\mathfrak{A}$ has the PAP, then $\mathfrak{A}$ has the strong PAP. Moreover, if $(\delta^{\mathfrak{A}}_n \mid n \in \omega)$ witness that $\mathfrak{A}$ has the PAP, then taking $\Delta^{\mathfrak{A}}_n = \delta^{\mathfrak{A}}_n$, $(n \in \omega)$ witnesses that $\mathfrak{A}$ has the strong PAP.*

Proof. (1): Consider L-structures $(\mathcal{A}_i)_{i \in I}$ in $\mathbb{M}^{\mathfrak{A}}$, for some index set I. There is no loss of generality to assume that for all $i \neq j \in I$, $A_i \cap A_j = \emptyset$. To show that $\mathfrak{A}$ has the strong JEP it is enough to show that the $L(\bigcup_{i \in I} A_i)$-theory

$$\Sigma^{\mathfrak{A}} \cup \bigcup_{i \in I} \mathrm{Diag}(\mathcal{A}_i)$$

is satisfiable, where $\Sigma^{\mathfrak{A}}$ is defined in Notation 3.19. This is immediate by compactness, using the fact that $\mathfrak{A}$ has the JEP.

(2): Consider L-structures $\mathcal{A}, (\mathcal{B}_i)_{i \in I}$ in $\mathbb{M}^{\mathfrak{A}}$ and embeddings $f_i : \mathcal{A} \to \mathcal{B}_i$ for some index set I. There is no loss of generality to assume that for all $i \neq j \in I$, $B_i \cap B_j = A$, and for all $i \in I$, f_i is the inclusion map. To prove that $\mathfrak{A}$ has the strong AP, it is enough to show that the $L(\bigcup_{i \in I} B_i)$-theory

$$\Sigma^{\mathfrak{A}} \cup \bigcup_{i \in I} \mathrm{Diag}(\mathcal{B}_i)$$

is satisfiable. This is immediate by compactness, using the fact that $\mathfrak{A}$ has the AP.

(3): It is enough to prove that for all $n \in \omega$, $\epsilon > 0$, and $p \in S^{\mathfrak{A}}_{n+1}$, there exists $\Delta^{\mathfrak{A}}_n(\epsilon, p) > 0$ which satisfies the condition in Definition 5.7. We will show that for all $n \in \omega$ we may choose $\Delta^{\mathfrak{A}}_n$ such that $\Delta^{\mathfrak{A}}_n = \delta^{\mathfrak{A}}_n$. Fix $n \in \omega$, $\epsilon > 0$, $p \in S^{\mathfrak{A}}_{n+1}$, $\mathcal{M} \in \mathbb{M}^{\mathfrak{A}}$, and $\bar{a} \subseteq M$ such that $\rho_n(p \restriction n, q) < \delta^{\mathfrak{A}}_n(\epsilon, p)$ where $q = \mathrm{qftp}_{\mathcal{M}}(\bar{a})$. Let $\bar{b}$ be a tuple which enumerates the elements of M such that $\bar{b} \restriction n = \bar{a}$. Set $r = \mathrm{qftp}_{\mathcal{M}}(\bar{b})$. To show that $\mathfrak{A}$ has the strong PAP it is enough to show that the following set of L-conditions is satisfiable:

$$\Sigma := \{\tau_p(x_1, \ldots, x_n, y) \dot{-} \epsilon = 0\} \cup q((x_i)_{i \leq n}) \cup r((x_i)_{i \in \alpha}) \cup \Sigma^{\mathfrak{A}}.$$

By compactness, it is enough to show that for every $l \in \omega$ with $l \geq n$ the following set of formulas is satisfiable:

$$\{\tau_p(x_1, \ldots, x_n, y) \dot{-} \epsilon = 0\} \cup q((x_i)_{i \leq n}) \cup r((x_i)_{i \leq l}) \cup \Sigma^{\mathfrak{A}}.$$

Clearly there exists $\mathcal{N} \in \mathfrak{A}$ and $\bar{c} \subseteq N$ with $\bar{c} \models r \restriction l$. Then clearly $\bar{c} \restriction n \models q$. Since $\mathfrak{A}$ has the PAP and $\rho_n(p \restriction n, q) < \delta_n^{\mathfrak{A}}(\epsilon, p)$, there exist $\mathcal{N}' \in \mathfrak{A}$, with $\mathcal{N} \subseteq \mathcal{N}'$, and $d \in N'$ such that $\rho_{n+1}(p, \mathrm{qftp}(\bar{c} \restriction n, d)) < \epsilon$. Therefore

$$\tau_p(\bar{c} \restriction n, d) \dot{-} \epsilon = 0,$$

and Σ is satisfiable. The fact that for all $n \in \omega$ we may choose $\Delta_n = \delta_n$ is immediate from the proof. □

Notation 5.10. Let $\mathfrak{A}$ be an age with the PAP. Set $\Psi^{\mathfrak{A}}$ to be the set of all closed conditions of the form

$$\sup_{\bar{x}} \min(\delta_n^{\mathfrak{A}}(\epsilon, p) \dot{-} \tau_{p \restriction n}(\bar{x}), \inf_{x_{n+1}} \tau_p(\bar{x}, x_{n+1}) \dot{-} \epsilon) = 0$$

for all $n \in \omega$, $\epsilon > 0$, and $p \in S_{n+1}^{\mathfrak{A}}$, where $(\delta_n^{\mathfrak{A}} \mid n \in \omega)$ witnesses the fact that $\mathfrak{A}$ has the PAP. Set

$$T^{\mathfrak{A}} = \Sigma^{\mathfrak{A}} \cup \Psi^{\mathfrak{A}}.$$

Remark 5.11. Note that at first glance the L-theory $T^{\mathfrak{A}}$ seems to depend on the the functions $\delta_n^{\mathfrak{A}}$. Nevertheless, in Proposition 5.15 we will show that if $\mathfrak{A}$ is a ρ-compact age with the PAP and $(\delta_n^{\mathfrak{A}} \mid n \in \omega)$ witness the fact that $\mathfrak{A}$ has the PAP, the theory $T^{\mathfrak{A}}$ has QE and satisfies $FS(T^{\mathfrak{A}}) = \mathfrak{A}$. By Proposition 4.19, there exists a unique (up to equivalence of theories) such theory and hence $T^{\mathfrak{A}}$ is independent of $(\delta_n^{\mathfrak{A}} \mid n \in \omega)$.

Proposition 5.12. *Let $\mathfrak{A}$ be a ρ-compact age with the PAP. Then $T^{\mathfrak{A}}$ is a satisfiable L-theory and $FS(T^{\mathfrak{A}}) = \mathfrak{A}$.*

We need the following lemmas.

Lemma 5.13. *Let $\mathfrak{A}$ be a ρ-compact age with the PAP. If $\mathcal{A}$ is an L-structure in $\mathbb{M}^{\mathfrak{A}}$, then there exists an L-structure $\mathcal{B} \supseteq \mathcal{A}$ in $\mathbb{M}^{\mathfrak{A}}$ such that for all $\sigma \in \Psi^{\mathfrak{A}}$, and for all finite tuples $\bar{a}$, if σ is the L-condition*

$$\sup_{\bar{x}} \min(\delta_n^{\mathfrak{A}}(\epsilon, p) \dot{-} \tau_{p \restriction n}(\bar{x}), \inf_{x_{n+1}} \tau_p(\bar{x}, x_{n+1}) \dot{-} \epsilon) = 0$$

for some $n \in \omega$, $\epsilon > 0$, $p \in S^{\mathfrak{A}}_{n+1}$, and if $\bar{a}$ is an n-tuple, then there exists $a_{n+1} \in B$ such that

$$\min(\delta^{\mathfrak{A}}_n(\epsilon, p) \dot{-} \tau_{p\restriction n}(\bar{a}), \tau_p(\bar{a}, a_{n+1}) \dot{-} \epsilon) = 0$$

is true in $\mathcal{B}$.

Proof. Since $\mathfrak{A}$ has the PAP and is ρ-compact, by Proposition 5.6 we deduce that $\mathfrak{A}$ has the AP. Then by Proposition 5.9 we deduce that $\mathfrak{A}$ has the strong PAP and the strong AP. Let Q be the set of all ordered pairs $(\sigma, \bar{a})$ where $\sigma \in \Psi^{\mathfrak{A}}$ and $\bar{a}$ is a tuple as in the statement of the lemma. Since $\mathfrak{A}$ has the strong PAP, for every $(\sigma, \bar{a}) \in Q$ there exist an L-structure $\mathcal{M}_{(\sigma,\bar{a})}$ such that $\mathcal{A} \subseteq \mathcal{M}_{(\sigma,\bar{a})}$, $\text{age}(\mathcal{M}_{(\sigma,\bar{a})}) \subseteq \mathfrak{A}$ and $b \in \mathcal{M}_{(\sigma,\bar{a})}$ with

$$\min(\delta^{\mathfrak{A}}_n(\epsilon, p) \dot{-} \tau_{p\restriction n}(\bar{a}), \tau_p(\bar{a}, b) \dot{-} \epsilon) = 0.$$

From the strong AP we obtain an L-structure $\mathcal{M}$ such that for all $(\sigma, \bar{a}) \in Q$, $\mathcal{A} \subseteq \mathcal{M}_{(\sigma,\bar{a})} \subseteq \mathcal{M}$, and $\text{age}(\mathcal{M}) \subseteq \mathfrak{A}$. □

Lemma 5.14. *Let $\mathfrak{A}$ be a ρ-compact age. Then, there is an L-structure $\mathcal{A}$ such that $\text{age}(\mathcal{A}) = \mathfrak{A}$.*

Proof. (of lemma) Since $\mathfrak{A}$ has the JEP and is ρ-compact, by Proposition 5.9 we deduce that $\mathfrak{A}$ has the strong JEP. Let $(\mathcal{A}_i)_{i\in I}$ be an enumeration of a set of representatives of $\mathfrak{A}$, up to isomorphism; that is, every element of $\mathfrak{A}$ is isomorphic to an element of $(\mathcal{A}_i)_{i\in I}$. By the strong JEP, there exists $\mathcal{B} \in \mathbb{M}^{\mathfrak{A}}$ such that for each $i \in I$, $\mathcal{A}_i$ can be embedded into $\mathcal{B}$. Clearly $\text{age}(\mathcal{B}) = \mathfrak{A}$. □

Proof. (of Proposition 5.12) By induction on $n \in \omega$ we define a chain of L-structures $(\mathcal{M}_n \mid n \in \omega)$ such that $\overline{\bigcup_{n\in\omega} \mathcal{M}_n} \models T^{\mathfrak{A}}$. For $n = 0$ we define $\mathcal{M}_0 = \mathcal{A}$ where $\mathcal{A}$ is an L-structure obtained by applying Lemma 5.14. Given the L-structure $\mathcal{M}_n$ we define $\mathcal{M}_{n+1}$ by applying Lemma 5.13 to the structure $\mathcal{M}_n$. Set $\mathcal{M} = \bigcup_{n\in\omega} \mathcal{M}_n$ and $\mathcal{N} = \overline{\mathcal{M}}$. Clearly $\mathcal{M} \models \Psi^{\mathfrak{A}}$. Also $\text{age}(\mathcal{M}) = \mathfrak{A}$ and so by Proposition 3.17 we have that $\mathcal{M} \models \Sigma^{\mathfrak{A}}$. Then, from Proposition 3.20 we deduce that $\mathcal{N} \models \Sigma^{\mathfrak{A}}$. We conclude that $\mathcal{N} \models T^{\mathfrak{A}}$.

Since $\text{age}(\mathcal{M}) = \mathfrak{A}$, we have $\mathfrak{A} \subseteq FS(T^{\mathfrak{A}})$ and since $\Sigma^{\mathfrak{A}} \subseteq T^{\mathfrak{A}}$ we have $FS(T^{\mathfrak{A}}) = \mathfrak{A}$.

□

Proposition 5.15. *Let $\mathfrak{A}$ be a ρ-compact age with the PAP.*

1. *$T^{\mathfrak{A}}$ has QE.*
2. *$T^{\mathfrak{A}}$ is a complete L-theory.*

Proof. (1): By Proposition 5.12 we have $FS(T^{\mathfrak{A}}) = \mathfrak{A}$. Therefore, by Proposition 4.3 it is enough to prove the following claim.

Claim 5.16. *Let $\mathcal{M} \models T^{\mathfrak{A}}$, where $\mathcal{M}$ is separable; consider $n \in \omega$, $p \in S^{\mathfrak{A}}_{n+1}$ and $\bar{a} \subseteq M$ with $\bar{a} \models p \restriction n$. Then there exists an $\aleph_1$-saturated elementary extension $\mathcal{N} \succeq \mathcal{M}$ and $a_{n+1} \in N$ such that $(\bar{a}, a_{n+1}) \models p$.*

Proof. (of claim) Fix a separable L-structure $\mathcal{M} \models T^{\mathfrak{A}}$, $n \in \omega$, $p \in S^{\mathfrak{A}}_n$ and $\bar{a} \subseteq \mathcal{M}$ with $\bar{a} \models p \restriction n$. For every $\epsilon > 0$ we have that

$$T \models \sup_{\bar{x}} \min(\delta^{\mathfrak{A}}_n(\epsilon, p) \dot{-} \tau_{p\restriction n}(\bar{x}), \inf_{x_{n+1}} \tau_p(\bar{x}, x_{n+1}) \dot{-} \epsilon) = 0.$$

So, for every $\epsilon > 0$ there exists $b_\epsilon \in N$ such that

$$\min(\delta^{\mathfrak{A}}_n(\epsilon, p) \dot{-} \tau_{p\restriction n}(\bar{a}), \tau_p(\bar{a}, b_\epsilon) \dot{-} 2\epsilon) = 0.$$

This implies that $\tau_p(\bar{a}, b_\epsilon) \leq 2\epsilon$. Equivalently we have that that $\rho_{n+1}(p, q) \leq 2\epsilon$ where $q = \mathrm{qftp}_{\mathcal{N}}(\bar{a}, b_\epsilon)$. If $\mathcal{N}$ is an $\aleph_1$-saturated extension of $\mathcal{M}$ clearly there exists $a_{n+1} \in N$ such that $(\bar{a}, a_{n+1}) \models p$. □

(2): We show that every $\aleph_1$-saturated model of $T^{\mathfrak{A}}$ realizes every element of $S^{\mathfrak{A}}_1$. Then by Proposition 4.5 we conclude that $T^{\mathfrak{A}}$ is a complete L-theory.

Let $\mathcal{M}$ be an $\aleph_1$-saturated model of $T^{\mathfrak{A}}$. Let q be any 1-type realized in $\mathcal{M}$, say by a, and $r \in S^{\mathfrak{A}}_1$. Since $\mathfrak{A}$ a ρ-compact age with the PAP, by Proposition 5.6 we deduce that $\mathfrak{A}$ has the AP. Therefore there exists $p \in S^{\mathfrak{A}}_2$ such that $p \restriction 1 = q$ and the restriction of p in the second variable is equal to r. For any $\epsilon > 0$,

$$T \models \sup_{x_1} \min(\delta^{\mathfrak{A}}_1(\epsilon, p) \dot{-} \tau_q(x_1), \inf_{x_2} \tau_p(x_1, x_2) \dot{-} \epsilon) = 0.$$

Applying the condition to $x_1 = a$ in M and noting that $\tau^{\mathcal{M}}_q(a) = 0$, we see that

$$\mathcal{M} \models \inf_{x_2} \tau_p(a, x_2) \leq \epsilon.$$

Since $\epsilon > 0$ was arbitrary,

$$\mathcal{M} \models \inf_{x_2} \tau_p(a, x_2) = 0.$$

Consequently, since $\mathcal{M}$ is $\aleph_1$-saturated, there exists $b \in M$ such that $(a, b) \models p$. Hence $b \models r$. That is, any $\aleph_1$-saturated model of $T^{\mathfrak{A}}$ realizes every element of $S^{\mathfrak{A}}_1$. □

We state again and prove Theorem 5.4.

Theorem 5.17. *Let $\mathfrak{A}$ be a ρ-compact age with the PAP. There exists a unique L-theory T which is complete, has QE and satisfies age$(T) = \mathfrak{A}$.*

Proof. The existence of an L-theory T with the prescribed properties is immediate from Propositions 5.12 and 5.15. The uniqueness of T comes from Proposition 4.19. $\square$

5.2 Axiomatization of complete QE theories

Definition 5.18. Let T be an L-theory. Set $\mathfrak{A} = FS(T)$. We say that T has the uniformly perturbed extension property if for all $n \in \omega$, $\epsilon > 0$ there exists $\delta_n^T(\epsilon) > 0$ such that for all $p \in S_{n+1}^{\mathfrak{A}}$ we have that

$$T \models \sup_{\bar{x}} \min(\delta_n^T(\epsilon) \dot{-} \tau_{p|n}(\bar{x}), \inf_{x_{n+1}} \tau_p(\bar{x}, x_{n+1}) \dot{-} \epsilon) = 0.$$

Proposition 5.19. *Let T be an L-theory. The following statements are equivalent:*

1. *T has QE.*

2. *T has the uniformly perturbed extension property.*

Proof. (2) $\Rightarrow$ (1): Similar to the proof of the fact that $T^{\mathfrak{A}}$ has QE in the proof of Theorem 5.15.

(1) $\Rightarrow$ (2): We will need the following definition and lemma.

Definition 5.20. Let T be an L-theory and $\mathfrak{A} = FS(T)$. We say that T has the perturbed extension property if for every $n \in \omega$, $\epsilon > 0$, and $p \in S_{n+1}^{\mathfrak{A}}$ there exists $\delta(n, \epsilon, p) > 0$ such that

$$T \models \sup_{\bar{x}} \min(\delta(n, \epsilon, p) \dot{-} \tau_{p|n}(\bar{x}), \inf_{x_{n+1}} \tau_p(\bar{x}, x_{n+1}) \dot{-} \epsilon) = 0.$$

Lemma 5.21. *Let T be an L-theory that has QE. Then T has the perturbed extension property.*

Proof. (of lemma) Set $\mathfrak{A} = FS(T)$. Assume that T does not have the perturbed extension property. This implies that there exists $n \in \omega$, $\epsilon > 0$, and $p \in S_{n+1}^{\mathfrak{A}}$ such that for all $\delta > 0$,

$$T \not\models \sup_{\bar{x}} \min(\delta \dot{-} \tau_{p|n}(\bar{x}), \inf_{x_{n+1}} \tau_p(\bar{x}, x_{n+1}) \dot{-} \epsilon) = 0.$$

Taking $\delta = 1/m$ for $m \geq 1$ we see that

$$T \cup \{\tau_{p|n}(\bar{x}) \leq \frac{1}{m} \mid m \geq 1\} \cup \{\inf_{x_{n+1}} \tau_p(\bar{x}, x_{n+1}) \geq \epsilon\}$$

is satisfiable. By model-theoretic compactness, there exist an L-structure $\mathcal{M} \models T$ and $\bar{a} \subseteq M$ with $\bar{a} \models p{\restriction}n$

such that

$$\inf_{x_{n+1}} \tau_p(\bar{a}, x_{n+1}) \geq \epsilon.$$

The above statement implies that there is no a_{n+1} in any elementary extension of $\mathcal{M}$ such that

$$\rho_n(p, q) < \frac{\epsilon}{2}$$

where $q = \mathrm{qftp}_{\mathcal{M}}(\bar{a}, a_{n+1})$. But T has QE and $p \in S^{\mathfrak{A}}_{n+1}$, so by Proposition 4.3 there exists $\mathcal{N} \succeq \mathcal{M}$ and $a_{n+1} \in N$ such that $(\bar{a}, a_{n+1}) \models p$ which is a contradiction. □

We continue the proof of Proposition 5.19. From Lemma 5.21, for every $n \in \omega$, $\epsilon > 0$, $p \in S^{\mathfrak{A}}_{n+1}$ there exists $\delta(n, \epsilon, p)$ such that

$$T \models \sup_{\bar{x}} \min(\delta(n, \epsilon, p) \dot{-} \tau_{p|n}(\bar{x}), \inf_{x_{n+1}} \tau_p(\bar{x}, x_{n+1}) \dot{-} \epsilon) = 0.$$

For every $n \geq 0$ we define a function

$$\delta_n : (0, 1] \times S^{\mathfrak{A}}_{n+1} \to (0, 1]$$

as follows:

$$\delta_n(\epsilon, p) = \sup\{\delta \in (0, 1] \quad | \quad T \models \sup_{\bar{x}} \min(\delta \dot{-} \tau_{p|n}(\bar{x}), \inf_{x_{n+1}} \tau_p(\bar{x}, x_{n+1}) \dot{-} \epsilon) = 0\}.$$

Claim 5.22. *For every $n \in \omega$, $\epsilon > 0$, $\delta_n(\epsilon, p)$ is a continuous function of $p \in (S^{\mathfrak{A}}_{n+1}, \rho_{n+1})$.*

Proof. (of claim) Fix $n \geq 1$, $\epsilon > 0$. To show that $\delta_n(\epsilon, p)$ is continuous it is enough to show that for every $p \in S^{\mathfrak{A}}_{n+1}$, $\epsilon' > 0$ there exists $\delta' > 0$ such that if $q \in S^{\mathfrak{A}}_{n+1}$ with $\rho_{n+1}(p, q) < \delta'$, then

$$|\delta_n(\epsilon, p) - \delta_n(\epsilon, q)| < \epsilon'.$$

Fix $p \in S^{\mathfrak{A}}_{n+1}$, $\epsilon' > 0$. We claim that it suffices to choose $\delta' < \epsilon'$. Indeed we have that

$$\delta_n(\epsilon, p) - \delta' < \delta_n(\epsilon, q) < \delta_n(\epsilon, p) + \delta' \Leftrightarrow$$

$$|\delta_n(\epsilon, p) - \delta_n(\epsilon, q)| < \delta' < \epsilon.$$

□

For every $n \in \omega$, $\epsilon > 0$ set

$$\delta_n^T(\epsilon) = \inf\{\delta_n(\epsilon, p) \mid p \in S_{n+1}^{\mathfrak{A}}\}.$$

The above infimum is actually a minimum because $\delta_n(\epsilon, p)$ is a continuous function of $p \in (S_{n+1}^{\mathfrak{A}}, \rho_{n+1})$ and $(S_{n+1}^{\mathfrak{A}}, \rho_{n+1})$ is compact. From Lemma 5.21, for every $n \geq 1$, $\epsilon > 0$, $p \in S_{n+1}^T(\epsilon) > 0$ we have that $\delta_n(\epsilon, p) > 0$. This implies that $\delta_n^T(\epsilon) > 0$. Clearly $\delta_n^T(\epsilon)$ satisfies the condition in Definition 5.18. □

Theorem 5.23. *Let T be a complete L-theory that has QE, and let $\mathfrak{A} = age(T)$. Then:*

1. *$\mathfrak{A}$ is a ρ-compact age with the UPAP.*
2. *T is axiomatized by $T^{\mathfrak{A}}$.*
3. *For every $n \in \omega$ let $\delta_n^{\mathfrak{A}}$, δ_n^T be as in Definition 5.2 and Definition 5.18 respectively. Then for every $n \in \omega$, given δ_n^T, we may choose $\delta_n^{\mathfrak{A}}$ such that $\delta_n^{\mathfrak{A}} = \delta_n^T$.*
4. *For every $n \in \omega$ let $\delta_n^{\mathfrak{A}}$, δ_n^T be as in Definition 5.2 and Definition 5.18 respectively. Then, for every $n \in \omega$, given $\delta_n^{\mathfrak{A}}$, we may choose δ_n^T such that $\delta_n^T = \delta_n^{\mathfrak{A}}$.*

Proof. (1): Since T is a complete L-theory, Proposition 3.16 implies that $\mathfrak{A}$ is a ρ-compact age. Next we show that $\mathfrak{A}$ has the UPAP.

We show that for every $n \in \omega$, $\epsilon > 0$, it suffices to choose $\delta_n^{\mathfrak{A}} = \delta_n^T$. Fix $n \in \omega$, $\epsilon > 0$, $\mathcal{M} \in \mathfrak{A}$, $p \in S_{n+1}^{\mathfrak{A}}$ $\bar{a} \subseteq M$ such that $\rho_n(p \restriction n, q) < \delta_n^T(\epsilon)$ where $q = \mathrm{qftp}_{\mathcal{M}}(\bar{a})$. Because $\mathcal{M} \in \mathfrak{A}$ we have that there exists an L-structure $\mathcal{A} \models T$ and $\mathcal{M}$ can be embedded in $\mathcal{A}$. Without losing generality we may assume that $\mathcal{M}$ is a substructure of $\mathcal{A}$. Because T has the uniformly perturbed extension property we have that

$$T \models \sup_{\bar{x}} \min(\delta_n^T(\epsilon) \dot{-} \tau_{p \restriction n}(\bar{x}), \inf_{x_{n+1}} \tau_p(\bar{x}, x_{n+1}) \dot{-} \epsilon) = 0.$$

This implies that

$$(\sup_{\bar{x}} \min(\delta_n^T(\epsilon) \dot{-} \tau_{p \restriction n}(\bar{x}), \inf_{x_{n+1}} \tau_p(\bar{x}, x_{n+1}) \dot{-} \epsilon))^{\mathcal{A}} = 0.$$

So we have that for every $\epsilon' > \epsilon$ there exists $b_{\epsilon'} \in \mathcal{A}$ such that $\tau_p(\bar{a}, b_{\epsilon'}) < \epsilon'$. Set $r_{\epsilon'} = \mathrm{qftp}_{\mathcal{A}}(\bar{a}, b_{\epsilon'})$. Because $(S_{n+1}^{\mathfrak{A}}, \rho_{n+1})$ is compact we have that

$$r = \lim_{\epsilon' \to \epsilon} r_{\epsilon'}$$

exists in $S_{n+1}^{\mathfrak{A}}$. Clearly we have that $\rho_n(p, r) \leq \epsilon$. Because $\mathfrak{A}$ has the AP we have that there exists $\mathcal{N} \in \mathcal{A}$ such that $\mathcal{M} \subseteq \mathcal{N}$ and $a_{n+1} \in \mathcal{N}$ such that $\rho_n(p, r) \leq \epsilon$ where $r = \mathrm{qftp}_{\mathcal{N}}(\bar{a}, a_{n+1})$.

(2): Since T, $T^{\mathfrak{A}}$ both have QE and $FS(T) = FS(T^{\mathfrak{A}}) = \mathfrak{A}$, by Proposition 4.19 we deduce that they are equivalent, namely $T^{\mathfrak{A}}$ axiomatizes T. □

(3): Immediate from the proof of (1).

(4): Immediate from (2).

Theorem 5.24. *Let $\mathfrak{A}$ be a ρ-compact age. The following statements are equivalent:*

1. *$\mathfrak{A}$ has the PAP.*

2. *$\mathfrak{A}$ has the UPAP.*

Proof. (1)⇒(2): By Theorem 5.4, let T be the unique complete L-theory that has QE and satisfies $\text{age}(T) = \mathfrak{A}$. Then by Theorem 5.23, $\mathfrak{A}$ has the UPAP.

(2)⇒(1): Trivial. □

Remark 5.25. It is natural to ask whether we can strengthen Theorem 5.4 in the following way:
Given an ρ-compact age with the AP, is there a complete L-theory T such that T has QE and satisfies $\text{age}(T) = \mathfrak{A}$?
The following counterexample due to C. Ward Henson answers this question above in the negative.

Let L be the signature of pure metric spaces with d taking values in $[0, 1]$. Let $\mathfrak{A}$ be the class of all finite L-structures $\mathcal{M}$ with the following properties:

1. If $x, y \in M$ are distinct, then $d^{\mathcal{M}}(x, y)$ is in the interval $[\frac{1}{2}, 1]$.

2. If $x, y, z \in M$ are distinct, then at least one of the distances $d^{\mathcal{M}}(x, y)$, $d^{\mathcal{M}}(y, z)$, $d^{\mathcal{M}}(x, z)$, equals 1.

It is clear that the class of L-structures $\mathcal{M}$ that satisfy $\text{age}(\mathcal{M}) \subseteq \mathfrak{A}$ is axiomatizable by the closed L-conditions,

$$\sup_{x,y,z} \min(d(x, y), d(x, z), d(y, z), 1 \dot{-} d(x, y), 1 \dot{-} d(x, z), 1 \dot{-} d(y, z)) = 0,$$

and

$$\sup_{x,y} \min(d(x, y), \frac{1}{2} \dot{-} d(x, y)) = 0.$$

Therefore $\mathfrak{A}$ is ρ-compact. Also $\mathfrak{A}$ has the AP: If $\mathcal{M}_1$, $\mathcal{M}_2$ are in $\mathfrak{A}$ with intersection $\mathcal{M}$, then for every $x \in M_1 \setminus M$ and $y \in M_2 \setminus M$, set $d(x, y) = 1$.

Now we prove that if T is a complete L-theory that satisfies $\text{age}(T) = \mathfrak{A}$, then T does not have QE. Set

$$S = \{p \in S_3^{\mathfrak{A}} \mid p \in [\max(d(x_1, x_3), d(x_2, x_3)) < 1]\}.$$

Note that S is open in $(S_3^{\mathfrak{A}}, \rho_3)$. Map S into $S_2^{\mathfrak{A}}$ by eliminating the x_3 variable. The image is $\{p\}$, where p is the unique quantifier free 2-type that contains the condition $d(x_1, x_2) = 1$. Note that $\{p\}$ is not open in $(S_2^{\mathfrak{A}}, \rho_2)$, since $(S_2^{\mathfrak{A}}, \rho_2)$ is isometric to $\{0\} \cup [\frac{1}{2}, 1]$ with the absolute value metric, with p mapped to 1. But by Theorem 5.5, if T had QE, then $\mathfrak{A}$ would have the PAP and therefore the image $\{p\}$ of the open set S should be open, which is a contradiction. We conclude that T does not have QE.

Chapter 6

Fraïssé Theorems for separable structures

In this chapter we fix a countable bounded continuous signature L without function symbols.

The main result of Chapter 6 is Theorem 6.28, which is shown to be an optimal result by its converse in 6.30. In Sections 6.1, 6.2, and 6.3 we provide background concepts and material that are needed in both Chapters 6 and 7. The concepts and results of Sections 6.4 and 6.5 are rather technical and they are introduced solely in order to prove Theorem 6.28.

6.1 Definitions and basic theorems

Definition 6.1. Let $\mathcal{M}, \mathcal{N}$ be L-structures, $\bar{a} \subseteq M$ and $\bar{b} \subseteq N$. Let $\mathcal{A}$ be the substructure of $\mathcal{M}$ generated by $\bar{a}$ and $\mathcal{B}$ the substructure of $\mathcal{N}$ generated by $\bar{b}$.

1. $f : \bar{a} \to \mathcal{N}$ is an embedding if it has an extension that is an embedding from $\mathcal{A}$ into $\mathcal{N}$.

2. $f : \bar{a} \to \bar{b}$ is an isomorphism if it has an extension that is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Definition 6.2. Let $\mathcal{M}$ be an L-structure and $\mathfrak{A} = \text{age}(\mathcal{M})$.

1. $\mathcal{M}$ is ω-qf-near-homogeneous if for every $\epsilon > 0$, all finite tuples $\bar{a} \subseteq \bar{b} \subseteq M$, where $\bar{a}$ is an n-tuple for some $n \geq 1$, and every embedding $f : \bar{a} \to \mathcal{M}$, there exists an embedding $g : \bar{b} \to \mathcal{M}$ such that

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(f(a_i), g(b_i)) < \epsilon.$$

2. $\mathcal{M}$ is strongly ω-qf-near-homogeneous if for every $n \in \omega$ and n-tuples $\bar{a}, \bar{b} \subseteq M$ such that $\text{qftp}_{\mathcal{M}}(\bar{a}) = \text{qftp}\mathcal{M}(\bar{b})$, there exists an automorphism f of $\mathcal{M}$ such that

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(a_i, f(b_i)) < \epsilon.$$

Proposition 6.3. *Let $\mathcal{M}$ be an L-structure and $\mathfrak{A} = age(\mathcal{M})$. The following statements are equivalent:*

1. *$\mathcal{M}$ is ω-qf-near-homogeneous.*

2. *For every $\epsilon > 0$, $n \in \omega$, $l \geq 1$, $p \in S^{\mathfrak{A}}_{n+l}$, and $\bar{a} = (a_1, \ldots, a_n) \subseteq M$ such that $\bar{a} \models p \restriction n$, there exists $\bar{b} = (b_1, \ldots, b_{n+l})$ in M such that $\bar{b} \models p$ and*

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(a_i, b_i) < \epsilon.$$

3. *For every $\epsilon > 0$, $n \in \omega$, $p \in S^{\mathfrak{A}}_{n+1}$, $\bar{a} = (a_1, \ldots, a_n) \subseteq M$ such that $\bar{a} \models p \restriction n$, there exists $\bar{b} = (b_1, \ldots, b_{n+1})$ in M such that $\bar{b} \models p$ and*

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(a_i, b_i) < \epsilon.$$

4. *For every $\epsilon > 0$, $p \in S^{\mathfrak{A}}_{\omega}$, $\bar{a} = (a_1, \ldots, a_n) \subseteq M$ such that $\bar{a} \models p \restriction n$, there exists $\bar{b} = (b_i)_{1 \leq i \in \omega} \subseteq M$ such that $\bar{b} \models p$ and*

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(a_i, b_i) < \epsilon.$$

Proof. (1)$\Leftrightarrow$(2): Immediate.

(2)$\Rightarrow$(3): Immediate.

(3)$\Rightarrow$(4): Fix $\epsilon > 0$, $p \in S^{\mathfrak{A}}_{\omega}$, $\bar{a} = (a_1, \ldots, a_n) \subseteq M$ such that $\bar{a} \models p \restriction n$. We define inductively a chain of finite tuples $(\bar{b}_i)_{i \in \omega}$ such that for all $i \in \omega$, $\bar{b}_i$ is an $(n+i)$-tuple which realizes $p \restriction (n+i)$ and

$$\max_{1 \leq j \leq n+i} d^{\mathcal{M}}(b_{i,j}, b_{i+1,j}) < \frac{\epsilon}{4^{i+1}}.$$

For $i = 0$ we set $\bar{b}_0 = \bar{a}$. Suppose we have defined a chain $(\bar{b}_i \mid i \leq \kappa)$ for some $k \in \omega$, such that the conditions above are satisfied. By hypothesis, there exists a tuple $\bar{b}$ such that $\bar{b} \models p \restriction (n+k+1)$ and

$$\max_{1 \leq i \leq n+k} d^{\mathcal{M}}(b_{k,j}, b_{k+1,j}) < \frac{\epsilon}{4^{k+1}}.$$

For all $j \geq 1$ $\{b_{n,j} \mid n \geq j\}$ is a Cauchy sequence. Since $\mathcal{M}$ is a complete metric space $\lim_{n \to \infty} b_{n+j}$ exists. Set $\lim_{n \to \infty} b_{n,j} = b_j$.

Clearly $(b_i)_{i \in \omega} \models p$.

Now we prove that $\max_{1\le j\le n} d^{\mathcal{M}}(a_j,b_j) < \epsilon$. By the triangle inequality, for all $1 \le i \le n$, $j \ge 1$

$$\begin{aligned} d^{\mathcal{M}}(a_j,b_{i,j}) &\le d^{\mathcal{M}}(b_{0,j},b_{1,j}) + \cdots + d^{\mathcal{M}}(b_{i-1,j},b_{i,j}) \\ &\le \frac{\epsilon}{4} + \cdots + \frac{\epsilon}{4^i} \\ &\le \sum_{k=1}^{i} \frac{\epsilon}{4^k} \end{aligned}$$

Then for all $1 \le j \le n$,

$$d^{\mathcal{M}}(a_j,b_j) = \lim_{i\to\infty} d^{\mathcal{M}}(a_j,b_{i,j}) \le \lim_{i\to\infty} \sum_{k=1}^{i} \frac{\epsilon}{4^k} = \frac{\epsilon}{3} < \epsilon.$$

(4)⇒(1): Immediate. □

Proposition 6.4. *1. Let $\mathcal{M}$ be a separable L-structure and $\mathcal{N}$ an ω-qf-near-homogeneous L-structure such that age($\mathcal{M}$) $\subseteq$ age($\mathcal{N}$). If $\bar{a}$ is a finite tuple in $\mathcal{M}$, and $f : \bar{a} \to \mathcal{N}$ is an embedding, then there exists an embedding $g : \mathcal{M} \to \mathcal{N}$ such that*

$$\max_i d^{\mathcal{N}}(f(a_i), g(b_i)) < \epsilon.$$

2. If $\mathcal{M}, \mathcal{N}$ are both separable, ω-qf-near-homogeneous L-structures, age($\mathcal{M}$) = age($\mathcal{N}$), $\bar{a} \subseteq \mathcal{M}$, $\bar{b} \subseteq \mathcal{N}$, both finite tuples, and $f : \bar{a} \to \bar{b}$ is an isomorphism, then there exists an isomorphism $g : \mathcal{M} \to \mathcal{N}$ such that

$$\max_i d^{\mathcal{N}}(f(a_i), g(b_i)) < \epsilon.$$

Proof. (1): Let D be a countable dense subset of M, $\bar{a}$ be an enumeration of A and $\bar{d} = (d_i)_{i\in\lambda}$ be an enumeration of D for some $\lambda \le \omega$ such that $\bar{a}$ is an initial segment of $\bar{d}$. Set $p = \mathrm{qftp}_{\mathcal{M}}(\bar{d})$. Then $p \in S^{\mathfrak{A}}_{\lambda}$ for some $\lambda \le \omega$. Since $\mathcal{N}$ is ω-qf-near-homogeneous, by Proposition 6.3 there exists $\bar{c} \subseteq N$ such that $\bar{c} \models p$ and

$$\max_i d^{\mathcal{N}}(f(a_i), c_i) < \epsilon.$$

Define a map $f' : D \to N$ such that for all $i \in \lambda$, $f'(d_i) = c_i$. Since D is dense in M, f' can be extended to an embedding $g : \mathcal{M} \to \mathcal{N}$. Clearly

$$\max_i d^{\mathcal{N}}(f(a_i), g(b_i)) < \epsilon.$$

(2): This can be proved by a back-and-forth version of the above argument.

□

Theorem 6.5. *1. Any strongly ω-qf-near-homogeneous structure is ω-qf-near-homogeneous.*

2. Any separable ω-qf-near-homogeneous structure is strongly ω-qf-near-homogeneous.

3. If $\mathcal{M}, \mathcal{N}$ are separable L-structures that are both ω-qf-near-homogeneous and $age(\mathcal{M}) = age(\mathcal{N})$ then $\mathcal{M}, \mathcal{N}$ are isomorphic.

Proof. (1): Obvious.

(2): Clear by Proposition 6.4(2) if we take $\mathcal{M} = \mathcal{N}$.

(3): Immediate from Proposition 6.4.

□

6.2 Near amalgamation property

Definition 6.6. Let $\mathfrak{A}$ be an age. We say that $\mathfrak{A}$ has the near amalgamation property (near-AP) if for every $\epsilon > 0$, $n, l \in \omega$, $\mathcal{M} \in \mathfrak{A}$, $p \in S^{\mathfrak{A}}_{n+l}$, and $\bar{a} = (a_1, \ldots, a_n) \subseteq M$ such that $\bar{a} \models p \restriction n$, there exists $\mathcal{N}$ in $\mathfrak{A}$ such that $\mathcal{M} \subseteq \mathcal{N}$ and $\bar{b} = (b_1, \ldots, b_n) \subseteq N$ such that $\bar{b} \models p$ and

$$\max_{1 \leq i \leq n} d^{\mathcal{N}}(a_i, b_i) < \epsilon.$$

The near-AP will play an important role only in this chapter. The reason is that, in the rest of the document, we will mostly consider ages of complete theories and in that case the following proposition applies.

Proposition 6.7. *1. If $\mathfrak{A}$ is a ρ-compact age with the near-AP, then $\mathfrak{A}$ has the AP.*

2. Let T be a complete L-theory and $\mathfrak{A} = age(T)$. If $\mathfrak{A}$ has the near-AP, then $\mathfrak{A}$ has the AP.

Proof. (1): This is immediate consequence of the fact that $\mathfrak{A}$ is ρ-compact.

(2): By Proposition 3.16 we deduce that $\mathfrak{A}$ is ρ-compact. Then (1) implies that $\mathfrak{A}$ has the AP. □

The following proposition shows that the near-AP is equivalent to the "1-point near-AP".

Proposition 6.8. *Let $\mathfrak{A}$ be an age. The following statements are equivalent:*

1. $\mathfrak{A}$ has the near-AP.

2. For every $\epsilon > 0$, $n \in \omega$, $\mathcal{M} \in \mathfrak{A}$, $p \in S^{\mathfrak{A}}_{n+1}$, and $\bar{a} = (a_1, \ldots, a_n) \subseteq M$ such that $\bar{a} \models p \restriction n$, there exists $\mathcal{N}$ in $\mathfrak{A}$ such that $\mathcal{M} \subseteq \mathcal{N}$ and $\bar{b} = (b_1, \ldots, b_{n+1}) \subseteq N$ such that $\bar{b} \models p$ and

$$\max_{1 \leq i \leq n} d^{\mathcal{N}}(a_i, b_i) < \epsilon.$$

Proof. (1)⇒(2): obvious.

(2)⇒(1): Fix $\epsilon > 0$, $n \in \omega$, $l \geq 1$, $\mathcal{M} \in \mathfrak{A}$, $p \in S_{n+l}$, and $\bar{a} \subseteq M$ with $\bar{a} \models p \restriction n$. We will define inductively a finite chain of finite structures $(\mathcal{N}_j \mid 0 \leq j \leq l)$ and finite tuples $(\bar{b}_j)_{0 \leq j \leq l}$ where for all $0 \leq j \leq l$

$$\bar{b}_j = (b_{j,1}, \ldots, b_{j,n+j})$$

such that the following properties are true:

- $\mathcal{N}_0 = \mathcal{M}$ and $\bar{b}_0 = \bar{a}$.
- For all $0 \leq j \leq j' \leq l$ we have that $\mathcal{N}_j \leq \mathcal{N}_{j'}$.
- For all $0 \leq j \leq l$ we have that $\bar{b}_j \subseteq \mathcal{N}_j$ and

$$\max_{1 \leq i \leq n+j} d^{\mathcal{N}_{j+1}}(b_{j+1,i}, b_{j,i}) < \frac{\epsilon}{l}.$$

For $n = 0$ we define $\mathcal{N}_0 = \mathcal{M}$ and $\bar{b}_0 = \bar{a}$. Suppose that for some $k < l$ we have defined $(\mathcal{N}_j \mid 0 \leq j \leq k)$ and $(\bar{b}_j \mid 0 \leq j \leq k)$ which satisfy the properties stated above for all $0 \leq j \leq j' \leq k$. Because $\mathfrak{A}$ has the near-AP there exist $\mathcal{N} \supseteq \mathcal{N}_k$ and $\bar{b} = (b_1, \ldots, b_{n+k+1}) \subseteq N$ such that $\bar{b} \models p \restriction k + 1$ and

$$\max_{1 \leq i \leq n+k} d^{\mathcal{N}}(b_i, b_{j,i}) < \frac{\epsilon}{l}.$$

Set $\mathcal{N}_{k+1} = \mathcal{N}$ and $\bar{b}_{k+1} = \bar{b}$. Then clearly for all $1 \leq i \leq n$

$$d^{\mathcal{N}(l)}(a_i, b_{l,i}) \leq d^{\mathcal{N}(l)}(b_{0,i}, b_{1,i}) + \cdots + d^{\mathcal{N}(l)}(b_{l-1,i}, b_{l,i}).$$

This implies that

$$\begin{aligned} \max_{1 \leq i \leq n} d^{\mathcal{N}(l)}(a_i, b_{l,i}) &\leq \max_{1 \leq i \leq n} d^{\mathcal{N}(l)}(b_{0,i}, b_{1,i}) + \cdots + \max_{1 \leq i \leq n} d^{\mathcal{N}(l)}(b_{l-1,i}, b_{l,i}) \\ &< l\frac{\epsilon}{l} = \epsilon. \end{aligned}$$

This suffices. $\square$

6.3 The $d^{\mathfrak{A}}$-metric

Definition 6.9. Let $\mathfrak{A}$ be an age. For every $n \in \omega$, and p, $q \in S_n^{\mathfrak{A}}$ we define

$$d_n^{\mathfrak{A}}(p,q) = \inf\{\max_{1\leq i\leq n} d^{\mathcal{M}}(a_i, b_i) \mid \mathcal{M} \in \mathfrak{A}, \quad \bar{a}, \bar{b} \subseteq M, \quad \mathcal{M} \models p(\bar{a}), \quad \mathcal{M} \models q(\bar{b})\}.$$

Remark 6.10. We note that if $\mathfrak{A}$ is an age, then for all $n \in \omega$, $p, q \in S_n^{\mathfrak{A}}$, we have that $d_n^{\mathfrak{A}}(p,q) < \infty$. This is an immediate consequence of the JEP.

Proposition 6.11. *Let $\mathfrak{A}$ be an age with the near-AP. Then for all $n \in \omega$, $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is a metric space.*

Proof. Fix $n \in \omega$. Clearly for every p, $q \in S_n^{\mathfrak{A}}$, $d_n(p,p) = 0$ and $d_n^{\mathfrak{A}}(p,q) = d_n^{\mathfrak{A}}(q,p)$. Clearly also $d^{\mathfrak{A}}(p,q) = 0$ implies that $p = q$.

Now we show that the triangle inequality holds. Suppose that p, q, r are in $S_n^{\mathfrak{A}}$. Set $d_n^{\mathfrak{A}}(p,q) = a$, $d_n^{\mathfrak{A}}(q,r) = b$. Fix $\epsilon > 0$. There exist $\mathcal{M} \in \mathfrak{A}$ and $\bar{a}$, $\bar{b} \subseteq M$ such that $\bar{a} \models p$, $\bar{b} \models q$ and

$$\max_{1\leq i\leq n} d^{\mathcal{M}}(a_i, b_i) < a + \epsilon.$$

Similarly there exist $\mathcal{N} \in \mathfrak{A}$ and $\bar{d}$, $\bar{e} \subseteq N$ such that $\bar{d} \models q$, $\bar{e} \models r$ and

$$\max_{1\leq i\leq n} d^{\mathcal{N}}(d_i, e_i) < b + \epsilon.$$

Set $s = \mathrm{qftp}_{\mathcal{N}}(\bar{d}, \bar{e})$. Note that $\mathcal{M} \models (s \restriction n)(\bar{b})$. Because $\mathfrak{A}$ has the near-AP, for the given $\epsilon > 0$ there exist $\mathcal{M}_1$ in $\mathfrak{A}$ with $\mathcal{M}_1 \supseteq \mathcal{M}$ and $\bar{f} = (f_1, \ldots, f_n)$, $\bar{g} = (g_1, \ldots, g_n) \subseteq M_1$ such that $(\bar{f}, \bar{g}) \models s$ and

$$\max_{1\leq i\leq n} d^{\mathcal{M}_1}(b_i, f_i) < \epsilon.$$

Note that $\bar{f} \models q$, $\bar{g} \models r$. Then, from the triangle inequality we have

$$\forall i \leq n \quad d^{\mathcal{M}_1}(a_i, g_i) \leq d^{\mathcal{M}_1}(a_i, b_i) + d^{\mathcal{M}_1}(b_i, f_i) + d^{\mathcal{M}_1}(f_i, g_i).$$

Combining the above inequalities we get

$$d_n^{\mathfrak{A}}(p,r) < (d_n^{\mathfrak{A}}(p,q) + \epsilon) + \epsilon + (d_n^{\mathfrak{A}}(q,r) + \epsilon).$$

The above is true for all $\epsilon > 0$. We deduce that

$$d_n^{\mathfrak{A}}(p,r) \leq d_n^{\mathfrak{A}}(p,q) + d_n^{\mathfrak{A}}(q,r).$$

We conclude that $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is a metric space. $\square$

Definition 6.12. Let $\mathfrak{A}$ be an age with the near-AP.

1. We say that $\mathfrak{A}$ is $d^{\mathfrak{A}}$-separable if for every $n \in \omega$, the metric space $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is separable.

2. We say that $\mathfrak{A}$ is $d^{\mathfrak{A}}$-complete if for every $n \in \omega$, the metric space $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is complete.

Proposition 6.13. *Let $\mathcal{M}$ be an L-structure which is ω-qf-near-homogeneous. Set $\mathfrak{A} = age(\mathcal{M})$. Then $\mathfrak{A}$ is a $d^{\mathfrak{A}}$-complete age with the near-AP.*

Proof. By Proposition 3.5 we have that $\mathfrak{A}$ is an age.

Now we show that $\mathfrak{A}$ has the near-AP. Fix $\epsilon > 0$ n, $l \in \omega$, $\mathcal{N} \in \mathfrak{A}$, $p \in S_{n+l}^{\mathfrak{A}}$, and $\bar{a} \subseteq N$ with $\bar{a} \models p \restriction n$. Because $\mathcal{M}$ is ω-qf-near-homogeneous we have that there exists $\bar{b} = (b_1, \ldots, b_{n+l})$ such that $\bar{b} \models p$ and

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(a_i, b_i) < \epsilon.$$

Let $\mathcal{N}' \supseteq \mathcal{N}$ be a finite L-substructure of $\mathcal{M}$ such that $\bar{b} \subseteq N'$.

Since $\mathfrak{A}$ has the near-AP, we deduce that for all $n \in \omega$, $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is a metric space. Now we show that $\mathfrak{A}$ is $d^{\mathfrak{A}}$-complete. We will need the following lemma.

Lemma 6.14. *Let $\mathcal{M}$ be an L-structure which is ω-qf-near-homogeneous. Put $\mathfrak{A} = age(\mathcal{M})$. If $\epsilon > 0$, p, q are in $S_n^{\mathfrak{A}}$, for some $n \in \omega$, with $d_n^{\mathfrak{A}}(p,q) < \epsilon$, and $\bar{a} \subseteq M$ is such that $\bar{a} \models p$, then there exists $\bar{b} \subseteq M$ such that $\bar{b} \models q$ and*

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(a_i, b_i) < \epsilon.$$

Proof. Fix $\epsilon >$, $n \in \omega$, $p, q \in S_n^{\mathfrak{A}}$, $\bar{a} \subseteq M$ with $\bar{a} \models p$. Because $d_n^{\mathfrak{A}}(p,q) < \epsilon$ we have that there exists $\mathcal{N} \in \mathfrak{A}$ and $\bar{c}, \bar{d} \subseteq N$ such that $\bar{c} \models p$, $\bar{d} \models q$ and

$$\max_{1 \leq i \leq n} d^{\mathcal{N}}(c_i, d_i) < \epsilon.$$

Set

$$\epsilon_1 = \max_{1 \leq i \leq n} d^{\mathcal{N}}(c_i, d_i) < \epsilon.$$

Because $\mathfrak{A} = \mathrm{age}(\mathcal{M})$ we may assume that $\mathcal{N} \subseteq \mathcal{M}$. Set $r = \mathrm{qftp}_{\mathcal{N}}(\bar{c}, \bar{d})$. Choose ϵ_2 such that

$$0 < \epsilon_2 < \epsilon - \epsilon_1.$$

Because $\bar{a} \models r \restriction n$, Proposition 6.3 implies that there exist $\bar{e} = (e_1, \ldots, e_{2n})$ such that $\bar{e} \models r$ and

$$\max_{1 \leq i \leq n} d^{\mathcal{N}}(a_i, e_i) < \epsilon_2.$$

Set $(b_1, \ldots, b_n) = (e_{n+1}, \ldots, e_{2n})$ and $\bar{b} = (b_1, \ldots, b_n)$. Clearly we have that $\bar{b} \models q$. For all $1 \leq i \leq n$ we have that

$$d^{\mathcal{M}}(a_i, b_i) \leq d^{\mathcal{M}}(a_i, e_i) + d^{\mathcal{M}}(e_i, b_i).$$

Therefore

$$d^{\mathcal{M}}(a_i, b_i) \leq d^{\mathcal{M}}(a_i, e_i) + d^{\mathcal{M}}(c_i, d_i).$$

We conclude that

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(a_i, b_i) \leq \max_{1 \leq i \leq n} d^{\mathcal{M}}(a_i, e_i) + \max_{1 \leq i \leq n} d^{\mathcal{M}}(c_i, d_i) < \epsilon_2 + \epsilon_1 < \epsilon.$$

□

Now we continue with the proof of Proposition 6.13 and we show that for all $n \in \omega$, $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is complete.

Fix $n \in \omega$. Let $(p_j \mid j \in \omega)$ be a Cauchy sequence in $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$. To prove completeness, we need only to consider sequences that satisfy

$$\forall j' \geq j \quad d_l^{\mathfrak{A}}(p_j, p_j') < \frac{1}{2^j}.$$

We define inductively a sequence $(\bar{a}_j \mid j \in \omega)$ in M such that the following properties are true.

1. For all $j \in \omega$, $\bar{a}_j \models p_j$.
2. For all $j \leq j'$,
$$\max_{1 \leq i \leq l} d^{\mathcal{N}}(a_{j,i}, a_{j',i}) < \frac{1}{2^j}.$$

For $j = 1$, pick $\bar{a}_1 \subseteq M$ such that $\bar{a}_1 \models p_1$. Assume we have defined $\bar{a}_j$ for all $j \leq k$ such that the above two conditions are true for all values of $j, j' \leq k$. We have that $\bar{a}(k) \models p_k$ and that

$$d(p_k, p_{k+1}) < \frac{1}{2^k}.$$

So from Lemma 6.14 there exists $\bar{a}_{k+1} \subseteq M$ such that

$$\max_{1\leq i\leq n} d^{\mathcal{N}}(a_{n+1,i}, a_{n,i}) < \frac{1}{2^k}.$$

This suffices for the the definition of the sequence $(\bar{a}_j \mid j \in \omega)$.

Then for every $0 \leq i \leq n$ consider the sequence $S_i = (a_{j,i} \mid j \in \omega)$. The second condition implies that S_i is a Cauchy sequence in $\mathcal{M}$. Because $\mathcal{M}$ is based on a complete metric space we have that $\lim_{j\to\infty} S_i$ exists in M. For all $0 \leq j \leq n$, set

$$b_i = \lim_{j\to\infty} S_i,$$

$\bar{b} = (b_1, \ldots, b_n)$ and $p = \mathrm{qftp}_{\mathcal{M}}(\bar{b})$. Clearly we have that $p \in S_n^{\mathfrak{A}}$ and that

$$\lim_{j\to\infty} d_n^{\mathfrak{A}}(p, p_j) = 0.$$

So $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is complete. □

Definition 6.15. Let (X, d_1), (X, d_2) be metric spaces.

1. We say that d_1 is finer than d_2 if for every $x \in X$, $\epsilon > 0$, there exists $\delta > 0$ such that for all $y \in X$,

$$d_1(x, y) < \delta \quad \text{implies} \quad d_2(x, y) < \epsilon.$$

2. We say that d_1 is uniformly finer than d_2 if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every x, $y \in X$

$$d_1(x, y) < \delta \quad \text{implies} \quad d_2(x, y) < \epsilon.$$

Proposition 6.16. *Let $\mathfrak{A}$ be an age with the near-AP. Then for all $n \in \omega$, $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is uniformly finer than $(S_n^{\mathfrak{A}}, \rho_n)$.*

Proof. It is enough to prove that for every $n \in \omega$ and $\epsilon > 0$ there exists $\delta > 0$ such that for every $p, q \in S_n^{\mathfrak{A}}$

$$d_n^{\mathfrak{A}}(p, q) < \delta \quad \text{implies} \quad \rho_n(p, q) < \epsilon.$$

Fix $n \in \omega$ and $\epsilon > 0$. Let m_P be the number of predicate symbols and m_C be the number of constant symbols in L.

For each $1 \leq i \leq m_P$, let Δ_{P_i} be the modulus of uniform continuity that L specifies for the predicate symbol P_i and let $a(i)$ be the arity of the predicate P_i. We will show that it suffices to choose $\delta > 0$ such that

$$\delta < \min\{\frac{\epsilon}{2}, \min_{1\leq i\leq m_P}\{\Delta_{P_i}(\epsilon)\}\}.$$

So assume that $p, q \in S_n^{\mathfrak{A}}$ and that

$$d_n^{\mathfrak{A}}(p,q) < \delta.$$

This implies that there exist $\mathcal{M} \in \mathfrak{A}$ and $\bar{a} = (a_1, \ldots, a_n) \subseteq M$, $\bar{b} = (b_1, \ldots, b_n) \subseteq M$ such that $\bar{a} \models p$, $\bar{b} \models q$ and

$$\max_{1\leq i\leq n} d^{\mathcal{M}}(a_i, b_i) < \delta.$$

Let $\bar{e} = (e_1, \ldots, e_{m_C+n}) = (c_1, \ldots, c_{m_C}, x_1, \ldots, x_n)$. Then there are finitely many atomic L-formulas in $\bar{x} = (x_1, \ldots, x_n)$ and they have the forms:

- $d(c_i, c_j)$ where $1 \leq i, j \leq m_C$
- $d(c_i, x_j)$ and $d(x_j, c_i)$ where $1 \leq i \leq m_C$, $1 \leq j \leq n$.
- $d(x_i, x_j)$ where $1 \leq i, j \leq n$.
- $P_l(e_{j_1}, \ldots, e_{j_{a(l)}})$ where $1 \leq l \leq m_P$ and $1 \leq j_1, \ldots, j_{a(l)} \leq m_C + n$.

To complete the proof we will need to prove some easy facts.

Claim 6.17. *1. For every i, j with $1 \leq i \leq m_C$, $1 \leq j \leq n$ we have that*

$$|d(c_i, x_j)^p - d(c_i, x_j)^q| < \epsilon$$

2. For every i, j with $1 \leq i, j \leq n$ we have that

$$|d(x_i, x_j)^p - d(x_i, x_j)^q| < \epsilon.$$

Proof. (of claim) Fix i, j with $1 \leq i \leq m_C$, $1 \leq j \leq n$ respectively. As above there are $\bar{a}, \bar{b} \subseteq M$, such that $\bar{a} \models p$, $\bar{b} \models q$ and

$$\max_{1\leq i\leq n} d^{\mathcal{M}}(a_i, b_i) < \delta \leq \frac{\epsilon}{2}.$$

(1): By the triangle inequality, for all $1 \leq i \leq m_C$, $1 \leq j \leq n$

$$d^{\mathcal{M}}(c_i^{\mathcal{M}}, a_j) \leq d^{\mathcal{M}}(c_i^{\mathcal{M}}, b_j) + d^{\mathcal{M}}(b_j, a_j) \Leftrightarrow$$

$$d^{\mathcal{M}}(c_i^{\mathcal{M}}, a_j) - d^{\mathcal{M}}(c_i^{\mathcal{M}}, b_j) < \frac{\epsilon}{2}.$$

Similarly, for all $1 \leq i \leq n$, $1 \leq j \leq m$,

$$d^{\mathcal{M}}(c_i^{\mathcal{M}}, b_j) - d^{\mathcal{M}}(c_i^{\mathcal{M}}, a_j) < \frac{\epsilon}{2}.$$

Combining the above inequalities we deduce that for all $1 \leq i \leq m_C$, $1 \leq j \leq n$,

$$|d^{\mathcal{M}}(c_i^{\mathcal{M}}, a_j) - d^{\mathcal{M}}(c_i^{\mathcal{M}}, b_j)| < \frac{\epsilon}{2}.$$

We deduce that

$$|d(c_i, x_j)^p - d(c_i, x_j)^q| < \frac{\epsilon}{2} < \epsilon.$$

(2): By the triangle inequality, for all $i, j \leq n$

$$d^{\mathcal{M}}(a_i, a_j) \leq d^{\mathcal{M}}(a_i, b_i) + d^{\mathcal{M}}(b_i, b_j) + d^{\mathcal{M}}(b_j, a_j) \quad \Leftrightarrow$$

$$d^{\mathcal{M}}(a_i, a_j) \leq \frac{\epsilon}{2} + d^{\mathcal{M}}(b_i, b_j) + \frac{\epsilon}{2} \quad \Leftrightarrow$$

$$d^{\mathcal{M}}(a_i, a_j) - d^{\mathcal{M}}(b_i, b_j) < \epsilon,$$

and similarly

$$d^{\mathcal{M}}(b_i, b_j) - d^{\mathcal{M}}(a_i, a_j) < \epsilon.$$

By combining the above two inequalities we deduce that for all $1 \leq i, j \leq n$,

$$|d^{\mathcal{M}}(a_i, a_j) - d^{\mathcal{M}}(b_i, b_j)| < \epsilon.$$

We conclude that

$$|d(x_i, x_j)^p - d(x_i, x_j)^q)| < \epsilon.$$

□

Claim 6.18. *For every $\epsilon > 0$, $1 \leq l \leq m_P$, $1 \leq j_1, \ldots, j_{a(l)} \leq m_C + n$,*

$$|P_l(e_{j_1}, \ldots, e_{j_{a(l)}})^p - P_l(e_{j_1}, \ldots, e_{j_{a(l)}})^q| < \epsilon.$$

Proof. (of claim) Fix $\epsilon > 0$, $1 \le l \le m_P$, $1 \le j_1, \dots, j_{a(l)} \le m_C + n$. We have that $\bar{a}, \bar{b} \subseteq M$, $\bar{a} \models p$, $\bar{b} \models q$ and that

$$\max_{1 \le i \le n} d^{\mathcal{M}}(a_i, b_i) < \delta \le \Delta_{P_l}(\epsilon).$$

Set

$$\bar{f} = (f_1, \dots, f_{m_C+n}) = (c_1, \dots, c_{m_C}, a_1, \dots, a_n),$$

$$\bar{g} = (g_1, \dots, g_{m_C+n}) = (c_1, \dots, c_{m_C}, b_1, \dots, b_n).$$

Clearly

$$\max_{1 \le i \le m_C+n} d^{\mathcal{M}}(f_i, g_i) < \Delta_{P_l}(\epsilon).$$

This implies that

$$|P_l^{\mathcal{M}}(f_{j_1}, \dots, f_{j_{a(l)}}) - P_l^{\mathcal{M}}(g_{j_1}, \dots, g_{j_{a(l)}})| < \epsilon,$$

which in turns implies that

$$|P_l(e_{j_1}, \dots, e_{j_{a(l)}})^p - P_l(e_{j_1}, \dots, e_{j_{a(l)}})^q| < \epsilon.$$

□

It is easy to see that Claim 6.17 and Claim 6.18 imply that $\rho_n(p, q) < \epsilon$. □

6.4 Completion of ω-qf-near-homogeneous metric prestructures

Proposition 6.19. *Let $\mathcal{M}$ be a metric L-prestructure. Set $\mathfrak{A} = age(\mathcal{M})$. Suppose that the following conditions are true:*

- *$\mathcal{M}$ is ω-qf-near-homogeneous.*
- *For every $l \ge 1$, if $(p_n \mid n \ge 1)$, $(q_n \mid n \ge 1)$ are Cauchy sequences in $(S_l^{\mathfrak{A}}, d_l^{\mathfrak{A}})$, then*

$$\lim_{n\to\infty} \rho_l(p_n, q_n) = 0 \quad \textit{implies} \quad \lim_{n\to\infty} d_l^{\mathfrak{A}}(p_n, q_n) = 0.$$

If we set $\overline{\mathcal{M}}$ to be the completion of $\mathcal{M}$, then $\overline{\mathcal{M}}$ is ω-qf-near-homogeneous.

Proof. Fix $\epsilon > 0$ and $l \in \omega$. Let $\bar{a} = (a_1, \dots, a_l)$, $\bar{b} = (b_1, \dots, b_{l+1})$ in $\overline{M}$ be such that $\bar{a} \models q \restriction l$ where $q = \text{qftp}_{\mathcal{M}}(\bar{b})$. For all $n \in \omega$ let $\bar{a}_n = (a_{n,1}, \dots, a_{n,l})$, $\bar{b}_n = (b_{n,1}, \dots, b_{n,l+1})$ be in M such that for all

$1 \leq i \leq l$, $\lim_{n\to\infty} a_{n,i} = a_i$, and for all $1 \leq i \leq l+1$, $\lim_{n\to n} b_{n,i} = b_i$. Set

$$p_n = \mathrm{qftp}_{\mathcal{M}}(a_{n,1}, \ldots, a_{n,l}), \quad q_n = \mathrm{qftp}_{\mathcal{M}}(b_{n,1}, \ldots, b_{n,l+1}).$$

For all $1 \leq i \leq l$, $(a_{n,i} \mid n \geq 1)$, $(b_{n,i} \mid n \geq 1)$ are Cauchy sequences in M. So $(p_n \mid n \geq 1)$ is a Cauchy sequence in $(S_l^{\mathfrak{A}}, d_l^{\mathfrak{A}})$ and $(q_n \mid n \geq 1)$ is a Cauchy sequence in $(S_{l+1}^{\mathfrak{A}}, d_{l+1}^{\mathfrak{A}})$. Clearly we have that $(q_n \restriction l \mid n \geq 1)$ is a Cauchy sequence in $(S_l^{\mathfrak{A}}, d_l^{\mathfrak{A}})$ and that

$$\lim_{n\to\infty} \rho_l(p_n, q_n \restriction l) = 0.$$

By the hypothesis we get

$$\lim_{n\to\infty} d_l^{\mathfrak{A}}(p_n, q_n \restriction l) = 0.$$

We define inductively a sequence of $(l+1)$-tuples $(\bar{e}_n \mid n \geq 1)$ in M, where for all $n \geq 1$ $\bar{e}_n = (e_{n,1}, \ldots, e_{n,l+1})$, such that if we set $s_n = \mathrm{qftp}_{\mathcal{M}}(\bar{e}_n)$ then the following conditions are true:

1. $(s_n \mid n \geq 1)$ is a subsequence of $(q_n \mid n \geq 1)$; there exists an increasing sequence of natural numbers $(n(k) \mid k \geq 1)$ such that for all $k \geq 1$, $s_k = q_{n(k)}$.

2. for every i, $j \geq n(k)$, $k \geq 1$,

$$d_{l+1}^{\mathfrak{A}}(q_i, q_j) < \frac{\epsilon}{2^{k+2}}.$$

3.

$$\max_{1\leq i\leq l} d^{\overline{\mathcal{M}}}(a_i, e_{1,i}) < \frac{\epsilon}{2} + \frac{\epsilon}{2^2}.$$

4. for all $n \geq 1$,

$$\max_{1\leq i\leq l} d^{\mathcal{M}}(e_{n,i}, e_{n+1,i}) < \frac{\epsilon}{2^{n+2}}.$$

Definition of $\bar{e}_1$: For all $i = 1, \ldots, l$, $\lim_{n\to\infty} a_{n,i} = a_i$, so there exists $n'(1) \geq 1$ such that for all $n \geq n'(1)$

$$\max_{1\leq i\leq l} d^{\overline{\mathcal{M}}}(a_i, a_{n,i}) < \frac{\epsilon}{2}. \tag{1}$$

Also,

$$\lim_{n\to\infty} d_n^{\mathfrak{A}}(p_n, q_n \restriction l) = 0,$$

so there exists $n''(1) \geq 1$ such that for all $n \geq n''(1)$

$$d_l^{\mathfrak{A}}(p_n, q \restriction l_n) < \frac{\epsilon}{2^2}.$$

Further, since $(q_n \mid n \geq 1)$ is a Cauchy sequence in $(S_{l+1}^{\mathfrak{A}}, d_{l+1}^{\mathfrak{A}})$, there exists $n'''(1) \geq 1$ so that for all $i, j \geq n'''(1)$

$$d_{l+1}^{\mathfrak{A}}(q_i, q_j) < \frac{\epsilon}{2^3}.$$

Set $n(1) = \max\{n'(1), n''(1), n'''(1)\}$. We have that $\bar{a}_{n(1)} \models p_{n(1)}$. Because

$$d_l^{\mathfrak{A}}(p_{n(1)}, q_{n(1)} \restriction l) < \frac{\epsilon}{2^2}$$

by Lemma 6.14 there exists $\bar{e}_1 = (e_{1,1}, \ldots, e_{1,l+1})$ in M such that $\bar{e}_1 \models q_{n(1)}$ and

$$\max_{1 \leq i \leq l} d^{\mathcal{M}}(a_{n(1),i}, e_{1,i}) < \frac{\epsilon}{2^2}. \tag{2}$$

From (1), (2) and the triangle inequality we deduce that

$$\max_{1 \leq i \leq l} d^{\overline{\mathcal{M}}}(a_i, e_{1,i}) < \frac{\epsilon}{2} + \frac{\epsilon}{2^2}.$$

Inductive step: Assume that we have defined $(l+1)$-tuples $\bar{e}_n$ in M for all $1 \leq n \leq k$ such that the conditions (1)-(4) are satisfied. We will define tuple $\bar{e}_{k+1}$. Since $(q_n \mid n \geq 1)$ is a Cauchy sequence in $(S_{l+1}^{\mathfrak{A}}, d_{l+1}^{\mathfrak{A}})$, there exists $n(k+1) \geq n(k)$ such that for all $i, j \geq n(k+1)$

$$d_{l+1}^{\mathfrak{A}}(q_i, q_j) < \frac{\epsilon}{2^{k+3}}.$$

Then by the induction hypothesis

$$d_{l+1}^{\mathfrak{A}}(q_{n(k+1)}, q_{n(k)}) < \frac{\epsilon}{2^{k+2}}.$$

Because $\bar{e}_k \models q_{n(k)}$, by Lemma 6.14 there exists a tuple $\bar{e}_{k+1}$ in M such that $\bar{e}_{k+1} \models q_{n(k+1)}$ and

$$\max_{1 \leq i \leq l+1} d^{\overline{\mathcal{M}}}(e_{k+1,i}, e_{k,i}) < \frac{\epsilon}{2^{k+2}}.$$

This suffices for the construction of the sequence $(\bar{e}_n \mid n \geq 1)$.

For all $1 \le i \le l+1$, $(e_{n,i} \mid n \ge 1)$ is a Cauchy sequence in M, and since $\overline{\mathcal{M}}$ is complete, $(e_{n,i} \mid n \ge 1)$ converges in $\overline{M}$. For all $1 \le i \le l$, let $e_i = \lim_{n\to\infty} e_{n,i}$ and $\bar{e} = (e_1, \ldots, e_{l+1})$. To complete the proof that $\overline{\mathcal{M}}$ is ω-qf-near-homogeneous, it is enough to prove the following claim:

Claim 6.20. *1. $qftp_{\overline{\mathcal{M}}}(\bar{e}) = q$.*

2. $\max_{1\le i\le l} d^{\overline{\mathcal{M}}}(a_i, e_i) \le \epsilon$.

We prove the claim.

(1): Set $r = \mathrm{qftp}_{\overline{\mathcal{M}}}(\bar{e})$. Then we have that

$$\lim_{n\to\infty} \rho_{l+1}(q, q_n) = \lim_{n\to\infty} \rho_{l+1}(r, q_n) = 0.$$

Therefore $\rho_{l+1}(q, r) = 0$, and because $(S^{\mathfrak{A}}_{l+1}, \rho_{l+1})$ is a metric space, we conclude that $q = r$.

(2): From conditions (3), (4) and the triangle inequality we have that for all $1 \le i \le l$, $n \ge 1$

$$d^{\overline{\mathcal{M}}}(a_i, e_{n,i}) \le d^{\overline{\mathcal{M}}}(a_i, e_{1,i}) + \sum_{j=1}^{n-1} d^{\overline{\mathcal{M}}}(e_{j,i}, e_{j+1,i}) < \sum_{j=1}^{n+1} \frac{\epsilon}{2^j}.$$

Therefore, for all $1 \le i \le l$,

$$d^{\overline{\mathcal{M}}}(a_i, e_i) = \lim_{n\to\infty} d^{\overline{\mathcal{M}}}(a_i, e_{n,i}) \le \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon.$$

We conclude that

$$\max_{1\le i\le l} d^{\overline{\mathcal{M}}}(a_i, e_i) \le \epsilon.$$

□

6.5 Fraïssé theory for separable structures

In this section our main goal is to prove Theorem 6.28, which is the main theorem of this chapter. First we prove some propositions.

Proposition 6.21. *Let $\mathfrak{A}$ be an age with the UPAP. If $\mathfrak{B}$ is the completion of $\mathfrak{A}$ (as defined in Definition 3.21), then $\mathfrak{B}$ is a ρ-compact age with the UPAP.*

Proof. The fact that $\mathfrak{B}$ is a ρ-compact age follows from Proposition 3.23.

We will show that for all $n \in \omega$ we may choose $\delta^{\mathfrak{B}}_n = \delta^{\mathfrak{A}}_n$ where $\delta^{\mathfrak{A}}_n$, $\delta^{\mathfrak{B}}_n$ are defined in Definition 5.2. Fix $n \in \omega$, $\epsilon > 0$, $p \in S^{\mathfrak{B}}_{n+1}$, $\mathcal{M} \in \mathfrak{B}$, and $\bar{a} \subseteq M$ such that $\rho_n(p \restriction n, q) < \delta^{\mathfrak{A}}_n(\epsilon)$ where $q = \mathrm{qftp}_{\mathcal{M}}(\bar{a})$. Set

$\rho_n(p\restriction n, q) = \delta$. Let $\bar{b}$ be an enumeration of $M \setminus \bar{a}$ and l the length of $\bar{b}$. Set $\bar{c} = \bar{b}\bar{a}$ and $s = \mathrm{qftp}_{\mathcal{M}}(\bar{c})$. Let $m_0 \geq 1$ be such that

$$\delta + \frac{2}{m_0} < \delta_n^{\mathfrak{A}}(\epsilon).$$

Since $\mathfrak{B}$ is the completion of $\mathfrak{A}$ and $s \in S_{l+n}^{\mathfrak{B}}$, $p \in S_{n+1}^{\mathfrak{B}}$, for every $m \geq m_0$ there exist $s_m \in S_{l+n}^{\mathfrak{A}}$ and $p_m \in S_{n+1}^{\mathfrak{A}}$ such that

$$\rho_{l+n}(s, s_m) < \frac{1}{m}, \quad \rho_{n+1}(p, p_m) < \frac{1}{m}.$$

Fix $m \geq m_0$. Let c_m be a realization of s_m in an L-structure $\mathcal{A}$ and let $\mathcal{M}_m$ be a the L-structure generated by c_m in $\mathcal{A}$. Then $\mathcal{M}_m \in \mathfrak{A}$. Set $q_m = \mathrm{qftp}_{\mathcal{M}_m}(c_{l+1}, \ldots, c_{l+n})$. We have that

$$\begin{aligned} \rho_n(p_m\restriction n, q_m) &\leq \rho_n(p_m\restriction n, p\restriction n) + \rho_n(p\restriction n, q) + \rho_n(q, q_m) \\ &< \frac{1}{m} + \delta + \frac{1}{m} \\ &< \delta_n^{\mathfrak{A}}(\epsilon). \end{aligned}$$

Since $\mathfrak{A}$ has the UPAP, there exist $\mathcal{N}_m \in \mathfrak{A}$ such that $\mathcal{M}_m \subseteq \mathcal{N}_m$ and $d_m \in N_m$ such that

$$\rho_{n+1}(p_m, r_m) \leq \epsilon,$$

where $r_m = \mathrm{qftp}_{\mathcal{M}_m}(c_{m,l+1}, \ldots, c_{m,l+n}, d_m)$. Without losing generality we may assume that $N_m = \bar{c}_m \cup \{d_m\}$. Set $t_m = \mathrm{qftp}_{\mathcal{M}_m}(\bar{c}_m, d_m)$. Then the sequence $\{t_m \mid m \geq m_0\}$ is in $S_{l+n+1}^{\mathfrak{A}} \subseteq S_{l+n+1}^{\mathfrak{B}}$ and since $(S_{l+n+1}^{\mathfrak{B}}, \rho_{l+n+1})$ is compact we deduce that it has a limit point t. Let $\bar{e} \models t$ and $\mathcal{N}$ be the L-structure generated by $\bar{e}$. Set $r = \mathrm{qftp}_{\mathcal{N}}(e_{l+1}, \ldots, e_{l+n+1})$. We note that

$$\begin{aligned} \rho_{n+1}(p, r_m) &\leq \rho_{n+1}(p, p_m) + \rho_{n+1}(p_m, r_m) \\ &< \frac{1}{m} + \epsilon \end{aligned}$$

We deduce that

$$\rho_{n+1}(p, r) = \lim_{m\to\infty} \rho_{n+1}(p, r_m) \leq \epsilon.$$

Note that $M = \bar{b} \cup \bar{a}$. We define an embedding $f : \mathcal{M} \to \mathcal{N}$ such that for all $1 \leq i \leq l$, $f(b_i) = e_i$; for all $1 \leq i \leq n$, $f(a_i) = e_{l+i}$. Then without losing generality we may assume that $\mathcal{M} \subseteq \mathcal{N}$, and as we noted above $\rho_{n+1}(p, r) \leq \epsilon$ where $r = \mathrm{qftp}_{\mathcal{N}}(e_{l+1}, \ldots, e_{l+n})$.

□

Proposition 6.22. *Let $\mathfrak{A}$ be an age. The following conditions are equivalent.*

1. *$\mathfrak{A}$ has the UPAP.*

2. *For every $n \in \omega$, $\epsilon > 0$ there exists $\delta_n^{\mathfrak{A}}(\epsilon) > 0$ such that for every $\mathcal{M} \in \mathfrak{A}$ whenever we are given*

   - $p(x) \in S_{n+1}^{\mathfrak{A}}$
   - $\bar{a} = (a_1, \ldots, a_n) \subseteq M$.

   then there exist $\mathcal{N}$ in $\mathfrak{A}$ such that $\mathcal{M} \subseteq \mathcal{N}$ and a_{n+1} in N such that

$$\min(\delta_n^{\mathfrak{A}}(\epsilon) \dot{-} \tau_{p\restriction n}(\bar{a}), \tau_p(\bar{a}, a_{n+1}) \dot{-} \epsilon) = 0.$$

   is true in $\mathcal{N}$.

Proof. Immediate from the Definition 5.2. □

Proposition 6.23. *Let $\mathfrak{A}$ be a $d^{\mathfrak{A}}$-separable age with the near-AP. There exists a countable metric L-prestructure $\mathcal{N}$ such that if we set $\mathfrak{B} = age(\mathcal{N})$, then the following properties are true:*

1. *$\mathfrak{B} \subseteq \mathfrak{A}$.*

2. *$\mathcal{N}$ is ω-qf-near-homogeneous.*

3. *For every $n \in \omega$, for every $p, q \in S_n^{\mathfrak{B}}$, $d_n^{\mathfrak{B}}(p, q) = d_n^{\mathfrak{A}}(p, q)$.*

4. *For every $n \in \omega$, $S_n^{\mathfrak{B}}$ is dense $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$.*

5. *If, in addition, L is a signature whose nonlogical symbols are a finite number of constant and predicate symbols, $\mathfrak{A}$ has the UPAP, and we set $\mathfrak{D}$ to be the completion of $\mathfrak{A}$, then $\mathcal{N}$ can be chosen such that $\mathcal{N} \models T^{\mathfrak{D}}$.*

Proof. For every $n \in \omega$, $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is separable. So there exists a countable $\mathfrak{C} \subseteq \mathfrak{A}$ such that for every $n \in \omega$, $S_n^{\mathfrak{C}}$ is dense in $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$. We note that $\mathfrak{C}$ is not necessarily an age. Let $(\mathcal{M}_n)_{n\in\omega}$ be an enumeration of a set of representatives of $\mathfrak{C}$, up to isomorphism; that is, every element of $\mathfrak{C}$ is isomorphic to an element of $(\mathcal{M}_n)_{n\in\omega}$. By induction we will define a chain of finite L-structures $(\mathcal{N}_n \mid n \in \omega)$ such that the following conditions are true:

1. For every $n \in \omega$, $\mathcal{M}_n$ can be embedded in $\mathcal{N}_{n+1}$.

2. For every $n \in \omega$, if $p, q \in S_l^{\text{age}(\mathcal{N}_n)}$ for some $l \in \omega$, then there are $\bar{a}, \bar{b} \subseteq N_{n+1}$ such that $\bar{a} \models p$, $\bar{b} \models q$ and

$$\max_{1 \le i \le l} d^{\mathcal{N}_{n+1}}(a_i, b_i) < d_l^{\mathfrak{A}}(p, q) + \frac{1}{n}.$$

3. For every $n \in \omega$, if $p \in S_{l+1}^{\text{age}(\mathcal{N}_n)}$ for some $l \in \omega$, and $\bar{a} \subseteq N_n$ with $\bar{a} \models p \restriction l$, then there exists $\bar{b} \subseteq N_{n+1}$ such that $\bar{b} \models p$ and

$$\max_{1 \le i \le l} d^{\mathcal{N}_{n+1}}(a_i, b_i) < \frac{1}{n}.$$

Now we define the chain of finite L-structures $(\mathcal{N}_n \mid n \in \omega)$. To do so we need some notation and some lemmas.

Notation 6.24. Let $\mathfrak{A}$, $(\mathcal{M}_n)_{n \in \omega}$ as above. Let $\mathcal{M} \in \mathfrak{A}$ and $n \in \omega$. Set $I_n(\mathcal{M})$ to be a structure in $\mathfrak{A}$ into which $\mathcal{M}$ and $\mathcal{M}_n$ can both be embedded. Such a structure exists since $\mathfrak{A}$ has the JEP.

Lemma 6.25. *Let $\mathfrak{A}$ with the near-AP and $\mathcal{M} \in \mathfrak{A}$. For every $n \in \omega$ there is a structure $E_n(\mathcal{M})$ in $\mathfrak{A}$ such that $\mathcal{M} \subseteq E_n(\mathcal{M})$ and such that if $p, q \in S_l^{age(\mathcal{M})}$ for some $l \in \omega$, then there exist $\bar{a}, \bar{b} \subseteq E_n(M)$ such that $\bar{a} \models p$, $\bar{b} \models q$ and*

$$\max_{1 \le i \le l} d^{E_n(\mathcal{M})}(a_i, b_i) < d_l^{\mathfrak{A}}(p, q) + \frac{1}{n}.$$

Proof. (of lemma) Fix $n \in \omega$. Let $(p_j, q_j)_{j \le k}$ be an enumeration of the set

$$\{(p, q) \mid \exists l \in \omega \quad with \quad p, q \in S_l^{\text{age}(\mathcal{M})}\},$$

for some $k \in \omega$. Note that the above set is finite because $\mathcal{M}$ is finite. By induction we define a chain of structures $(\mathcal{A}_j \mid j \le k+1)$ in $\mathcal{A}$ by

$$\mathcal{A}_0 = \mathcal{M}$$

$$\mathcal{A}_{j+1} = \mathcal{B} \in \mathfrak{A}$$

where $\mathcal{B} \supseteq \mathcal{A}_j$ is such that there exists $\bar{a}$, $\bar{b}$ in B with $\bar{a} \models p_j$, $\bar{b} \models q_j$, and (assuming that p_j, q_j are l-types)

$$\max_{1 \le i \le l} d^{\mathcal{B}}(a_i, b_i) < d_l^{\mathfrak{A}}(p, q) + \frac{1}{n}.$$

Note that $\mathcal{A}_{j+1}$ always exists from the definition of the metric $d_l^{\mathfrak{A}}$ and the fact that $\mathfrak{A}$ satisfies the JEP. Set

$$E_n(\mathcal{M}) = \bigcup_{j \le k+1} \mathcal{A}_j = \mathcal{A}_{k+1}.$$

□

Lemma 6.26. *Let $\mathfrak{A}$ be an age with the near-AP and $\mathcal{M} \in \mathfrak{A}$. For every $n \in \omega$ there is a structure $H_n(\mathcal{M})$ in $\mathfrak{A}$ such that $\mathcal{M} \subseteq H_n(\mathcal{M})$, and such that if $p \in S_{l+1}^{age(\mathcal{M})}$ for some $l \in \omega$ and $\bar{a} \subseteq M$ with $\bar{a} \models p \restriction l$, then there exists $\bar{b} \subseteq H_n(M)$ such that $\bar{b} \models p$ and*

$$\max_{i \leq i \leq l} d^{H_N(\mathcal{M})}(a_i, b_i) < \frac{1}{n}.$$

Proof. (of lemma) Fix $n \in \omega$. Let $((p_j, \bar{a}_j))_{j \leq k}$ be an enumeration of the set

$$\{(p, \bar{a}) \mid \exists l \in \omega \quad with \quad p \in S_{l+1}^{\text{age}(\mathcal{M})} \quad and \quad \bar{a} \models p \restriction l\},$$

for some $k \in \omega$. We write $\bar{a}_j$ for $(a_{j,1}, \ldots, a_{j,l})$. Note that the above set is finite because $\mathcal{M}$ is finite. By induction we define a finite chain of finite structures $(\mathcal{A}_j \mid j \leq k+1)$ in $\mathfrak{A}$ by

$$\mathcal{A}_0 = \mathcal{M}$$

$$\mathcal{A}_{j+1} = \mathcal{B} \in \mathfrak{A}$$

where $\mathcal{B} \supseteq A_j$ is such that there exists $\bar{b} \subseteq B$ with $\bar{b} \models p_j$, and (assuming p_j is an $(l+1)$-type)

$$\max_{1 \leq i \leq l} d^{\mathcal{B}}(a_{j,i}, b_i) < \frac{1}{n}.$$

Note that $\mathcal{A}_{j+1}$ always exists because of the near-AP of $\mathfrak{A}$. Set

$$H_n(M) = \bigcup_{j \leq k+1} \mathcal{A}_j = \mathcal{A}_{k+1}.$$

□

Now we define by induction on n the chain $(\mathcal{N}_n)_{n \in \omega}$ announced at the beginning of the proof of Proposition 6.23. Let $\mathcal{M} \in \mathfrak{A}$. Set

$$\mathcal{N}_0 = \mathcal{M}$$

$$\mathcal{N}_{n+1} = H_n E_n I_n(\mathcal{N}_n).$$

Then $(\mathcal{N}_n)_{n\in\omega}$ clearly satisfy the three conditions stated above. Put

$$\mathcal{N} = \bigcup_{n\in\omega} \mathcal{N}_n$$

and $\mathfrak{B} = \mathrm{age}(\mathcal{N})$. Clearly $\mathcal{N}$ is countable.

Now we prove that $\mathcal{N}$ satisfies the conditions in Proposition 6.23.

(1): Clearly, by construction $\mathfrak{B} \subseteq \mathfrak{A}$.

(2): Suppose that $p \in S^{\mathfrak{B}}_{l+1}$ for some $l \in \omega$ and $\bar{a} \subseteq N$ with $\bar{a} \models p\restriction l$. Then there exists $j_1 \in \omega$ such that $\bar{a} \subseteq N(j_1)$. Pick $j_2 \in \omega$ such that $\frac{1}{j_2} < \epsilon$. Set $j = \max\{j_1, j_2\}$. Then $\bar{a} \subseteq N(j)$ and there exists $\bar{b} \in N_{j+1}$ such that $\bar{b} \models p$ and $\max_{1\leq i\leq l} d^{\mathcal{N}}(a_i, b_i) < \epsilon$. This proves that $\mathcal{N}$ is ω-qf-near-homogeneous.

(3): We have that (2) implies that $\mathfrak{B}$ is an age with the near-AP. Because $\mathfrak{B} \subseteq \mathfrak{A}$ we have that for every $l \in \omega$, for every $p, q \in S^{\mathfrak{B}}_l$

$$d^{\mathfrak{B}}_l(p, q) \geq d^{\mathfrak{A}}_l(p, q).$$

By condition (3) we get

$$d^{\mathfrak{B}}_l(p, q) = d^{\mathfrak{A}}_l(p, q).$$

(4): This is evident from the first condition at the beginning of the proof of Proposition 6.23 that the chain $(\mathcal{N}_n \mid n \in \omega)$ satisfies.

(5): Let $\mathfrak{D}$ be the completion of $\mathfrak{A}$. From Proposition 6.21 we have that $\mathfrak{D}$ is a ρ-compact age with the UPAP. To show that $\mathcal{N} \models T^{\mathfrak{D}}$ it is enough to show that $\mathcal{N} \models \Sigma^{\mathfrak{D}}$ and $\mathcal{N} \models \Psi^{\mathfrak{D}}$. Since $\mathrm{age}(\mathcal{N}) \subseteq \mathfrak{B} \subseteq \mathfrak{D}$ and $\mathfrak{D}$ is a ρ-compact age, by Proposition 3.17 we deduce that $\mathcal{N} \models \Sigma^{\mathfrak{D}}$. Therefore it suffices to modify the construction above to ensure that in addition $\mathcal{N} \models \Psi^{\mathfrak{D}}$.

Let Σ be the set of closed L-conditions

$$\sup_{\bar{x}} \min(\delta^{\mathfrak{D}}_m(1/l) \dot{-} \tau_{p\restriction m}(\bar{x}), \inf_{x_{m+1}} \tau_p(\bar{x}, x_{m+1}) \dot{-} (1/l)) = 0$$

for $l, m \in \omega$, and $p \in S^{\mathfrak{D}}_m$, where $\delta^{\mathfrak{D}}_m$ is defined in Definition 5.2. Then Σ is countable and is clearly equivalent to the theory $\Psi^{\mathfrak{D}}$. Let $(\sigma)_{i\in\omega}$ be an enumeration of the set Σ and set $\Sigma_n = \{\sigma_i \mid i \leq n\}$.

To ensure that $\mathcal{N} \models \Sigma$ we will define, as above, a chain of finite L-structures $(\mathcal{N}_n \mid n \in \omega)$ which satisfies the three conditions stated at the beginning of the proof of Proposition 6.23 and, in addition, also satisfies the following condition:

- For every $n \in \omega$, $\sigma \in \Sigma_n$, and $\bar{a} \subseteq N_n$, if σ is the condition

$$\sup_{\bar{x}} \min(\delta_m^{\mathfrak{D}}(1/l) \dot{-} \tau_{p|m}(\bar{x}), \inf_{x_{m+1}} \tau_p(\bar{x}, x_{m+1}) \dot{-} (1/l)) = 0$$

for some $l, m \in \omega$, $p \in S_m^{\mathfrak{A}}$, and $\bar{a}$ is an m-tuple, then there exists $a_{m+1} \in N_{n+1}$ such that

$$\min(\delta_m^{\mathfrak{D}}(1/l) \dot{-} \tau_{p|m}(\bar{a}), \inf_{x_{m+1}} \tau_p(\bar{a}, a_{m+1}) \dot{-} (1/l)) = 0$$

is true in $\mathcal{N}$.

We will need the following lemma.

Lemma 6.27. *Let $\mathfrak{A}$ be an age which satisfies the near-AP and UPAP. Let Σ be as defined above, and take $\mathcal{M} \in \mathfrak{A}$. For every $n \in \omega$, there is an L-structure $G_n(\mathcal{M}) \in \mathfrak{A}$ such that for every $\sigma \in \Sigma_n$, and $\bar{a} \subseteq M$, if σ is the sentence*

$$\sup_{\bar{x}} \min(\delta_m^{\mathfrak{A}}(1/l) \dot{-} \tau_{p|m}(\bar{x}), \inf_{x_{m+1}} \tau_p(\bar{x}, x_{m+1}) \dot{-} (1/l)) = 0$$

for some $l, m \in \omega$, $p \in S_m^{\mathfrak{A}}$, and $\bar{a}$ is an m-tuple, then there exists $a_{m+1} \in G_n(M)$ such that

$$\min(\delta_m^{\mathfrak{A}}(1/l) \dot{-} \tau_{p|m}(\bar{a}), \inf_{x_{m+1}} \tau_p(\bar{a}, a_{m+1}) \dot{-} (1/l)) = 0$$

is true in $G_n(\mathcal{M})$.

Proof. The lemma is implied by Proposition 6.22 and the near-AP. The details of this proof are similar to the details of the proof of Proposition 5.13. □

Now we prove (5). We define by induction the chain $(\mathcal{N}_n)_{n\in\omega}$. Let $\mathcal{M} \in \mathfrak{A}$ and set

$$\mathcal{N}_0 = \mathcal{M}$$

$$\mathcal{N}_{n+1} = G_n H_n E_n I_n(\mathcal{N}_n).$$

Then $(\mathcal{N}_n)_{n\in\omega}$ clearly satisfy conditions (1)-(4) stated the beginning of the proof 6.23. Set $\mathcal{N} = \bigcup_{n\in\omega} \mathcal{N}_n$. Then $\mathcal{N}$ is a metric L-prestructure and satisfies conditions (1)-(4) of Proposition 6.23. □

Here is the main theorem of this chapter.

Theorem 6.28. *Let $\mathfrak{A}$ be a $d^{\mathfrak{A}}$-complete, $d^{\mathfrak{A}}$-separable age with the near-AP. Then there exists a unique separable L-structure $\mathcal{M}$ which satisfies the following properties:*

1. $\mathcal{M}$ *is strongly* ω*-qf-near-homogeneous.*

2. *age*$(\mathcal{M}) = \mathfrak{A}$*.*

3. *If, in addition,* L *is a signature whose only nonlogical symbols are a finite number of constant and predicate symbols, and* $\mathfrak{A}$ *has the UPAP, then Th*$(\mathcal{M})$ *has QE.*

Proof. By Proposition 6.23 we have that there exists a metric L-prestructure $\mathcal{N}$ such that if we set $\mathfrak{B} =$ age$(\mathcal{N})$ then the following properties are true:

- $\mathcal{N}$ is countable.
- $\mathcal{N}$ is ω-qf-near-homogeneous.
- $\mathfrak{B} \subseteq \mathfrak{A}$.
- For every $n \in \omega$, $p, q \in S_n^{\mathfrak{B}}$, $d_n^{\mathfrak{B}}(p,q) = d_n^{\mathfrak{A}}(p,q)$.

Set $\mathcal{M} = \overline{\mathcal{N}}$. The following obvious fact will be useful.

Fact 6.29. *If* $\mathfrak{A}$*,* $\mathfrak{B}$ *are ages with the near-AP and* $\mathfrak{B} \subseteq \mathfrak{A}$*, then for every* $n \in \omega$ *and* $p, q \in S_n^{\mathfrak{A}}$ *we have that* $d_n^{\mathfrak{B}}(p,q) \geq d_n^{\mathfrak{A}}(p,q)$*.*

(1): We will use Proposition 6.19. By definition $\mathcal{N}$ is ω-qf-near-homogeneous. Now we show that $\mathcal{N}$ satisfies the second condition in Proposition 6.19.

Fix $l \in \omega$ and let $S = (p_n \mid n \geq 1)$, $S' = (q_n \mid n \geq 1)$ be Cauchy sequences in $(S_l^{\mathfrak{B}}, d_l^{\mathfrak{B}})$ such that $\lim_{n\to\infty} \rho_l(p_n, q_n) = 0$. By Fact 6.29 we deduce that S and S' are Cauchy sequences in $(S_l^{\mathfrak{A}}, d_l^{\mathfrak{A}})$. Since $(S_l^{\mathfrak{A}}, d_l^{\mathfrak{A}})$ is complete and $\lim_{n\to\infty} \rho_l(p_n, q_n) = 0$ we deduce that there exists $r \in S_l^{\mathfrak{A}}$ such that

$$\lim_{n\to\infty} d_l^{\mathfrak{A}}(p_n, r) = \lim_{n\to\infty} d_l^{\mathfrak{A}}(q_n, r) = 0.$$

So we have that $\lim_{n\to\infty} d_l^{\mathfrak{A}}(p_n, q_n) = 0$. Since for all $p, q \in S_l^{\mathfrak{B}}$ $d_l^{\mathfrak{A}}(p,q) = d_l^{\mathfrak{B}}(p,q)$, we deduce that $\lim_{n\to\infty} d_l^{\mathfrak{B}}(p_n, q_n) = 0$. Therefore $\mathcal{N}$ satisfies the second condition in Proposition 6.19. We conclude that $\mathcal{M}$ is ω-qf-near-homogeneous.

(2): Set $\mathfrak{C} =$ age$(\mathcal{M})$. To prove that $\mathfrak{A} = \mathfrak{C}$, it is enough to prove that for all $l \in \omega$, $S_l^{\mathfrak{A}} = S_l^{\mathfrak{C}}$.

Fix $l \in \omega$. First we prove that $S_l^{\mathfrak{C}} \subseteq S_l^{\mathfrak{A}}$. Let $p \in S_l^{\mathfrak{C}}$. Then there exists $\bar{a} = (a_1, \ldots, a_l) \subseteq M$ such that $\bar{a} \models p$. Because N is dense in M there exists a sequence of l-tuples $(\bar{a}_j \mid j \in \omega)$ such that for all $j \in \omega$, $\bar{a}_j = (a_{j,1}, \ldots, a_{j,l}) \subseteq N$ and for all $1 \leq i \leq l$

$$\lim_{j\to\infty} d^{\mathcal{M}}(a_{j,i}, a_i) = 0.$$

Set $p_j = \mathrm{qftp}_{\mathcal{M}}(\bar{a}_j)$. Clearly $(p_j \mid j \in \omega)$ is a Cauchy sequence in $(S_l^{\mathfrak{B}}, d_l^{\mathfrak{B}})$ and

$$\lim_{j\to\infty} d_l^{\mathfrak{C}}(p, p_j) = 0. \tag{6.1}$$

By Proposition 6.16 and (6.1) we deduce that

$$\lim_{j\to\infty} \rho_l(p, p_j) = 0. \tag{6.2}$$

Since $(p_j \mid j \in \omega)$ is a Cauchy sequence in $(S_l^{\mathfrak{B}}, d_l^{\mathfrak{B}})$, by Fact 6.29 we deduce that it is also Cauchy sequence in $(S_l^{\mathfrak{A}}, d_l^{\mathfrak{A}})$. Because $(S_l^{\mathfrak{A}}, d_l^{\mathfrak{A}})$ is a complete metric space, there exists $r \in S_l^{\mathfrak{A}}$ such that

$$\lim_{j\to\infty} d_j^{\mathfrak{A}}(r, p_j) = 0. \tag{6.3}$$

By Proposition 6.16 and (6.3) we deduce that

$$\lim_{j\to\infty} \rho_l(r, p_j) = 0. \tag{6.4}$$

By (6.2) and (6.4) we deduce that $p = r$. We conclude that p is in $S_n^{\mathfrak{A}}$.

Now we prove that $S_l^{\mathfrak{A}} \subseteq S_l^{\mathfrak{C}}$. Let $p \in S_l^{\mathfrak{A}}$. We have that $S_l^{\mathfrak{B}}$ is dense in $(S_l^{\mathfrak{A}}, d_l^{\mathfrak{A}})$. So there exists a sequence $(p_j \mid j \in \omega)$ in $S_l^{\mathfrak{B}}$ such that

$$\lim_{j\to\infty} d_l^{\mathfrak{A}}(p, p_j) = 0.$$

By Proposition 6.16 we have that

$$\lim_{j\to\infty} \rho_l(p, p_j) = 0. \tag{6.5}$$

Clearly $(p_j \mid j \in \omega)$ is a Cauchy sequence in $(S_l^{\mathfrak{A}}, d_l^{\mathfrak{A}})$. Since for all $p, q \in S_l^{\mathfrak{B}}$

$$d_l^{\mathfrak{B}}(p, q) = d_n^{\mathfrak{A}}(p, q),$$

we deduce that $(p_j \mid j \in \omega)$ is a Cauchy sequence in $(S_l^{\mathfrak{B}}, d_l^{\mathfrak{B}})$. Since $\mathfrak{B} \subseteq \mathfrak{C}$, by Fact 6.29 we deduce that $(p_j \mid j \in \omega)$ is a Cauchy sequence in $(S_l^{\mathfrak{C}}, d_l^{\mathfrak{C}})$. By Proposition 6.13, $(S_l^{\mathfrak{C}}, d_l^{\mathfrak{C}})$ is a complete metric space. So there exists $r \in S_l^{\mathfrak{C}}$ such that

$$\lim_{j\to\infty} d_l^{\mathfrak{C}}(r, p_j) = 0.$$

By Proposition 6.16

$$\lim_{j\to\infty} \rho_l(r, p_j) = 0 \tag{6.6}$$

From (6.5), (6.6) we have that $p = r$. This implies that $p \in \mathfrak{C}$.

The uniqueness of the structure $\mathcal{M}$ is implied by Theorem 6.5(3).

(3): Let $\mathfrak{D}$ be the completion of the age $\mathfrak{A}$. By Proposition 6.23 we have $\mathcal{N} \models T^{\mathfrak{D}}$ and consequently $\mathcal{M} \models T^{\mathfrak{D}}$, where $\mathfrak{D}$ is a ρ-compact age with UPAP and therefore, by Proposition 5.15, $T^{\mathfrak{D}}$ has QE. We conclude that Th($\mathcal{M}$) has QE. □

The converse of Theorem 6.28 is the following theorem.

Theorem 6.30. *Let $\mathcal{M}$ be a separable L-structure which is ω-qf-near-homogeneous. Set $\mathfrak{A} = age(\mathcal{M})$. Then $\mathfrak{A}$ is a $d^{\mathfrak{A}}$-complete, $d^{\mathfrak{A}}$-separable age with the near-AP.*

Proof. By Proposition 6.13, $\mathfrak{A}$ is a $d^{\mathfrak{A}}$-complete age with the near-AP. Since $\mathcal{M}$ is separable, we deduce that $\mathfrak{A}$ is $d^{\mathfrak{A}}$-separable. □

Chapter 7

Fraïssé Theorems for complete, ω-categorical theories with QE

In this chapter we fix a continuous signature L whose nonlogical symbols are a finite number of constant and predicate symbols.

In Chapter 7 we prove a generalization to the continuous setting of the Fraïssé Theorems for complete, ω-categorical theories that have QE.

Definition 7.1. Let $\mathfrak{A}$ be an age. We say that $\mathfrak{A}$ is totally bounded if for all $\epsilon > 0$ there exists $n \geq 1$ such that for all $\mathcal{M} \in \mathfrak{A}$, $\mathcal{M}$ has an ϵ-net of size $\leq n$.

The following remark is clear.

Remark 7.2. Let $\mathcal{M}$ be an L-structure and $\mathfrak{A} = \text{age}(\mathcal{M})$. Then, $\mathcal{M}$ is compact iff $\mathfrak{A}$ is totally bounded. Therefore, if T is a complete theory that is ω-categorical, then age(T) is not totally bounded.

Definition 7.3. Let $\mathfrak{A}$ be an age with the near-AP. We say that $\mathfrak{A}$ is $d^{\mathfrak{A}}$-compact if for every $n \in \omega$, the metric space $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is compact.

Remark 7.4. In Proposition 7.10 we will show that if $\mathfrak{A}$ is a $d^{\mathfrak{A}}$-compact age with the near-AP, then $\mathfrak{A}$ has the AP.

Our main goal in this chapter is to prove the following theorems, which we call Fraïssé Theorems for complete, QE, ω-categorical theories.

Theorem 7.5. *Let T be a complete theory that has QE and is ω-categorical. Then age(T) is a $d^{\mathfrak{A}}$-compact age that has the UPAP and is not totally bounded.*

A strong converse of Theorem 7.5 is the following theorem.

Theorem 7.6. *Let $\mathfrak{A}$ be a $d^{\mathfrak{A}}$-compact age that has the AP and is not totally bounded. Then there exists a unique complete L-theory T that has QE, is ω-categorical and satisfies age$(T) = \mathfrak{A}$.*

First we will prove Theorem 7.5.

Proposition 7.7. *Let T be a complete L-theory that has QE and let $\mathfrak{A} = age(T)$. The following statements are equivalent:*

1. *T is ω-categorical.*

2. *$\mathfrak{A}$ is $d^{\mathfrak{A}}$-compact age that is not totally bounded.*

Proof. (1)⇒(2): Because T is ω-categorical, from Theorem 4.23 we conclude that $(S_n(T), d)$ is compact. Since T is a complete L-theory and has QE, by Theorem 5.5, $\mathfrak{A}$ is a ρ-compact age with the UPAP, and by Proposition 5.6, $\mathfrak{A}$ has the AP. So for all $n \in \omega$, $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is a metric space. For all $n \in \omega$ the space of types $S_n(T)$ can be identified with the space of quantifier-free types types $S_n^{\mathfrak{A}}$, and in addition, $(S_n(T), d)$ can identified with $(S_n^{\mathfrak{A}}, d^{\mathfrak{A}})$. Then, for all $n \in \omega$, $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is compact. We conclude that $\mathfrak{A}$ is $d^{\mathfrak{A}}$-compact.

Since T is ω-categorical, by definition T has a unique separable noncompact model $\mathcal{M}$. Clearly age$(\mathcal{M}) = \mathfrak{A}$. Then by Remark 7.2 we conclude that $\mathfrak{A}$ is not totally bounded.

(2)⇒(1): As in the implication (1)⇒(2), for all $n \in \omega$, $(S_n(T), d)$ can identified with $(S_n^{\mathfrak{A}}, d^{\mathfrak{A}})$. This implies that for all $n \in \omega$, $(S_n(T), d)$ is a compact metric space. Since $\mathfrak{A}$ is not totally bounded, by the compactness theorem we deduce that T has a noncompact model. By Theorem 4.23 we conclude that T is ω-categorical. □

Proof of Theorem 7.5. Since T is a complete L-theory that has QE, by Theorem 5.5 we deduce that $\mathfrak{A}$ is a ρ-compact age with the UPAP. By the implication (1)⇒(2) in Proposition 7.7 we conclude that $\mathfrak{A}$ is not totally bounded and $d^{\mathfrak{A}}$-compact. □

Now we prove Theorem 7.6. We will first need to prove some propositions.

Definition 7.8. Let (Y, d_1), (Y, d_2) be metric spaces. We say that d_1, d_2 are uniformly equivalent if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in Y$,

$$d_1(x, y) < \delta \quad \text{implies} \quad d_2(x, y) < \epsilon,$$

and

$$d_2(x, y) < \delta \quad \text{implies} \quad d_1(x, y) < \epsilon.$$

In this case we write $(Y, d_1) \sim^u (Y, d_2)$.

Proposition 7.9. *Let (Y, d_1), (Y, d_2) be metric spaces. Suppose that (Y, d_1) is compact and that (Y, d_1) is finer than (Y, d_2). Then $(Y, d_1) \sim^u (Y, d_2)$.*

Proof. Let $I_Y : (Y, d_1) \to (Y, d_2)$ be the identity map on Y. Since (Y, d_1) is compact and I_Y is continuous and bijective, we deduce that I_Y is a homeomorphism. We conclude that I_Y and its inverse are both uniformly continuous. □

Proposition 7.10. *Let $\mathfrak{A}$ be a $d^{\mathfrak{A}}$-compact age with the near-AP. Then:*

1. *$\mathfrak{A}$ is ρ-compact.*

2. *$\mathfrak{A}$ has the AP.*

Proof. (1): Fix $n \in \omega$. By Proposition 6.16, $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})$ is finer than $(S_n^{\mathfrak{A}}, \rho_n)$. By Proposition 7.9 we deduce that $(S_n^{\mathfrak{A}}, d_n^{\mathfrak{A}})) \sim^u (S_n^{\mathfrak{A}}, \rho_n)$. We conclude that $(S_n^{\mathfrak{A}}, \rho_n)$ is compact.

(2): Clear. □

Proposition 7.11. *Let $\mathfrak{A}$ be an $d^{\mathfrak{A}}$-compact age with the AP. Then $\mathfrak{A}$ has the UPAP.*

Proof. Since $\mathfrak{A}$ is $d^{\mathfrak{A}}$-compact, by Proposition 7.9, for all $n \in \omega$, $(S_n^{\mathfrak{A}}, \rho_n) \sim^u (S_n^{\mathfrak{A}}, d^{\mathfrak{A}})$. So there exist functions

$$\Delta_n : (0, 1] \to (0, 1]$$

$$\Delta_n^{'} : (0, 1] \to (0, 1]$$

such that for every $\epsilon \in (0, 1]$

$$\rho_n(p, q) < \Delta_n(\epsilon) \quad \text{implies} \quad d_n^{\mathfrak{A}}(p, q) < \epsilon$$

and

$$d_n^{\mathfrak{A}}(p, q) < \Delta_n^{'}(\epsilon) \quad \text{implies} \quad \rho_n(p, q) < \epsilon.$$

Fix $n \in \omega$ and $\epsilon > 0$. To show that $\mathfrak{A}$ has the UPAP it is enough to find $\delta_n^{\mathfrak{A}}(\epsilon)$ as in Definition 5.2. We claim that we may choose $\delta_n^{\mathfrak{A}}(\epsilon)$ such that

$$\delta_n^{\mathfrak{A}}(\epsilon) = \Delta_n(\Delta_{n+1}^{'}(\epsilon)).$$

To prove the claim, fix $\mathcal{M} \in \mathfrak{A}$, $p \in S_{n+1}^{\mathfrak{A}}$, $\bar{a} = (a_1, \ldots, a_n) \subseteq M$ such that if we set $q = \mathrm{qftp}_{\mathcal{M}}(\bar{a})$ then

$$\rho_n(p{\restriction}n, q) < \Delta_n(\Delta_{n+1}^{'}(\epsilon)).$$

This implies that

$$d_n^{\mathfrak{A}}(p{\restriction}n, q) < \Delta_{n+1}^{'}(\epsilon).$$

From Lemma 6.14, there exist $\mathcal{A}_1 \in \mathfrak{A}$ such that $\mathcal{M} \subseteq \mathcal{A}_1$ and $\bar{b} = (b_1, \ldots, b_n) \subseteq A_1$ such that $\bar{b} \models p \restriction n$, and

$$\max_{1 \le i \le n} d^{\mathcal{A}_1}(a_i, b_i) < \Delta'_{n+1}(\epsilon).$$

Because $\mathfrak{A}$ has the AP, by Proposition 3.8 there exists $\mathcal{A}_2 \in \mathfrak{A}$ such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $c \in A_2$ such that $(\bar{a}, c) \models p$. Set

$$\bar{d} = (d_1, \ldots, d_{n+1}) = (a_1, \ldots, a_n, c),$$

$$\bar{e} = (e_1, \ldots, e_{n+1}) = (b_1, \ldots, b_n, c).$$

Set $r = \text{qftp}_{\mathcal{A}_2}(\bar{d})$. Clearly, $\bar{e} \models p$ and

$$\max_{1 \le i \le n+1} d^{\mathcal{A}_2}(d_i, e_i) < \Delta'_{n+1}(\epsilon).$$

This implies that

$$d^{\mathfrak{A}}_{n+1}(r, p) < \Delta'_{n+1}(\epsilon),$$

which in turn implies that

$$\rho_{n+1}(r, p) < \epsilon.$$

□

Now we restate and prove Theorem 7.6.

Theorem 7.12. *Let $\mathfrak{A}$ be a $d^{\mathfrak{A}}$-compact age that has the AP and is not totally bounded. Then there exists a unique complete L-theory T that has QE, is ω-categorical and satisfies age$(T) = \mathfrak{A}$.*

Proof. Since $\mathfrak{A}$ is $d^{\mathfrak{A}}$-compact, by Proposition 7.10 and Proposition 7.11 we deduce that $\mathfrak{A}$ is ρ-compact and that it has the UPAP respectively. By Theorem 5.4, there exists a unique complete L-theory T that has QE. Since $\mathfrak{A}$ is a $d^{\mathfrak{A}}$-compact age that is not totally bounded, by Proposition 7.7 we conclude that T is ω-categorical. □

Remark 7.13. A natural question which arises is the following: If L is a continuous signature whose only nonlogical symbols are a finite number of constants and predicate symbols, are there any L-theories that have QE but are not ω-categorical?

The answer is no in the first-order setting: if an L-theory T has QE and L is a finite relational first-order signature, then for every $n \in \omega$, $S_n(T)$ is finite, and therefore T is ω-categorical.

The situation is different in the continuous setting. Here is an example of a structure $\mathcal{M}$ such that $\mathrm{Th}(\mathcal{M})$ has QE but is not ω-categorical. Let L be a signature where the only nonlogical symbol is a constant c. Let $M = \{a, b_0, b_1, \dots\}$. We define an L structure $\mathcal{M}$ on M as follows:

- We set $c^{\mathcal{M}} = a$.
- For all $i \in \omega$, we set $d^{\mathcal{M}}(a, b_i) = 1 - 1/(i+3)$.
- For all $i < j \in \omega$, we set $d^{\mathcal{M}}(b_i, b_j) = 1$.

Set $T = \mathrm{Th}(\mathcal{M})$.

Claim 7.14. *T has QE.*

To prove the claim we will use Proposition 4.3. Set $\mathfrak{A} = \mathrm{age}(\mathcal{M})$. Let $\mathcal{M} \models T$, $n \in \omega$, $p \in S^{\mathfrak{A}}_{n+1}$, $\bar{a} \models M$ with $\bar{a} \models p \restriction n$. Note that, since $\mathcal{M}$ is discrete we have that for all $\epsilon > 0$,

$$\sup_{\bar{x}} \min(\epsilon \dot{-} \tau_{p \restriction n}(\bar{x}), \inf_{x_{n+1}} \tau_p(\bar{x}, x_{n+1}) \dot{-} \epsilon) = 0$$

is true in $\mathcal{M}$. Hence for all $\epsilon > 0$,

$$\min(\epsilon \dot{-} \tau_{p \restriction n}(\bar{a}), \inf_{x_{n+1}} \tau_p(\bar{a}, x_{n+1}) \dot{-} \epsilon) = 0$$

is true in $\mathcal{M}$. We deduce that there exists $\mathcal{M}' \succeq \mathcal{M}$ and $b \in M'$ such that $(\bar{a}, b) \models p$.

Claim 7.15. *T is not ω-categorical.*

It suffices to find a model of T which realizes a type that is not realized in $\mathcal{M}$. Set $A = \{a, e, b_0, b_1 \dots\}$, and let

$$T' = T \cup \{d(e, a) = 1\} \cup \{d(e, b_i) = 1 \mid i \in \omega\}.$$

Then T' is satisfiable by the compactness theorem, since every finite subset of T' is satisfied in $\mathcal{M}$. Every model of T' is a model of T and realizes a type which is not realized in $\mathcal{M}$. We conclude that T is not ω-categorical.

Chapter 8

Macpherson's Theorem

In this chapter we prove a generalization of the following theorem of H. D. Macpherson.

Theorem 8.1. *Let L be a countable first-order signature and $\mathcal{M}$ be a countably infinite L-structure such that $Th(\mathcal{M})$ is ω-categorical. Set $G = Aut(\mathcal{M})$. Then there exists a dense subgroup $F \leq G$ which is freely generated by countably many elements, where G is equipped with the pointwise convergence topology.*

Proof. See [8, Theorem 3.1]. □

For the remainder of the chapter, L will denote a countable bounded continuous signature. Also, all the automorphism groups of L-structures will be equipped with the topology of pointwise convergence, and subgroups of such groups will be equipped with the subspace topology.

The following theorem is a generalization of Theorem 8.1 to the metric setting.

Theorem 8.2. *Let $\mathcal{M}$ be a separable L-structure which is strongly ω-homogeneous, noncompact and such that $Th(\mathcal{M})$ is ω-categorical. Set $G = Aut(\mathcal{M})$. Then there exists a dense subgroup $F \leq G$ which is freely generated by countably many elements.*

The following proposition is a strengthening of the implication (1)⇒(2) in Theorem 4.25.

Proposition 8.3. *Let $\mathcal{M}$ be a separable L-structure such that $Th(\mathcal{M})$ is ω-categorical and let G be a dense subgroup of $Aut(\mathcal{M})$. Then for every $\epsilon > 0$ and $n \geq 1$ there exist n-tuples $\bar{a}_1 = (a_{1,1}, \ldots, a_{1,n}), \ldots, \bar{a}_l = (a_{l,1}, \ldots, a_{l,n}) \subseteq M$ for some $l \geq 1$ such that for every n-tuple $\bar{b} \subseteq M$ there exist $h \in G$ and $1 \leq j \leq n$ such that*

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(b_i, h(a_{j,i})) < \epsilon.$$

Proof. Fix $\epsilon > 0$ and $\bar{b} = (b_1, \ldots, b_n) \subseteq M$. From the implication (1)⇒(2) in Theorem 4.25, there exists n-tuples $\bar{a}_1 = (a_{1,1}, \ldots, a_{1,n}), \ldots, \bar{a}_l = (a_{l,1}, \ldots, a_{l,n}) \subseteq M$ for some $l \geq 1$, $g \in Aut(\mathcal{M})$ and $1 \leq j \leq n$ such that

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(b_i, g(a_{j,i})) < \frac{\epsilon}{2}. \tag{8.1}$$

Because G is dense in $Aut(\mathcal{M})$, there exists $h \in G$ such that

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(g(a_{j,i}), h(a_{j,i})) < \epsilon. \tag{8.2}$$

Equations (8.1), (8.2) and the triangle inequality imply

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(b_i, h(a_{j,i})) < \epsilon.$$

□

We will need also the following theorem.

Theorem 8.4. *Let $\mathcal{M}$ be an L-structure such that $Th(\mathcal{M})$ is ω-categorical. Let $G \leq Aut(\mathcal{M})$ be dense in $Aut(\mathcal{M})$. Then the union of the totally bounded orbits of G, as it acts on $\mathcal{M}$, is totally bounded.*

Proof. Let K be the union of the totally bounded orbits of G. Fix $\epsilon > 0$. Then by Proposition 8.3 we have that there exist $a_1, \ldots, a_l \in \mathcal{M}$ for some $l \geq 1$ such that for every $b \in \mathcal{M}$ there exist $h \in G$ and $1 \leq i \leq l$ such that

$$d^{\mathcal{M}}(b, h(a_i)) < \frac{\epsilon}{3}.$$

For each $b \in K$ there exists $1 \leq i \leq l$ such that $b \in G \cdot B(a_i, \epsilon/3)$. Set

$$I = \{i \mid G \cdot B(a_i, \frac{\epsilon}{3}) \cap K \neq \emptyset\}.$$

Then we have that

$$K \subseteq \bigcup_{i \in I} G \cdot B(a_i, \frac{\epsilon}{3}). \tag{8.3}$$

For each $i \in I$ we pick b_i such that

$$b_i \in G \cdot B(a_i, \frac{\epsilon}{3}) \cap K.$$

Then $G \cdot b_i$ is totally bounded. For each $1 \leq i \leq l$, let $N_i \subseteq G \cdot b_i$ be a finite $\epsilon/3$-net for $G \cdot b_i$. Then, for each $1 \leq i \leq l$ we have that

$$G \cdot B(b_i, \frac{\epsilon}{3}) \subseteq \bigcup_{d \in N_i} B(d, \epsilon). \tag{8.4}$$

Then from (8.3) and (8.4) we have that

$$K \subseteq \bigcup_{i\in I} G \cdot B(a_i, \frac{\epsilon}{3}) \subseteq \bigcup_{i\in I} \bigcup_{d\in N_i} B(d, \epsilon).$$

□

A consequence of the Theorem 8.4 is the following theorem.

Theorem 8.5. *Let $\mathcal{M}$ be a separable L-structure such that $Th(\mathcal{M})$ is ω-categorical. Then the set of algebraic elements of $\mathcal{M}$ over the empty set is compact.*

Proof. Set $G = Aut(\mathcal{M})$. Let K be the union of the totally bounded orbits of G and K' the union of the algebraic elements of $\mathcal{M}$ over the empty set.

First we prove that K is closed. To do so, it is enough to prove that K includes the set of its limit points. Let a be a limit point of K. we show that $G \cdot a$ is totally bounded. Fix $\epsilon > 0$. We show that $G \cdot a$ has a finite ϵ-net. Let $b \in K$ such that $d^{\mathcal{M}}(a, b) < \epsilon/4$. Since $b \in K$, $G \cdot b$ is totally bounded. Let F be a finite $\epsilon/4$-net for $G \cdot b$.

Claim 8.6. *For every $x \in G \cdot a$ there exists $y \in F$ such that $d^{\mathcal{M}}(x, y) < \epsilon/2$.*

Proof. (of claim) Let $g \cdot a \in G \cdot a$ for some $g \in G$. Then $d^{\mathcal{M}}(g \cdot a, g \cdot b) < \epsilon/4$. Let $c \in F$ such that $d^{\mathcal{M}}(c, g \cdot b) < \epsilon/4$. By the triangle inequality we have $d^{\mathcal{M}}(g \cdot a, c) < \epsilon/2$. □

Claim 8.7. *$G \cdot a$ contains a finite subset F' that is ϵ-dense.*

Proof. (of claim) Set

$$F'' = \{c \in F \mid \text{For some } x \in G \cdot a \quad d^{\mathcal{M}}(c, x) < \epsilon/2\}.$$

Note that F'' is finite. For each $c \in F''$ choose $x(c) \in G \cdot a$ such that $d(c, x(c)) < \epsilon/2$. Set

$$F' = \{x(c) \mid c \in F''\}.$$

It is easy to see that F' is a finite ϵ-net for $G \cdot a$. Indeed, if $x \in G \cdot a$ then by Claim 8.6 there exists $c \in F''$ such that $d^{\mathcal{M}}(c, x) < \epsilon/2$. Then $d^{\mathcal{M}}(c, x(c)) < \epsilon/2$ and by the triangle inequality we get that $d^{\mathcal{M}}(x, x(c)) < \epsilon$. □

Since K is closed and, by Theorem 8.4, totally bounded we deduce that K is compact. To complete the proof of the Theorem 8.5 it is enough to prove that $K = K'$. It is enough to prove that a is algebraic over the empty set iff $G \cdot a$ is totally bounded. If a is algebraic over the empty set then by [1, Exercise 10.8] the set of realizations of $p := \text{tp}_{\mathcal{M}}(a)$ is compact. By Proposition 4.24 the set of realizations of p is the set $\overline{G \cdot a}$. We conclude that $G \cdot a$ is totally bounded. For the other direction, if a is not algebraic over the empty set then by [1, Exercise 10.8] there exists an ω_1-saturated $\mathcal{N} \succeq \mathcal{M}$ such that the set of realization of p in in $\mathcal{N}$ has density$\geq \omega_1$. By the Downward Löwenheim-Skolem Theorem there exists a separable L-structure $\mathcal{N}'$ such that $\mathcal{M} \preceq \mathcal{N}' \preceq \mathcal{N}$, and the set of realizations of p in $\mathcal{N}'$ is noncompact. Since $Th(\mathcal{M})$ is ω-categorical, we have $\mathcal{M} \cong \mathcal{N}'$. We deduce that the set of realizations of p in $\mathcal{M}$ is noncompact. As above, the set of realizations of p in $\mathcal{M}$ is $\overline{G \cdot a}$. We conclude that $G \cdot a$ is not totally bounded. □

Proposition 8.8. *Let $\mathcal{M}$ be a separable L-structure which is noncompact. If $G \leq Aut(\mathcal{M})$ is such that for every $\bar{a} \subseteq M$, the union of the totally bounded orbits of $G_{\bar{a}}$ is totally bounded, then there is a dense subgroup $F \leq G$ which is freely generated by countably many elements and F is dense in G.*

Proof. We will define by finite approximations a subset $\{g_i \mid i \in \omega\}$ of G which will freely generate F. Let A be a countable dense subset of $\mathcal{M}$. Because $\mathcal{M}$ is separable, $Aut(\mathcal{M})$ is separable and so G is separable. Let H be a countable dense subgroup of G. Set $B = H \cdot A$ and

$$C = \{(\bar{a}, \bar{b}) \mid \bar{a}, \bar{b} \subset B \wedge \exists g \in H \quad g(\bar{a}) = \bar{b}\}.$$

Note that $\text{card}(B) = \text{card}(C) = \omega$ and that B is not totally bounded. Let $(\bar{a}_n, \bar{b}_n)_{n\in\omega}$ be an enumeration of C. Let $(w_n)_{n\in\omega}$ be an enumeration of the reduced words in $\{g_j, g_j^{-1} \mid j \in \omega\}$ so that w_n does not involve g_j or g_j^{-1} if $j > n$. Let $(x_n)_{n\in\omega}$ be an enumeration of B.

Suppose that up to step $3n-1$ we have defined finite partial automorphisms $g_{0,3n-1}, \dots, g_{n-1,3n-1}$ on B such that each of them is a restriction of an automorphism of $\mathcal{M}$ which is in H.

Step $3n$: Define a partial automorphism $g_{n,3n}$ by the rule $g_{n,3n}(\bar{a}_n) = \bar{b}_n$. For every $i < n$, we put $g_{i,3n} = g_{i,3n-1}$.

Step $3n+1$: Let $w_n = g_{i_1}^{\epsilon_1} \dots g_{i_r}^{\epsilon_r}$ where $i_1, \dots, i_r$ such that $1 \leq i_1, \dots, i_r \leq n$ and $\epsilon_1, \dots, \epsilon_r \in \{\pm 1\}$, for some $r \geq 1$.

Claim 8.9. *There are $x \in B$ and extensions $g_{i_1,3n+1}, \dots, g_{i_r,3n+1}$, of $g_{i_1,3n}, \dots, g_{i_r,3n}$ respectively, which are restrictions of automorphisms of $\mathcal{M}$ in H, such that:*

- $g_{i_1,3n+1}^{\epsilon_1} \cdots g_{i_r,3n+1}^{\epsilon_r}(x)$ *is defined.*
- *Given s with $1 < s \leq r$ and the definitions of $g_{i_r,3n+1}^{\epsilon_r}(x), \dots, g_{i_s,3n+1}^{\epsilon_s} \cdots g_{i_r,3n+1}^{\epsilon_r}(x) \in B$ we have that*

$$O_s := H_{\bar{a}_{s-1}, \bar{b}_{s-1}} \cdot (g_{i_s,3n+1}^{\epsilon_s} \cdots g_{i_r,3n+1}^{\epsilon_r}(x))$$

is a nonempty, not totally bounded subset of B, where $\bar{a}_{s-1}$ is a finite tuple which enumerates the elements of B where the partial automorphism $g^{\epsilon_{s-1}}_{i_{s-1},3n}$ is defined, $\bar{b}_{s-1} = g^{\epsilon_{s-1}}_{i_{s-1},3n}(\bar{a}_{s-1})$, and $H_{\bar{a}_{s-1},\bar{b}_{s-1}}$ is the pointwise stabilizer of the tuple $(\bar{a}_{s-1}, \bar{b}_{s-1})$.

Note that O_s can be considered as the set of the possible 1-point extensions of $g^{\epsilon_s}_{i_s,3n}$ at the point $g^{\epsilon_s}_{i_s,3n+1} \cdots g^{\epsilon_r}_{i_r,3n+1}(x)$, to partial automorphisms of B, which are restrictions of automorphisms of $\mathcal{M}$ in H .

Proof. (of claim) This is proved by induction on s. Note that the induction goes backwards, from r toward 1, and that at the start of the induction we use the fact that B is not totally bounded. Suppose that we have defined $g^{\epsilon_s}_{i_s,3n+1} \cdots g^{\epsilon_r}_{i_r,3n+1}(x) \in B$, for some $1 < s \leq r$, such that O_s is a nonempty, not totally bounded subset of B. We claim that for some $y \in B$ we have that

$$H_{\bar{a}_{s-2},\bar{b}_{s-2}} \cdot y$$

is a nonempty, not totally bounded subset of B, where $\bar{a}_{s-2}$ is a finite tuple which enumerates the elements of B where the partial automorphism $g^{\epsilon_{s-2}}_{i_{s-2},3n}$ is defined and $\bar{b}_{s-2} = g^{\epsilon_{s-2}}_{i_{s-2},3n}(\bar{a}_{s-2})$. Let K be the union of the totally bounded orbits of $H_{\bar{a}_{s-2},\bar{b}_{s-2}}$ as it acts on B. Let K' be the the union of the totally bounded orbits of $G_{\bar{a}_{s-2},\bar{b}_{s-2}}$ as it acts on $\mathcal{M}$. From hypothesis we have that that K' is totally bounded. Because $K \subseteq K'$, K is totally bounded. Set

$$P := H_{\bar{a}_{s-2},\bar{b}_{s-2}} \cdot O_s.$$

Clearly,

$$O_s \subseteq P \subseteq B.$$

Since by induction hypothesis O_s is not totally bounded, we deduce that P is not totally bounded. We conclude that $P \setminus K$ is a nonempty, not totally bounded subset of B. This implies that there exists $y \in O_s$ such that

$$H_{\bar{a}_{s-2},\bar{b}_{s-2}} \cdot y$$

is not totally bounded. If we set

$$g^{\epsilon_{s-1}}_{i_{s-1},3n+1} \cdots g^{\epsilon_r}_{i_r,3n+1}(x) = y,$$

then

$$O_{s-1} = H_{\bar{a}_{s-2},\bar{b}_{s-2}} \cdot y$$

and O_{s-1} is not totally bounded.

$\square$

From the claim we conclude that we may find $x \in B$ and define $g^{\epsilon_r}_{i_r,3n+1}(x), \ldots, w_n(x)$ in such a way that $w_n(x) \neq x$. For every i such that $1 \leq i \leq n$, if $i \in \{i_1, \ldots, i_r\}$ we extend $g_{i,3n}$ to $g_{i,3n+1}$ appropriately, otherwise we set $g_{i,3n+1} = g_{i,3n}$.

Step $3n+2$: For every $i \leq n$, we extend $g_{i,3n+1}$ to $g_{i,3n+2}$ to ensure that $g_{i,3n+2}$ and $g^{-1}_{i,3n+2}$ are defined on x_n. We can do this in such a way such that for every $i \leq n$, $g_{i,3n+2}$ is a restriction of an automorphism of $\mathcal{M}$ in H.

For each $i \in \omega$, we set

$$g_i = \bigcup_{n \in \omega} g_{i,n}.$$

We have that for all $i \in \omega$, g_i is a restriction of an element in H. Because for every $i \in \omega$, g_i is defined on D, which is dense in $\mathcal{M}$, we have that g_i can be extended uniquely to an $f_i \in H$. Set

$$F = < f_i \mid i \in \omega > .$$

Claim 8.10. *F is dense in H.*

It is enough to show that given $h \in H$, $\epsilon > 0$, $c_1, \ldots, c_n \in M$ for some $n \in \omega$, we have that there exists $f_j \in F$ such that

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(h(c_i), f_j(c_i)) < \epsilon.$$

Fix $h \in H$, $\epsilon > 0$, $c_1, \ldots, c_n \in \mathcal{M}$. Because D is dense in $\mathcal{M}$ there exists $d_1, \ldots, d_n \in D$ such that

$$\max_{1 \leq i \leq n} d^{\mathcal{M}}(c_i, d_i) < \frac{\epsilon}{2}.$$

Set $\bar{d} = (d_1, \ldots, d_n)$. We have that $(\bar{d}, h(\bar{d})) \in C$. So there exists $j \in \omega$ such that $(\bar{d}, h(\bar{d})) = (\bar{a}_j, \bar{b}_j)$. So we have that

$$\forall i \leq n \quad f_j(d_i) = h(d_i).$$

Because h, f_j are isometries we have that for $i \leq n$

$$d^{\mathcal{M}}(h(c_i), h(d_i)) = d^{\mathcal{M}}(f_j(c_i), f_j(d_i)) = d^{\mathcal{M}}(c_i, d_i) < \frac{\epsilon}{2}.$$

From the triangle inequality we have that

$$d^{\mathcal{M}}(h(c_i), f_j(c_i)) \le d^{\mathcal{M}}(h(c_i), f_j(d_i)) + d^{\mathcal{M}}(f_j(d_i), f_j(c_i)).$$

Hence

$$d^{\mathcal{M}}(h(c_i), f_j(c_i)) \le d^{\mathcal{M}}(h(c_i), h(d_i)) + d^{\mathcal{M}}(f_j(d_i), f_j(c_i)).$$

Therefore

$$\max_{1 \le i \le n} d^{\mathcal{M}}(h(c_i), f_j(c_i)) \le \max_{1 \le i \le n} d^{\mathcal{M}}(h(c_i), h(d_i)) + \max_{1 \le i \le n} d^{\mathcal{M}}(f_j(d_i), f_j(c_i)) \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Theorem 8.11. *Let $\mathcal{M}$ be a separable L-structure that is strongly ω-homogeneous, noncompact and such that $Th(\mathcal{M})$ is ω-categorical. Let $G \le Aut(\mathcal{M})$ be such that for every finite tuple $\bar{a} \subseteq M$, $G_{\bar{a}}$ is dense in $Aut_{\bar{a}}(\mathcal{M})$. Then there exists a dense subgroup $F \le G$ which is freely generated by countably many elements.*

Proof. Fix a finite tuple $\bar{a}$ in M. Because $\mathrm{Th}(\mathcal{M})$ is ω-categorical and $\mathcal{M}$ is ω-homogeneous, Proposition 4.27 implies that $\mathrm{Th}(\mathcal{M}, \bar{a})$ is ω-categorical. By hypothesis we have that $G_{\bar{a}}$ is dense in $\mathrm{Aut}(\mathcal{M}, \bar{a}) = \mathrm{Aut}_{\bar{a}}(\mathcal{M})$. From Theorem 8.4 we have that the union of the totally bounded orbits of $G_{\bar{a}}$ is totally bounded. Then by Proposition 8.8 the result follows. □

Theorem 8.2 easily follows from Theorem 8.11 if we take $G = \mathrm{Aut}(\mathcal{M})$.

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Author's Biography

Konstantinos Schoretsanitis was born in Athens, Greece. After he obtained a Bachelor of Science in Mathematics from the University of Patras, Greece, he decided to pursue graduate studies in Mathematics at the University of Illinois at Urbana-Champaign. He obtained a Master of Science in Mathematics, and he will be granted the Doctor of Philosophy in 2007.

Fraïssé Theory for Metric Structures

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In 1954, Roland Fraïssé published a paper that answered the following questions: Given a first-order signature L and a class $\mathfrak{A}$ of finite L-structures that is closed under isomorphism:

1. find necessary and sufficient conditions on $\mathfrak{A}$ that guarantee the existence of a "homogeneous" L-structure $\mathcal{M}$ such that the class of L-structures that are isomorphic to finite L-substructures of $\mathcal{M}$ is $\mathfrak{A}$;

2. find necessary and sufficient conditions on $\mathfrak{A}$ that guarantee the existence of an L-structure $\mathcal{M}$ such that $\mathrm{Th}(\mathcal{M})$ has QE and is ω-categorical, and such that the class of L-structures that are isomorphic to finite L-substructures of $\mathcal{M}$ is $\mathfrak{A}$.

In this thesis we generalize Fraïssé's results to the setting of bounded continuous logic for metric structures. This logic was presented in 2004 by Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, and it may be considered as a generalization of first-order logic.

We also prove a theorem, in the setting of continuous model theory, that is a generalization of a theorem of H. D. Macpherson about the automorphism groups of ω-categorical structures.