1. Introduction

Section 2 of this summary contains basic definitions and facts about \( \mathbb{R} \)-trees. In Section 3 it is indicated why the theory of \( \mathbb{R} \)-trees is axiomatizable in a suitable continuous signature. In addition, a model companion to this theory is found which is complete and superstable, among other properties. Section 4 describes generic isometries of \( \mathbb{R} \)-trees. Section 5 begins to consider group actions on \( \mathbb{R} \)-trees, and outlines some areas of possible further research. Many details are glossed over, and “proofs” given are fragments or outlines of such.

2. \( \mathbb{R} \)-tree background

2.1. Definition. A metric space \((M, d)\) is geodesic if for any \(x, y \in M\) there exists an isometric embedding \(\gamma: [0, d(x, y)] \to M\) with \(\gamma(0) = x\) and \(\gamma(d(x, y)) = y\), where \([0, d(x, y)] \subseteq \mathbb{R}\). A geodesic segment is the image of such a path.

2.2. Definition. An \(\mathbb{R}\)-tree is a metric space \(X\) such that between any two points in \(X\) there is a unique arc and that arc is a geodesic segment. In an \(\mathbb{R}\)-tree we will use \([a, b]\) to denote the unique geodesic segment between \(a\) and \(b\).

2.3. Definition. Let \(M\) be an \(\mathbb{R}\)-tree and \(a \in M\). Call the connected components of \(M \setminus \{a\}\) branches at \(a\). The height of a branch \(B\) at \(a\) is \(\sup\{d(a, x) | x \in B\}\) if that supremum exists, and is \(\infty\) otherwise.

Items 2.4 through 2.8 are facts about \(\mathbb{R}\)-trees. They can be found in [6].

2.4. Lemma. If \(M\) is an \(\mathbb{R}\)-tree and \(E, F\) are disjoint, closed, non-empty subtrees of \(M\), then there exists a unique shortest geodesic path with its initial point in \(E\) and its terminal point in \(F\). Moreover, for all \(c \in E\) and \(b \in F\), the geodesic segment from \(c\) to \(b\) must contain this segment.

2.5. Lemma. If \(M\) is an \(\mathbb{R}\)-tree and \(a, b, c \in M\), then

\[\begin{align*}
(1) \quad & d(a, b) + d(b, c) = d(a, c) + 2 \text{dist}(b, [a, c]). \\
(2) \quad & b \in [a, c] \text{ if and only if } d(a, c) = d(a, b) + d(b, c). \\
(3) \quad & b \in [a, c] \text{ if and only if } a \text{ and } c \text{ are on different branches at } b.
\end{align*}\]
2.6. **Definition** (Gromov product). For a metric space \( M \) and \( x, y, w \in M \), define \( (x \cdot y)_w = \frac{1}{2}[d(x, w) + d(y, w) - d(x, y)] \).

2.7. **Definition.** Let \( \delta > 0 \). A metric space \( M \) is \( \delta \)-hyperbolic if, for all \( x, y, z, w \in M \)

\[
\min\{(x \cdot z)_w, (y \cdot z)_w\} - \delta \leq (x \cdot y)_w.
\]

If \( M \) is geodesic, this is equivalent to the condition: given \( a, b, c \in M \) and any geodesic segments \([a, b], [b, c], \) and \([c, a]\), the segment \([a, b] \subseteq ([b, c] \cup [c, a])^\delta\).

A metric space is 0-hyperbolic if it is \( \delta \)-hyperbolic for all \( \delta > 0 \).

2.8. **Lemma.** Any \( \mathbb{R} \)-tree is 0-hyperbolic. Moreover, any 0-hyperbolic metric space embeds isometrically in an \( \mathbb{R} \)-tree.

2.9. **Definition.** If \( A \subseteq M \) is a subset of the \( \mathbb{R} \)-tree \( M \) with basepoint \( p \), we let \( E_A \) denote \( \bigcup\{[a_1, a_2] \mid a_1, a_2 \in A \cup \{p\}\} \). Denote the closure of \( E_A \) by \( \overline{E_A} \).

2.10. **Definition.** Let \( h > 0 \). An \( \mathbb{R} \)-tree \( M \) is \( h \)-richly branching if,

\[
B_h := \{b \in M \text{ at } b \text{ there are at least } 3 \text{ branches of height } \geq h\}
\]

is dense in \( M \). If \( M \) is \( h \)-richly branching for some \( h \), we say it is *richly branching*.

It is straightforward to show that if \( M \) is \( h \)-richly branching for some \( h > 0 \), then it is \( h \)-richly branching for every \( h > 0 \).

### 3. \( \mathbb{R} \)-tree results

In the thesis it is shown that the class of \( \mathbb{R} \)-trees is axiomatizable in a suitable continuous signature (Lemma 3.1). The theory of this class, denoted \( \mathbb{RT} \), has amalgamation over substructures (Lemma 3.3). In order for \( M \models \mathbb{RT} \) to be an e.c. model, there must be enough branching to allow the (approximate) embedding of any finitely generated \( \mathbb{R} \)-tree anywhere in \( M \). In fact, an \( \mathbb{R} \)-tree is e.c. if and only if it is richly branching. Definition 3.5 defines the theory \( \text{rb}\mathbb{RT} \), axiomatizing the class of richly branching \( \mathbb{R} \)-trees. The theory \( \text{rb}\mathbb{RT} \) is the model companion of \( \mathbb{RT} \) (Theorem 3.10). In the rest of the section properties of \( \text{rb}\mathbb{RT} \) are explored.

An \( \mathbb{R} \)-tree is a geodesic space and hence balls are definable over their centers. To talk about unbounded \( \mathbb{R} \)-trees, one may use a many-sorted structure with a sort for each ball of integer radius. Let \( L \) be a continuous signature with sorts \( \{(M_n, d_n)| n \in \mathbb{N}^{>0}\} \), a constant symbol \( p_n \) in each sort, and function symbols \( I_{m,n}: M_m \to M_n \) for all \( m < n \). The bound on the metric \( d_n \) on \( M_0 \) is 2n. All \( I_{m,n} \) for \( m, n \in \mathbb{N}^{>0} \) have modulus of uniform continuity \( \Delta(\epsilon) = \epsilon \). Let \( \sup_{n,x} \) stand for the supremum over the sort \( (M_n, d_n) \) and likewise for \( \inf_{n,x} \). Given a complete pointed metric space \( (M, d, p) \), we make a corresponding \( L \)-structure \( M \). In \( M \), \( (M_n, d_n)^M = B_n(p) \), the closed ball in \( M \) of radius \( n \), with metric \( d_n^M = d \) restricted to that ball, \( p_n^M = p \), and \( I_{m,n}^M : M_m \to M_n \) is the inclusion map from \( B_m(p) \).
to $B_n(p)$. Any $L$-theory would include axioms to ensure the inclusion maps are interpreted correctly, but they are not stated here. For the sake of neatness and readability, in the formulas below any instances of $I_{m,n}$ have been omitted.

3.1. **Lemma.** The class of complete, pointed $\mathbb{R}$-trees is axiomatizable in the signature $L$ via the following theory $\mathbb{R}T$. Let $\mathbb{R}T$ consist of:

- **Approximate midpoint property (AMP):**
  For each $n \in \mathbb{N}$,
  \[
  \sup_{n,x} \sup_{n,y} \inf_{n,z} \max \{|d(x, z) - \frac{d(x,y)}{2}|, |d(y, z) - \frac{d(x,y)}{2}|\} = 0
  \]

- **0-hyperbolic property:**
  For each $n \in \mathbb{N}$
  \[
  \sup_{n,x} \sup_{n,y} \sup_{n,z} \sup_{n,w} \left( \min \{|(x \cdot z)_w, (y \cdot z)_w\} - (x \cdot y)_w\right) = 0
  \]

**Proof.** If a structure $\mathcal{M} \models T$, then $\mathcal{M}$ is 0-hyperbolic, and thus embeds isometrically in an $\mathbb{R}$-tree. Property AMP can be shown to imply that $\mathcal{M}$ is path connected. \hfill \Box

3.2. **Lemma.** The function $\mu: M \times M \to M$ where $\mu(x_1, x_2) = \text{midpoint of } [x_1, x_2]$ is a uniformly 0-definable function in models of $\mathbb{R}T$.

**Proof.** Let $M \models \mathbb{R}T$. The distance $d(\mu(x_1, x_2), y)$ is given by the formula
\[
|d(x_1, y) + d(x_2, y) - d(x_1, x_2)| + \max\{d(x_1, y) - \frac{d(x_1, x_2)}{2}, d(x_2, y) - \frac{d(x_1, x_2)}{2}\}
\]
Thus, the midpoint function is 0-definable via this quantifier-free formula in any model of $\mathbb{R}T$. \hfill \Box

3.3. **Theorem.** The $L$-theory $\mathbb{R}T$ has the amalgamation property for substructures of models. That is, if
\[
\mathcal{M}_0 = (M_0, d_0, p_0), \quad \mathcal{M}_1 = (M_1, d_1, p_1), \quad \mathcal{M}_2 = (M_2, d_2, p_2)
\]
are substructures of models of $\mathbb{R}T$ and $f_1: \mathcal{M}_0 \to \mathcal{M}_1$, $f_2: \mathcal{M}_0 \to \mathcal{M}_2$ are embeddings, then there exists an $\mathbb{R}$-tree $N = (N, d, p, \mu)$ and embeddings $g_i: \mathcal{M}_i \to N$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

**Proof.** Any such substructure generates a unique submodel, so we may assume $\mathcal{M}_0, \mathcal{M}_1$ and $\mathcal{M}_2$ are all submodels. To construct $N$, take a disjoint union of $\mathcal{M}_1$ and $\mathcal{M}_2$ and then identify the common submodel. \hfill \Box

3.4. **Definition.** Given $n \in \mathbb{N}$ and $m \in \mathbb{N}$, let $\varphi_{m,n}$ be the formula
\[
\inf_{2n,y_1} \ldots \inf_{2n,y_m} \max_{1 \leq i < j \leq m} \{ |(d(x, y_i) + d(x, y_j)) - d(y_i, y_j)|, |d(x, y_i) - \frac{1}{2}| \}
\]
Let $\varphi_{m,n}$ be the closed formula $\sup_{n,x} \varphi_{m,n}$.

3.5. **Definition.** Let $\text{rb}\mathbb{R}T = \mathbb{R}T \cup \{\varphi_{3,n} = 0 \mid n \in \mathbb{N}\}$.

3.6. **Theorem.** The $L$-theory $\text{rb}\mathbb{R}T$ axiomatizes the class of richly branching $\mathbb{R}$-trees.
3.7. Lemma. Any model of rb$\mathbb{R}T$ also satisfies
\[ \mathbb{R}T \cup \{ \varphi_{m,n} = 0 \mid m \in \mathbb{N}, n \in \mathbb{N} \}. \]

3.8. Lemma. Assume $M \models rb\mathbb{R}T$.

(1) If $M$ is $\kappa$-saturated, then there are at least $\kappa$ branches at every point.

(2) For any $a \in M$ and $B$ a branch at $a$, the height of $B$ is $\infty$.

Proof. If $M \models \mathbb{R}T \cup \{ \varphi_{m,n} = 0 \mid n \in \mathbb{N}, m \in \mathbb{N} \}$ is $\omega$-saturated then for any point $a \in M$, $\varphi_{m,n}(a) = 0$ is actually realized by $b_1, \ldots, b_m$. Then, by Lemma 2.5, it is clear that for all $1 \leq i, j \leq m$, $a \in [b_i, b_j]$, and $d(a, b_i) = \frac{1}{2}$. So, all of the $b_i$ are distinct and no two $b_i, b_j$ are on the same branch at $a$. Thus there must be at least $m$ branches at $a$. Statement (1) now follows from a saturation argument. \hfill \Box

The following lemma is the main tool for proving that rb$\mathbb{R}T$ is the model companion of $\mathbb{R}T$.

3.9. Lemma. Let $M$ be an $\omega$-saturated model of rb$\mathbb{R}T$. Let $K$ be a non-empty, finitely generated $\mathbb{R}$-tree with basepoint $p$. For any $e \in M$ and any collection $\{B_i \mid i \in \omega \}$ of branches at $e$, there exists an isometric embedding $f$ of $K$ into $M$ sending $p$ to $e$ such that $f(K) \cap (\cup_{i \in \omega} B_i) = \{ e \}$.

3.10. Theorem. The theory rb$\mathbb{R}T$ is the model companion of $\mathbb{R}T$.

Proof. Take $M \models rb\mathbb{R}T$. To show $M$ is an e.c. model of $\mathbb{R}T$ it suffices to show that an $\omega_1$-saturated elementary extension of $M$ is e.c. So, from now on assume $M$ is $\omega_1$-saturated.

Let $N \models \mathbb{R}T$ be an extension of $M$. Let $a_1, \ldots, a_k \in M$, and for simplicity assume $p = a_1$. Since the only atomic $L$-formulas are $d(t_1, t_2)$, the value of any $q$-free formula in any $L$-structure $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_k)$ is determined by the set of distances $d(x_i, x_j), d(y_i, y_j), d(x_i, y_j)$ in that structure. So, to ensure that
\[ \inf_{m_1, y_1} \ldots \inf_{m, y_n} \varphi(a_1, \ldots, a_k, y_1, \ldots, y_n)^N = \inf_{m_1, y_1} \ldots \inf_{m, y_n} \varphi(a_1, \ldots, a_k, y_1, \ldots, y_n)^M. \]

for any inf-formula, it is sufficient to show that given any $b_1, \ldots, b_n \in N$, we may find $c_1, \ldots, c_n \in M$ such that $d(c_j, a_i) = d(b_j, a_i)$ for $1 \leq i \leq k, 1 \leq j \leq n$ and $d(c_i, c_j) = d(b_i, b_j)$ for $1 \leq i \leq n, 1 \leq j \leq n$.

This is done using Lemma 3.9. \hfill \Box

3.11. Lemma. The L-theory rb$\mathbb{R}T$ has QE and is complete.

3.12. Lemma. If $M \models \mathbb{R}T$ and $b \in M$, then the type of $b$ over the parameter set $A \subseteq M$ (for convenience assume the basepoint $p \in A$) is determined by

(1) the unique point $e \in E_A$ which is closest to $b$ and

(2) the distance $\text{dist}(a, E_A) = d(b, e)$.

3.13. Lemma. A model of rb$\mathbb{R}T$ is $\kappa$-saturated if and only if the underlying space has at least $\kappa$ branches at every point.
Proof. We have already seen that a $\kappa$-universal $\mathbb{R}$-tree must have at least $\kappa$-branches at every point. Now, assume $M$ is an $\mathbb{R}$-tree with $\kappa$-branches at every point, and let $A \subseteq M$ have cardinality less than $\kappa$. Clearly $M \models \text{rb} \mathbb{R}T$. Take a 1-type over $A$. By Lemma 3.12, this type is determined by a closest point $e \in E_A$ and a distance $r \in \mathbb{R}^\geq 0$. Only $|A|$-many branches at $e$ can intersect $E_A$ at a point other than $e$. So, there are branches at $e$ in $M$ which do not intersect $E_A$ except at $e$. On one of these branches, take $b$ with distance $r$ from $e$. This $b$ satisfies the type.\hfill \Box

3.14. Example. (See [9] and [10].) Let $\mu$ be a cardinal. An $\mathbb{R}$-tree $M$ is called $\mu$-universal if any $\mathbb{R}$-tree with $\leq \mu$ branches at each point can be isometrically embedded in $M$.

In [10], the authors construct a $\mu$-universal $\mathbb{R}$-tree $A_\mu$ for any cardinal $\mu \geq 2$. Their space $A_\mu$ is a complete $\mathbb{R}$-tree which is homogeneous, with $\mu$ branches at every point. They show that any complete $\mathbb{R}$-tree with $\mu$-branches at each point is isometric to $A_\mu$.

3.15. Theorem. The theory $\text{rb} \mathbb{R}T$ is stable, but it is not superstable, and it is not $\omega$-stable. Indeed, when $\kappa$ is an infinite cardinal, $\text{rb} \mathbb{R}T$ is $\kappa$-stable if and only if $\kappa$ satisfies $\kappa^\omega = \kappa$.

Proof. Let $\kappa$ be an infinite cardinal. Let $M \models \text{rb} \mathbb{R}T$ be sufficiently saturated. First, assume $\kappa = \kappa^\omega$. Let $|A| = \kappa$. Then

$$|E_A| \leq |A \times A|^\omega = \kappa^2 \omega \leq \kappa^\omega 2^\omega = \kappa^\omega = \kappa.$$  

Counting possible 1-types using Lemma 3.12 shows

$$|S_1(A)| \leq |E_A^c \times \mathbb{R}^\geq 0| = |E_A| 2^\omega \leq |E_A|^\omega 2^\omega = \kappa^\omega 2^\omega = \kappa^\omega = \kappa.$$  

Thus the theory is $\kappa$-stable.

For the other direction, note that given any infinite $\kappa$ we may construct, via a tree construction, a subset $A$ of $M$ with $|A| = \kappa$ and $|E_A| = \kappa^\omega$. Now assume that $\kappa^\omega > \kappa$, and let $A$ be such a subset. For each $e \in E_A$, choose $b_e$ on a branch out of $e$ which intersects $E_A$ only at $e$ with $d(e, b_e) = 1$. We may always find such a branch provided our model is saturated enough. The set $\{b_e \mid e \in E_A\}$ has cardinality $\kappa^\omega$, and for any $e \neq f$ in $E_A$ it is straightforward to show $d(\text{tp}(b_e/A), \text{tp}(b_f/A)) \geq 2$. Since $\kappa^\omega > \kappa$, this implies the theory is not $\kappa$-stable.\hfill \Box

Let $\kappa$ be a cardinal greater than than $\text{card}(L)$. Let $U$ be a $\kappa$-universal domain for $\text{rb} \mathbb{R}T$.

3.16. Definition. Let $A, B$ and $C$ be small subsets of $U$. Say $A$ is $*$-independent from $B$ over $C$, denoted $A \not\vdash^*_C B$, if and only if for all $a \in A$ $\text{dist}(a, E_{C \cup B}) = \text{dist}(a, E_C)$.

3.17. Theorem. The $*$-independence relation is the model theoretic (non-dividing) independence relation for $\text{rb} \mathbb{R}T$.  

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Proof. The \( ^* \)-independence relation has all the properties (symmetry, transitivity, extension, finite character etc.) given in [1] Theorem 14.12, so we may apply [1] Theorem 14.14 which says any independence relation with these properties must be exactly the non-dividing relation. \( \square \)

3.18. Theorem. The theory \( rbRT \) is not categorical in any cardinal.

Proof. The distribution of the points with \( \geq 3 \) branches and the number of branches at each of these points can vary widely. Any isomorphism between models must be a homeomorphism, and thus would have to preserve branching. \( \square \)

In addition to these results, the thesis characterizes the metrics on type spaces. It also says more about definability, definable closures and algebraic closures.

4. Generic isometries of \( \mathbb{R} \)-trees

4.1. Definition. If an isometry \( f \) of an \( \mathbb{R} \)-tree \( M \) has a fixed point it is called elliptic, otherwise it is hyperbolic. The quantity \( ||f|| := \inf_{x \in M} d(x, f(x)) \) is called the translation distance of \( f \).

4.2. Lemma. If \( ||f|| = 0 \) then \( f \) is elliptic; if not, then \( f \) is hyperbolic and acts as a translation along a copy of \( \mathbb{R} \) in \( M \), with points on this axis being moved by exactly distance \( ||f|| \).

To consider structures with an added isometry, it is convenient to shift the setting to one in which we use gauged metric structures as in [3]. When finding a signature to describe \( \mathbb{R} \)-trees with an isometry, we need a uniform upper bound on how far the basepoint is moved in the structures we consider. We may not use a class of structures where the basepoint allowed to move an arbitrarily large distance, because in an ultraproduct of such structures the ultralimit of \( d(p, f(p)) \) need not exist. Also, when considering \( \mathbb{R} \)-trees with an isometry, in order to find model companions we must split into separate hyperbolic and elliptic cases. To find axioms for classes of \( \mathbb{R} \)-trees with only hyperbolic isometries, a positive lower bound on \( ||f|| \) must be specified. Otherwise there would be an ultraproduct of structures in the class where the isometry is elliptic.

4.3. Definition. For \( s \in \mathbb{R}_{\geq 0} \), let \( L_s \) be a gauged continuous signature with function symbol \( f \) and constant symbol \( p \). Let \( v(x) = d(x, p) \) be the gauge for this signature. Since \( f \) must be bounded on gauge-bounded sets, let \( \beta_f(y) := y + s \) be such that for any interpretation of \( f \), \( v(f(x)) \leq \beta(v(x)) \). This is called the \( v \)-bound for \( f \). We also specify a \( v \)-modulus which tells us how \( f \) is uniformly continuous on \( v \)-bounded sets. For \( f \) let this be \( \delta(\epsilon) = \epsilon \). For convenience, include a symbol (along with appropriate \( v \)-bound and \( v \)-modulus) for the midpoint function \( \mu \) in our signature as well. Above we showed that the midpoint function was \( q \)-free definable in \( L \), and it will also
be in $L_s$. Thus, adding this symbol (e.g. via extension by definitions) does not change the expressive power of the signature.

Note that the $L$-conditions in $\mathbb{R}T$ can be translated into $L_s$-conditions, and thus we may find axioms for the class of complete, pointed $\mathbb{R}$-trees in this signature as well. Call this $L_s$-theory $\mathbb{R}T$.

4.1. Hyperbolic Isometries. For $s > 0$, and $0 < r \leq s$, let $K_{r,s}$ be the class of $L_s$-structures $(M, d, p, f, v)$ where $(M, d, p)$ is a complete, pointed $\mathbb{R}$-tree and $f$ is a hyperbolic isometry of $M$ such that $\inf_x d(x, f(x)) \geq r$. For convenience, require $p$ to be on the axis of $f$. Let $\mathbb{H}\mathbb{R}T_{r,s}$ be the $L_s$-theory of $K_{r,s}$. This theory is axiomatizable via $L_s$-conditions which express the following properties:

- the underlying metric space is an $\mathbb{R}$-tree
- the function $f$ is a surjective isometry of $M$
- the point $p$ is on the axis (that is, $d(p, f(p)) = ||f||$)
- the translation distance $||f|| \geq r$
- the gauge $v(x) = d(p, x)$

4.4. Theorem. The $L_s$ theory $\mathbb{H}\mathbb{R}T_{r,s}$ has the amalgamation property over substructures of models.

4.5. Theorem. An $L_s$-structure $M = (M, d, p, f, v)$ is an e.c. model of $\mathbb{H}\mathbb{R}T_{r,s}$ if and only if $M$ is a richly branching $\mathbb{R}$-tree.

4.6. Corollary. For $0 < r \leq s$, the theory $\mathbb{H}\mathbb{R}T_{r,s}$ has a model companion.

Proof. Let $C$ be the class of e.c. models of $\mathbb{H}\mathbb{R}T_{r,s}$. Theorem 4.5 implies the class $C$ is axiomatizable. Hence $\mathbb{H}\mathbb{R}T_{r,s}$ has a model companion, namely, the theory of $C$. \qed

Call the model companion theory $\text{rb}\mathbb{H}\mathbb{R}T_{r,s}$.

4.2. Elliptic isometries. Let $L_0$ be $L_s$ as defined above for $s = 0$. Let $K_0$ be the class of $L_0$-structures $(M, d, p, f, v)$ where $M$ is an $\mathbb{R}$-tree, $f$ is an elliptic isometry of $M$ and $p$ is a fixed point in $M$. Let $\mathbb{E}\mathbb{R}T_s$ be the theory of $K_0$. The axioms for $\mathbb{E}\mathbb{T}_s$ are those for $\mathbb{H}\mathbb{R}T_{r,s}$ with the axiom $d(p, f(p)) = 0$ replacing the axioms which say $||f|| \geq r$.

4.7. Theorem. The $L_0$-theory $\mathbb{E}\mathbb{R}T_s$ has the amalgamation property over substructures of models.

4.8. Lemma. Let $(M, d, p, f, v) \models \mathbb{E}\mathbb{R}T_s$. For any $m \in \mathbb{N}$, the set

$$\text{fix}(f^m) := \{x \mid d(x, f^m(x)) = 0\}$$

is a definable set in $(M, d, p, f, v)$.

4.9. Lemma. Let $(M, d, p, f, v) \models \mathbb{E}\mathbb{R}T_s$. Let $k \in \mathbb{N}$ and $x \in M$. Let $\mu_k(x) := \mu(x, f^k(x))$, the midpoint of $[x, f^k(x)]$. This point $\mu_k(x)$ is the unique point in $\text{fix}(f^k)$ which is closest to $x$. 

We may find a $L_0$-theory $\text{rbERT}_s$ such that in an $\omega_1$-saturated model of the theory the following properties are true.

- the underlying metric space $M$ is an e.c. $\mathbb{R}$-tree
- $f$ is an isometry with dense image in $M$
- given $r \in \mathbb{Q}$, and $l, m, n \in \mathbb{N}^+$, and any $x \in M$ with order $m$ there exist points $y_1, ..., y_l$ such that
  1. $x$ is on $[y_i, y_j]$ for $i, j \in \{1, ..., l\}$
  2. $d(x, y_i) = r$ for $i \in \{1, ..., l\}$,
  3. for $i \in \{1, ..., l\}$, for all $k \leq n - 1$ we have $x = \mu_{km}(y_i)$ that is, $x$ is the closest point in $fix(f^{km})$ to $y_i$ and,
  4. $y_i$ has order $mn$ for $i \in \{1, ..., l\}$.
- the distance $d(p, f(p)) = 0$
- the gauge $v(x) = d(x, p)$.

4.10. Lemma. Let $(M, d, p, f, v)$ be an $\omega_1$-saturated model of $\text{rbERT}_s$.

1. For any $a$ with order $m$ and any $r \in \mathbb{R}_0^+$ and $l \in \mathbb{N}$ there are distinct points $b_1, ..., b_l$ on branches at a such that $d(a, b_i) = r$ and all the points in $[a, b_i] \setminus a$ have order $mn$.
2. For any $a$ with order $m$ and any $r \in \mathbb{R}_0^+$ there are uncountably many distinct points $b$ on branches at a such that $d(a, b) = r$ and all the points in $[a, b] \setminus a$ have order $mn$.
3. Let $r \in \mathbb{R}_0^+$. At every point $a$ in $M$, there are distinct points $\{b_i | i \in \mathbb{N}\}$ with disjoint orbits such that $d(a, b_i) = r$, and all the points in $[a, b_i] \setminus a$ have infinite order. In fact, we get uncountably many such points.

Proof. Statement (1) follows from the axioms. Statement (2) follows from (1) via a saturation argument, as does (3). □

4.11. Lemma. Let $(M, d, p, f, v)$ be an $\omega_1$-saturated model of $\text{rbERT}_s$. Given:

- $e \in M$
- a collection of branches $\{B_i | i \in \omega\}$ at $e$
- finitely generated $(K, d, q, g, v) \models \text{ERT}_s$ with $q$ a generator, where none of the generators are related by $g$.
- $(K', d, q, g, v)$ such that:
  1. $(K', d, q, g, v) \subseteq (K, d, q, g, v)$ is the substructure generated by $q$.
  2. every generator besides $q$ is on a branch at $q$ which intersects $K'$ exactly at $q$.
  3. there exists an isometric embedding $\hat{\varphi} : K' \rightarrow M$ sending $q$ to $e$ such that $f \circ \hat{\varphi} = \hat{\varphi} \circ g$.

there exists an isometric embedding $\varphi$ of $K$ extending $\hat{\varphi}$ into $M$ such that $\varphi(K) \cap (\cup_{i \in \omega} B_i) = \{e\}$, and such that $f \circ \varphi = \varphi \circ g$.

4.12. Theorem. The $L_0$-theory $\text{rbERT}_s$ is the model companion of $\text{ERT}_s$.

All the model companion theories discussed above have QE. The $L_s$-theory $\text{rbHRT}_{r,s}$ can be made complete by adding an axiom specifying the
translation distance exactly. The \( L_0 \)-theory \( rbE_R T_s \) is already complete. These theories are not \( \omega \)-stable, but are \( \kappa \)-stable if and only if \( \kappa = \kappa^\omega \). Their independence relations are analogous to the independence relation for \( rbT \).

5. Group actions on \( \mathbb{R} \)-trees and future work

The thesis also begins to investigate the model theory of isometric actions of arbitrary finitely generated groups on \( \mathbb{R} \)-trees. One possible signature to use is,

\[
[(M, d, p, v); \{f_i | i = 1, ..., 2n\}]
\]

where the \( f_i \) will be interpreted by isometries on the \( \mathbb{R} \)-tree \((M, d, p, v)\), with \( f_{n+i} \) interpreted as the inverse function of \( f_i \) for \( i = 1, ..., n \).

Model companions are found in the thesis for two extreme types of group actions on \( \mathbb{R} \)-trees. One extreme is when all of the group elements correspond to hyperbolic isometries with a uniform lower bound on their translation distances. In this case a theorem in [11] implies that the group involved must be a free group. The other extreme is when all the group elements correspond to elliptic isometries. This is what geometric group theorists call a “trivial” action. The model companions in these cases are analogous to those in the case of a single isometry.

Group actions on \( \mathbb{R} \)-trees comprise one area of further research. What can be said about actions of finitely generated groups which have a mixture of elliptic and hyperbolic isometries? For example, if we fix a group \( G \) and specify data such as a translation distance \( ||g|| \) for each \( g \in G \), is there a model companion? If so, what are the axioms and what properties does it have? Also, very generally, what connections to Geometric Group Theory can be made?

Perturbations are another area of future exploration. Since any perturbation between \( \mathbb{R} \)-trees will be a homeomorphism, it is already clear that \( rbT \) will not be categorical up to perturbations. What perturbations of \( \mathbb{R} \)-trees might be and what properties \( rbT \) has up to perturbation are left to be investigated.

The original inspiration for the contents of the thesis was the investigation of the continuous model theory of asymptotic cones of finitely generated groups.

5.1. Definition. Let \((M, d)\) be a metric space, \((p_n)\) a sequence of points in \( M \), \((d_n)\) a sequence of positive real numbers with \( \lim_{n \to \infty} d_n = \infty \) and \( U \) a non-principal ultrafilter on \( \mathbb{N} \). Let

\[
K := \{(x_n)|x_n \in M \text{ and there is } c \in \mathbb{R}^{\geq 0} \text{ such that } \forall n \frac{d(p_n, x_n)}{d_n} \leq c\}.
\]

Define \( \rho((x_n), (y_n)) := \lim_U \frac{d(x_n, y_n)}{d_n} \), which is a pseudometric on \( K \). The quotient of \( K \) with respect to \( \rho \) is called the asymptotic cone of \( M \), with respect to \((p_n), (d_n)\) and \( U \).
Note that this definition is an ultraproduct of a certain family of gauged metric spaces.

The Cayley graph of a group $G$ with respect to a generating set $A$, is a graph with a vertex for each $g \in G$, where for each $g \in G$ and $a \in A$ the vertices $g$ and $ga$ are joined by an edge. If we make each edge a copy of $[0,1]$, then the Cayley graph is a geodesic metric space.

5.2. **Definition.** A finitely generated group is called *hyperbolic* if its Cayley graph with respect to some generating set is a $\delta$-hyperbolic metric space for some $\delta > 0$. A hyperbolic group is *elementary* if it contains a cyclic subgroup of finite index.

Any asymptotic cone of any non-elementary hyperbolic group is isometric to the $2^{2\omega}$-universal $\mathbb{R}$-tree $A_{2\omega}$ mentioned earlier (see [9].) The study of asymptotic cones of finitely generated groups is a rich area of potential applications for continuous model theory.

**References**