Continuous logic and model theory of metric structures

C. Ward Henson
University of Illinois

July 28, 2009
UCLA Logic Center
Summer School for Undergraduates
Many classes of metric structures arising in analysis and geometry are well-behaved model theoretically, although they are not axiomatizable in the classical sense. Examples:

- (unit balls of) Hilbert spaces and other Banach spaces
- probability measure algebras
- (projective spaces of) nonarchimedean valued fields, with valuations in $\mathbb{R}$

Continuous logic is an attempt to apply model theoretic tools to such classes.

1 Structures and syntax

(bounded) metric structures

- A bounded metric structure $\mathcal{M}$ is based on a bounded metric space $(M, d)$. $\mathcal{M}$ is also equipped with distinguished elements, functions (mapping $M^n$ to $M$), and predicates (mapping $M^n$ to a bounded interval in $\mathbb{R}$, such as $[0,1]$).
- The functions and predicates must be uniformly continuous.

Replace $\{T,F\}$ with $[0,1]$

- The basic idea of continuous logic is: replace the space of truth values $\{T,F\}$ by a compact interval in $\mathbb{R}$, such as $[0,1]$.
- Quantifiers $\forall x$ and $\exists x$ are replaced by $\sup_x$ and $\inf_x$.
- Connectives are continuous functions.

Symbols and signatures

- A signature $\mathcal{L}$ for continuous logic consists of symbols for constants, functions, and predicates, as usual.
  - constant symbols: interpreted as distinguished elements of $M$.
  - $n$-ary function symbols: interpreted as functions $M^n \to M$.
  - $n$-ary predicate symbols: interpreted as functions $M^n \to [0,1]$.
- $\mathcal{L}$ specifies a modulus of uniform continuity for each function symbol and predicate symbol. (e.g.: 1-Lipschitz.)
- The metric is considered as a (logical) binary predicate (exactly as equality is used in classical logic).
Terms and atomic formulas

- **Terms** of $\mathcal{L}$ are defined inductively using variables, constant symbols, and function symbols, as usual.

- **Atomic formulas** of $\mathcal{L}$ are expressions of the form $P(t_1, \ldots, t_n)$, where $P$ is an $n$-ary predicate symbol of $\mathcal{L}$ and $t_1, \ldots, t_n$ are terms of $\mathcal{L}$.

Example: probability algebras

The signature of probability algebras is $\mathcal{L}_{Pr} = \{0, 1, ^c, \cap, \cup, \mu\}$.

- 0, 1 are constant symbols.
- $^c$ (complement) is a unary function symbol.
- $\cup, \cap$ (union, intersection) are binary function symbols.
- $\mu$ (probability) is a unary predicate symbol.

Thus:

- $z, x \cap y^c, x \triangle y$ are terms (values in the algebra).
- $\mu(x), \mu(x \cap y^c)$ are atomic formulas (values in $[0, 1]$).

Formulas

The **formulas** of a continuous signature $\mathcal{L}$ are built inductively starting from the atomic formulas of $\mathcal{L}$, as follows:

- If $\varphi_1, \ldots, \varphi_m$ are formulas and $u: [0, 1]^m \to [0, 1]$ is continuous, then $u(\varphi_1, \ldots, \varphi_m)$ is a formula.

- If $\varphi$ is a formula and $x$ is a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are formulas.

Probability algebras, continued.

- Some useful connectives:

\[
\begin{align*}
\min(a, b) &\quad \max(a, b) \\
|a - b| &\quad a \div b
\end{align*}
\]

(Here we define $a \div b := \max(a - b, 0)$.)

- Thus (in $\mathcal{L}_{Pr}$) $|\mu(x) - \mu(y)|$ is a quantifier-free formula, and

- $\sup_x \inf_y |\mu(x \cap y) - \frac{1}{2}\mu(x)|$ is a formula (it is in fact a sentence, since it has no free variables.)
Equality...?

In classical logic the symbol $=$ always satisfies:

\[
\begin{align*}
  x &= x \\
  (x = y) &\rightarrow (x = z) \rightarrow (y = z)
\end{align*}
\]  

(ER)

Replacing “$x = y$” with “$d(x, y)$” and “$\varphi \rightarrow \psi$” with “$(\psi \sim \varphi)$”:

\[
\begin{align*}
  d(x, x) &= 0 \\
  \left( (d(y, z) \sim d(x, z)) \sim d(x, y) \right) &= 0
\end{align*}
\]

That is, $d$ is a pseudo-metric:

\[
\begin{align*}
  d(x, x) &= 0 \\
  d(y, z) &\leq d(x, z) + d(x, y)
\end{align*}
\]  

(PM)

Similarly, in classical logic $=$ is a congruence relation:

\[
(x = y) \rightarrow \left( P(x, \bar{z}) \rightarrow P(y, \bar{z}) \right)
\]  

(CR)

This translates to:

\[
\left( (P(y, \bar{z}) \sim P(x, \bar{z})) \sim d(x, y) \right) = 0
\]

which says that $P$ is 1-Lipschitz:

\[
P(y, \bar{z}) - P(x, \bar{z}) \leq d(x, y).
\]  

(1L)

So, a literal translation of these statements from classical logic requires all predicate and function symbols to be 1-Lipschitz in $d$. (We can relax this to “uniformly continuous”.)

\textit{\textbf{\textit{L}}-Structures}

\textbf{Definition.} An \textit{\textbf{\textit{L}}}-pre-structure $\mathcal{M}$ is a set $M$, equipped with a pseudo-metric $d^\mathcal{M}$ and interpretations $c^\mathcal{M}$, $f^\mathcal{M}$, $P^\mathcal{M}$ of all symbols $c, f, P \in \mathcal{L}$ such that every $f^\mathcal{M}$ and $P^\mathcal{M}$ satisfies the modulus specified by $\mathcal{L}$.

$\mathcal{M}$ is an \textit{\textbf{\textit{L}}-structure} if, in addition, $d^\mathcal{M}$ is a complete metric.

- In classical logic $=^\mathcal{M}$ is a congruence relation; thus the quotient of $M$ by $=^\mathcal{M}$ is well-defined and it cannot be distinguished from $\mathcal{M}$ by the logic.

- Similarly, in continuous logic a prestructure $\mathcal{M}$ is logically indistinguishable from the completion $\hat{\mathcal{N}}$ of the quotient $\mathcal{N} = \mathcal{M}/\sim_d$. 

\[
(a \sim_d b \iff d(a, b) = 0)
\]

- We call $\mathcal{M}/\sim_d$ the \textit{completion} of the pre-structure $\mathcal{M}$. 

4
Probability algebras, redux

- Let \((\Omega, \mathcal{B}, \mu)\) be a probability space.

- Let \(\mathcal{I}_0 \leq \mathcal{B}\) be the ideal of \(\mu\)-null sets, and \(\hat{\mathcal{B}} = \mathcal{B}/\mathcal{I}_0\). Then \(\hat{\mathcal{B}}\) is a Boolean algebra and \(\mu\) induces \(\hat{\mu} : \hat{\mathcal{B}} \to [0,1]\). The pair \((\hat{\mathcal{B}}, \hat{\mu})\) is a probability algebra.

- \(\hat{\mathcal{B}}\) admits a complete metric: \(d(a,b) = \hat{\mu}(a \triangle b)\).

- \(\hat{\mu}\) and the Boolean operations are 1-Lipschitz.

- \((\mathcal{B}, 0, 1, \cap, \cup, \cdot, \mu)\) is a pre-structure; \((\hat{\mathcal{B}}, 0, 1, \cap, \cup, \cdot, \hat{\mu})\) is its completion (in particular, it is a structure).

2 Semantics

Semantics

As usual, the notation \(\phi(x_1, \ldots, x_n)\) [or simply \(\phi(\bar{x})\)] means that the free variables of \(\phi\) are among \(x_1, \ldots, x_n\), which are distinct.

- If \(\mathcal{M}\) is a structure and \(\bar{a} \in M^n\), we define the (truth) value \(\phi^\mathcal{M}(\bar{a}) \in [0,1]\) inductively, in the “obvious way”.

- Each function \(\phi^\mathcal{M} : M^n \to [0,1]\) is uniformly continuous (by induction on \(\phi\)).

Example. Let \(\mathcal{M} = (M, 0, 1, \cap, \cup, \cdot, \mu)\) be a probability algebra, and take \(\phi(x)\) to be the formula \(\inf_y |\mu(x \cap y) - \frac{1}{2}\mu(x)|\).

- If \(a \in M\) is an atom, then \(\phi^\mathcal{M}(a) = \frac{1}{2}\mu(a)\).

- If \(a\) has no atoms below it then \(\phi^\mathcal{M}(a) = 0\).

Various “elementary” notions

- Elementary equivalence (denoted \(\mathcal{M} \equiv \mathcal{N}\)): If \(\mathcal{M}, \mathcal{N}\) are two structures then \(\mathcal{M} \equiv \mathcal{N}\) if \(\phi^\mathcal{M} = \phi^\mathcal{N} \in [0,1]\) for every sentence \(\phi\).

  Equivalently: \(\phi^\mathcal{M} = 0 \iff \phi^\mathcal{N} = 0\) for all sentences \(\phi\).

- Elementary extension (denoted \(\mathcal{M} \preceq \mathcal{N}\)): This holds if \(\mathcal{M} \subseteq \mathcal{N}\) and \(\phi^\mathcal{M}(\bar{a}) = \phi^\mathcal{N}(\bar{a})\) for every formula \(\phi(\bar{x})\) and \(\bar{a} \in M\). It implies \(\mathcal{M} \equiv \mathcal{N}\).

Lemma (Elementary chains). The union of an elementary chain \(\mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \ldots\) is an elementary extension of each \(\mathcal{M}_i\).

Caution: by the union of an increasing chain we mean the completion of its set-theoretic union.
3 Ultraproducts

Ultraproducts

- $(\mathcal{M}_i \mid i \in I)$ are $\mathcal{L}$-structures, $\mathcal{U}$ an ultrafilter on $I$.
- We let $N_0 = \prod_{i \in I} M_i$ as a set.
- We interpret the symbols:
  
  \[
  c^{N_0} = \left( c^{M_i} \mid i \in I \right),
  f^{N_0}(\langle a_i \mid i \in I \rangle, \ldots) = \left( f^{M_i}(a_i, \ldots) \mid i \in I \right) \in N_0,
  P^{N_0}(\langle a_i \mid i \in I \rangle, \ldots) = \lim_{i, \mathcal{U}} P^{M_i}(a_i, \ldots) \in [0, 1]
  \]

- This makes $N_0$ a pre-structure. We define the ultraproduct to be $\hat{N}_0$ (the completion) and denote it by $\prod_{i \in I} M_i / \mathcal{U}$.
- The image of $(\bar{a}) \in N_0$ in $\hat{N}_0$ is denoted $(\bar{a})_{\mathcal{U}}$; note that
  
  \[
  (\bar{a})_{\mathcal{U}} = (\bar{b})_{\mathcal{U}} \iff 0 = \lim_{i, \mathcal{U}} d(a_i, b_i) = d^{N_0}((\bar{a}), (\bar{b})).
  \]

Properties of ultraproducts

- *Loś’s Theorem*: for every formula $\varphi(x, y, \ldots)$ and elements $(\bar{a})_{\mathcal{U}}, (\bar{b})_{\mathcal{U}}, \ldots \in N = \prod M_i / \mathcal{U}:
  
  $\varphi^N((\bar{a})_{\mathcal{U}}, (\bar{b})_{\mathcal{U}}, \ldots) = \lim_{i, \mathcal{U}} \varphi^{M_i}(a_i, b_i, \ldots)$.

- [Easy] $\mathcal{M} \equiv \mathcal{N}$ ($\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent) if and only if $\mathcal{M}$ admits an elementary embedding into an ultrapower of $\mathcal{N}$.

- [Deeper: generalising Keisler & Shelah] $\mathcal{M} \equiv \mathcal{N}$ if and only if $\mathcal{M}$ and $\mathcal{N}$ have ultrapowers that are isomorphic.

4 Theories

Theories

- A *theory* $T$ is a set of sentences (closed formulas).
- $\mathcal{M}$ is a *model* of $T$ (written $\mathcal{M} \models T$) if
  
  $\varphi^\mathcal{M} = 0$ for all $\varphi \in T$. 

• We sometimes write $T$ as a set of conditions “$\varphi = 0$”. We may also allow as conditions things of the form “$\varphi \leq r$”, “$\varphi \geq r$”, “$\varphi = r$”, etc.

• If $M$ is any structure then its theory is
  \[ \text{Th}(M) = \{ \varphi \mid \varphi^M = 0 \}. \]

  Theories of this form are called complete.

Compactness

Theorem (Compactness). A theory is satisfiable if and only if it is finitely satisfiable.

Note that:
\[ T \equiv \{ \text{"$\varphi \leq 2^{-n}$"} \mid n < \omega \text{ and "$\varphi = 0$"} \in T \}. \]

Corollary. Assume that $T$ is approximately finitely satisfiable. Then $T$ is satisfiable.

Löwenheim-Skolem Theorem

Notation: $|| \cdot ||$ denotes the metric density character.

$L$ is a signature with $\leq \kappa$ symbols.

$M$ is an $L$-structure and $A \subseteq M$ has $||A|| \leq \kappa$.

Theorem. There exists an elementary substructure $N$ of $M$ such that $A \subseteq N$ and $||N|| \leq \kappa$.

Elementary (axiomatizable) classes

A class $C$ of $L$-structures is elementary or axiomatizable if there is an $L$-theory $T$ such that $C$ is the class of all models of $T$. When this holds we call $T$ a set of axioms for $C$.

Theorem. Let $C$ be a class of $L$-structures. Then $C$ is axiomatizable iff $C$ is closed under isomorphisms, ultraproducts, and ultraroots.

Here: $M$ is an ultraroot of $N$ if $N$ is isomorphic to some ultrapower of $M$.

Universal theories

• A theory consisting solely of sentences of the form $\sup_{\bar{x}} \varphi(\bar{x})$, where $\varphi$ is quantifier-free, is called universal. Universal theories are those stable under substructures.

• We may write $\left( \sup_{\bar{x}} \varphi \right) = 0$ as $\forall \bar{x} (\varphi = 0)$.

• Similarly, we may write $\left( \sup_{\bar{x}} |\varphi - \psi| \right) = 0$ as $\forall \bar{x} (\varphi = \psi)$.

• And if $\sigma, \tau$ are terms: we may write $\left( \sup_{\bar{x}} d(\sigma, \tau) \right) = 0$ as $\forall \bar{x} (\sigma = \tau)$.  

The (universal) theory of probability algebras

The class of probability algebras is axiomatized by the following set $Pr$ of conditions:

The equational axioms for Boolean algebras

$$\forall xy \left( d(x, y) = \mu(x \triangle y) \right)$$

$$\forall xy \left( \mu(x) + \mu(y) = \mu(x \cap y) + \mu(x \cup y) \right)$$

$$\mu(1) = 1$$

The model companion of $Pr$ is the $\forall \exists$-theory $APr$ of atomless probability algebras, consisting of $Pr$ together with:

$$\sup_x \inf_y |\mu(x \cap y) - \frac{1}{2} \mu(x)| = 0.$$ 

The theory $APr$ is $\omega$-categorical, and it is therefore complete; it also satisfies quantifier elimination.

Probability algebras with an automorphism

An interesting research topic concerns the metric structures $(M, \tau)$ where $M$ is a probability algebra and $\tau$ is an automorphism of $M$. These all arise from measure preserving automorphisms of a probability space.

The fact that $\tau$ is an automorphism (of a model of $Pr$) can be axiomatized in a suitable signature $L_{Pr, \tau}$ extending $L_{Pr}$, by a theory $Pr_{\tau}$. For example

$$\sup_x |\mu(\tau(x)) - \mu(x)| = 0$$

expresses the fact that $\tau$ is measure preserving and

$$\sup_y \inf_x d(y, \tau(x)) = 0$$

expresses the fact that $\tau$ is surjective (given that $\tau$ is isometric).

Of special interest are the $(M, \tau) \models Pr_{\tau}$ that arise from an aperiodic automorphism $S$ of an atomless probability space $(\Omega, \mathcal{B}, \mu)$; this means that for each $n \in \mathbb{N}$ the set $\{\omega \in \Omega \mid S^n(\omega) = \omega\}$ has measure 0. This property can be axiomatized (over $Pr_{\tau}$) by the conditions (for $n \geq 1$)

$$\inf_x \max \left( 1/n - \mu(x), \mu(x \cap \tau(x)), \ldots, \mu(x \cap \tau^{n-1}(x)) \right) = 0.$$ 

The theory consisting of $Pr_{\tau}$ and these conditions is: $APr_A$.

**Theorem.** (Berenstein and WH) The theory $APr_A$ is the model companion of $Pr_{\tau}$; it is complete, has quantifier elimination, and is stable. Further (Ben Yaacov), $APr_A$ is not superstable; thus it is $\kappa$-stable iff $\kappa^+ = \kappa$. 


Model theory for metric structures, based on continuous logic


  [http://math.uiuc.edu/~henson/](http://math.uiuc.edu/~henson/)


Probability algebras

