Continuous logic and model theory of metric structures

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Many classes of *metric structures* arising in analysis and geometry are well-behaved model theoretically, although they are not axiomatizable in the classical sense. Examples:

- (unit balls of) Hilbert spaces and other Banach spaces
- probability measure algebras
- (projective spaces of) nonarchimedean valued fields, with valuations in $\mathbb{R}$

Continuous logic is an attempt to apply model theoretic tools to such classes.

1 Structures and syntax

(bounded) metric structures

- A bounded *metric structure* $\mathcal{M}$ is based on a bounded metric space (M, d).

  $\mathcal{M}$ is also equipped with distinguished elements, functions (mapping M^n to M), and predicates (mapping M^n to a bounded interval in $\mathbb{R}$, such as $[0, 1]$).

- The functions and predicates must be uniformly continuous.

Replace $\{T, F\}$ with $[0, 1]$

- The basic idea of continuous logic is: replace the space of truth values $\{T, F\}$ by a compact interval in $\mathbb{R}$, such as $[0, 1]$.
- Quantifiers $\forall x$ and $\exists x$ are replaced by $\sup_x$ and $\inf_x$.
- Connectives are continuous functions.

Symbols and signatures

- A *signature* $\mathcal{L}$ for continuous logic consists of symbols for constants, functions, and predicates, as usual.
  - constant symbols: interpreted as distinguished elements of M.
  - n-ary function symbols: interpreted as functions $M^n \to M$.
  - n-ary predicate symbols: interpreted as functions $M^n \to [0, 1]$.
- $\mathcal{L}$ specifies a *modulus of uniform continuity* for each function symbol and predicate symbol. (e.g.: 1-Lipshitz.)
- The *metric* is considered as a (logical) binary predicate
  (exactly as equality is used in classical logic).

Terms and atomic formulas

- *Terms* of $\mathcal{L}$ are defined inductively using variables, constant symbols, and function symbols, as usual.

- *Atomic formulas* of $\mathcal{L}$ are expressions of the form $P(t_1, \dots, t_n)$, where P is an n-ary predicate symbol of $\mathcal{L}$ and $t_1, \dots, t_n$ are terms of $\mathcal{L}$.

Example: probability algebras

The signature of probability algebras is $\mathcal{L}_{Pr} = \{0, 1, \cdot^c, \cap, \cup, \mu\}$.

- 0, 1 are constant symbols.
- $\cdot^c$ (complement) is a unary function symbol.
- $\cup$, $\cap$ (union, intersection) are binary function symbols.
- μ (probability) is a unary predicate symbol.

Thus:

- $z, x \cap y^c, x \triangle y$ are *terms* (values in the algebra).
- $\mu(x), \mu(x \cap y^c)$ are *atomic formulas* (values in $[0, 1]$).

Formulas

The *formulas* of a continuous signature $\mathcal{L}$ are built inductively starting from the atomic formulas of $\mathcal{L}$, as follows:

- If $\varphi_1, \dots, \varphi_m$ are formulas and $u\colon [0,1]^m \to [0,1]$ is continuous, then $u(\varphi_1, \dots, \varphi_m)$ is a formula.

- If φ is a formula and x is a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are formulas.

Probability algebras, continued.

- Some useful connectives:
$$\begin{array}{cc} \min(a, b) & \max(a, b) \\ |a - b| & a \dot{-} b \end{array}$$
(Here we define $a \dot{-} b := \max(a - b, 0)$.)

- Thus (in $\mathcal{L}_{Pr}$) $|\mu(x) - \mu(y)|$ is a *quantifier-free* formula, and

- $\sup_x \inf_y \left|\mu(x \cap y) - \frac{1}{2}\mu(x)\right|$ is a formula (it is in fact a *sentence*, since it has no *free variables*.

Equality...?

In classical logic the symbol $=$ always satisfies:

$$x = x \qquad\qquad (x = y) \to \Big((x = z) \to (y = z)\Big) \tag{ER}$$

Replacing "$x = y$" with "$d(x, y)$" and "$\varphi \to \psi$" with "$(\psi \dot- \varphi)$":

$$d(x, x) = 0 \qquad\qquad \Big(\big(d(y, z) \dot- d(x, z)\big) \dot- d(x, y)\Big) = 0$$

That is, d is a pseudo-metric:

$$d(x, x) = 0 \qquad\qquad d(y, z) \leq d(x, z) + d(x, y) \tag{PM}$$

Similarly, in classical logic $=$ is a congruence relation:

$$(x = y) \to \Big(P(x, \bar{z}) \to P(y, \bar{z})\Big) \tag{CR}$$

This translates to:

$$\Big(\big(P(y, \bar{z}) \dot- P(x, \bar{z})\big) \dot- d(x, y)\Big) = 0$$

which says that P is 1-Lipschitz:

$$P(y, \bar{z}) - P(x, \bar{z}) \leq d(x, y). \tag{1L}$$

So, a literal translation of these statements from classical logic requires all predicate and function symbols to be 1-Lipschitz in d. (We can relax this to "uniformly continuous".)

$\mathcal{L}$-Structures

Definition. An *$\mathcal{L}$-pre-structure* $\mathcal{M}$ is a set M, equipped with a *pseudo*-metric $d^{\mathcal{M}}$ and interpretations $c^{\mathcal{M}}$, $f^{\mathcal{M}}$, $P^{\mathcal{M}}$ of all symbols $c, f, P \in \mathcal{L}$ such that every $f^{\mathcal{M}}$ and $P^{\mathcal{M}}$ satisfies the modulus specified by $\mathcal{L}$.

$\mathcal{M}$ is an *$\mathcal{L}$-structure* if, in addition, $d^{\mathcal{M}}$ is a complete metric.

- In classical logic $=^{\mathcal{M}}$ is a congruence relation; thus the quotient of M by $=^{\mathcal{M}}$ is well-defined and it cannot be distinguished from $\mathcal{M}$ by the logic.
- Similarly, in continuous logic a prestructure $\mathcal{M}$ is logically indistinguishable from the completion $\widehat{\mathcal{N}}$ of the quotient $\mathcal{N} = \mathcal{M}/{\sim_d}$. $\qquad (a \sim_d b \iff d(a, b) = 0)$
- We call $\widehat{\mathcal{M}/{\sim_d}}$ the *completion* of the pre-structure $\mathcal{M}$.

Probability algebras, redux

- Let $(\Omega, \mathfrak{B}, \mu)$ be a probability space.
- Let $\mathfrak{I}_0 \leq \mathfrak{B}$ be the ideal of μ-null sets, and $\widehat{\mathfrak{B}} = \mathfrak{B}/\mathfrak{I}_0$. Then $\widehat{\mathfrak{B}}$ is a Boolean algebra and μ induces $\widehat{\mu}\colon \widehat{\mathfrak{B}} \to [0,1]$. The pair $(\widehat{\mathfrak{B}}, \widehat{\mu})$ is a *probability algebra*.
- $\widehat{\mathfrak{B}}$ admits a complete metric: $d(a,b) = \widehat{\mu}(a \triangle b)$.
- $\widehat{\mu}$ and the Boolean operations are 1-Lipschitz.
- $(\mathfrak{B}, 0, 1, \cap, \cup, \cdot^c, \mu)$ is a pre-structure;
  $(\widehat{\mathfrak{B}}, 0, 1, \cap, \cup, \cdot^c, \widehat{\mu})$ is its completion (in particular, it is a structure).

2 Semantics

Semantics

As usual, the notation $\varphi(x_1, \ldots, x_n)$ [or simply $\varphi(\bar{x})$] means that the free variables of φ are among $x_1, \ldots, x_n$, which are distinct.

- If $\mathcal{M}$ is a structure and $\bar{a} \in M^n$, we define the *(truth) value* $\varphi^{\mathcal{M}}(\bar{a}) \in [0,1]$ inductively, in the "obvious way".
- Each function $\varphi^{\mathcal{M}}\colon M^n \to [0,1]$ is uniformly continuous (by induction on φ).

Example. Let $\mathcal{M} = (M, 0, 1, \cdot^c, \cup, \cap, \mu)$ be a probability algebra, and take $\varphi(x)$ to be the formula $\inf_y \left|\mu(x \cap y) - \frac{1}{2}\mu(x)\right|$.

- If $a \in M$ is an atom, then $\varphi^{\mathcal{M}}(a) = \frac{1}{2}\mu(a)$.
- If a has no atoms below it then $\varphi^{\mathcal{M}}(a) = 0$.

Various "elementary" notions

- *Elementary equivalence* (denoted $\mathcal{M} \equiv \mathcal{N}$): If $\mathcal{M}, \mathcal{N}$ are two structures then $\mathcal{M} \equiv \mathcal{N}$ if $\varphi^{\mathcal{M}} = \varphi^{\mathcal{N}} \in [0,1]$ for every sentence φ.

  Equivalently: $\varphi^{\mathcal{M}} = 0 \iff \varphi^{\mathcal{N}} = 0$ for all sentences φ.
- *Elementary extension* (denoted $\mathcal{M} \preceq \mathcal{N}$): This holds if $\mathcal{M} \subseteq \mathcal{N}$ and $\varphi^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{N}}(\bar{a})$ for every formula $\varphi(\bar{x})$ and $\bar{a} \in M$. It implies $\mathcal{M} \equiv \mathcal{N}$.

Lemma (Elementary chains)**.** *The union of an elementary chain $\mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \ldots$ is an elementary extension of each $\mathcal{M}_i$.*

Caution: by the *union* of an increasing chain we mean the completion of its set-theoretic union.

3 Ultraproducts

Ultraproducts

- $(\mathcal{M}_i \mid i \in I)$ are $\mathcal{L}$-structures, $\mathscr{U}$ an ultrafilter on I.
- We let $N_0 = \prod_{i \in I} M_i$ as a set.
- We interpret the symbols:

$$\begin{array}{lll} c^{\mathcal{N}_0} & = \big(c^{\mathcal{M}_i} \mid i \in I\big) & \\ f^{\mathcal{N}_0}\big((a_i \mid i \in I), \ldots\big) & = \big(f^{\mathcal{M}_i}(a_i, \ldots) \mid i \in I\big) & \in N_0 \\ P^{\mathcal{N}_0}\big((a_i \mid i \in I), \ldots\big) & = \lim_{i,\mathscr{U}} P^{\mathcal{M}_i}(a_i, \ldots) & \in [0,1] \end{array}$$

- This makes $\mathcal{N}_0$ a pre-structure. We define the *ultraproduct* to be $\widehat{\mathcal{N}_0}$ (the completion) and denote it by $\prod_{i \in I} \mathcal{M}_i / \mathscr{U}$.
- The image of $(\bar{a}) \in N_0$ in $\widehat{N_0}$ is denoted $(\bar{a})_{\mathscr{U}}$; note that

$$(\bar{a})_{\mathscr{U}} = (\bar{b})_{\mathscr{U}} \Longleftrightarrow 0 = \lim_{i,\mathscr{U}} d(a_i, b_i) \quad \Big[= d^{\mathcal{N}_0}\big((\bar{a}), (\bar{b})\big)\Big].$$

Properties of ultraproducts

- *Łoś's Theorem*: for every formula $\varphi(x, y, \ldots)$ and elements $(\bar{a})_{\mathscr{U}}, (\bar{b})_{\mathscr{U}}, \ldots \in \mathcal{N} = \prod \mathcal{M}_i / \mathscr{U}$:

$$\varphi^{\mathcal{N}}\big((\bar{a})_{\mathscr{U}}, (\bar{b})_{\mathscr{U}}, \ldots\big) = \lim_{i,\mathscr{U}} \varphi^{\mathcal{M}_i}(a_i, b_i, \ldots).$$

- [Easy] $\mathcal{M} \equiv \mathcal{N}$ ($\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent) if and only if $\mathcal{M}$ admits an elementary embedding into an ultrapower of $\mathcal{N}$.
- [Deeper: generalising Keisler & Shelah] $\mathcal{M} \equiv \mathcal{N}$ if and only if $\mathcal{M}$ and $\mathcal{N}$ have ultrapowers that are isomorphic.

4 Theories

Theories

- A *theory* T is a set of sentences (closed formulas).
- $\mathcal{M}$ is a *model* of T (written $\mathcal{M} \vDash T$) if
  $\varphi^{\mathcal{M}} = 0$ for all $\varphi \in T$.

- We sometimes write T as a set of *conditions* "$\varphi = 0$". We may also allow as conditions things of the form "$\varphi \leq r$", "$\varphi \geq r$", "$\varphi = r$", etc.
- If $\mathcal{M}$ is any structure then its *theory* is
$$\mathrm{Th}(\mathcal{M}) = \{\varphi \mid \varphi^{\mathcal{M}} = 0\}.$$
Theories of this form are called *complete*.

Compactness

Theorem (Compactness)**.** *A theory is satisfiable if and only if it is finitely satisfiable.*

Note that:
$$T \equiv \{\text{“}\varphi \leq 2^{-n}\text{”} \mid n < \omega \text{ and “}\varphi = 0\text{”} \in T\}.$$

Corollary. *Assume that T is approximately finitely satisfiable. Then T is satisfiable.*

Löwenheim-Skolem Theorem

Notation: $\|\cdot\|$ denotes the *metric* density character.
$\mathcal{L}$ is a signature with $\leq \kappa$ symbols.
$\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$ has $\|A\| \leq \kappa$.

Theorem. *There exists an elementary substructure $\mathcal{N}$ of $\mathcal{M}$ such that $A \subseteq N$ and $\|N\| \leq \kappa$.*

Elementary (axiomatizable) classes

A class $\mathcal{C}$ of $\mathcal{L}$-structures is *elementary* or *axiomatizable* if there is an $\mathcal{L}$-theory T such that $\mathcal{C}$ is the class of all models of T. When this holds we call T a set of *axioms* for $\mathcal{C}$.

Theorem. *Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. Then $\mathcal{C}$ is axiomatizable iff $\mathcal{C}$ is closed under isomorphisms, ultraproducts, and ultraroots.*

Here: $\mathcal{M}$ is an *ultraroot* of $\mathcal{N}$ if $\mathcal{N}$ is isomorphic to some ultrapower of $\mathcal{M}$.

Universal theories

- A theory consisting solely of sentences of the form $\sup_{\bar{x}} \varphi(\bar{x})$, where φ is quantifier-free, is called *universal*. Universal theories are those stable under substructures.
- We may write $\big(\sup_{\bar{x}}\varphi\big) = 0$ as $\forall\bar{x}\big(\varphi = 0\big)$.
- Similarly, we may write $\big(\sup_{\bar{x}}|\varphi - \psi|\big) = 0$ as $\forall\bar{x}\big(\varphi = \psi\big)$.
- And if σ, τ are terms: we may write $\big(\sup_{\bar{x}}d(\sigma, \tau)\big) = 0$ as $\forall\bar{x}(\sigma = \tau)$.

The (universal) theory of probability algebras

The class of probability algebras is axiomatized by the following set Pr of conditions:

The equational axioms for Boolean algebras

$$\forall xy \Big(d(x,y) = \mu(x \triangle y) \Big)$$

$$\forall xy \Big(\mu(x) + \mu(y) = \mu(x \cap y) + \mu(x \cup y) \Big)$$

$$\mu(1) = 1$$

The *model companion* of Pr is the $\forall\exists$-theory APr of *atomless* probability algebras, consisting of Pr together with:

$$\sup_x \inf_y \left| \mu(x \cap y) - \tfrac{1}{2}\mu(x) \right| = 0.$$

The theory APr is ω-categorical, and it is therefore complete; it also satisfies quantifier elimination.

Probability algebras with an automorphism

An interesting research topic concerns the metric structures $(\mathcal{M}, \tau)$ where $\mathcal{M}$ is a probability algebra and τ is an automorphism of $\mathcal{M}$. These all arise from measure preserving automorphisms of a probability space.

The fact that τ is an automorphism (of a model of Pr) can be axiomatized in a suitable signature $\mathcal{L}_{Pr,\tau}$ extending $\mathcal{L}_{Pr}$, by a theory Pr_τ. For example

$$\sup_x \left| \mu(\tau(x)) - \mu(x) \right| = 0$$

expresses the fact that τ is measure preserving and

$$\sup_y \inf_x \, d(y, \tau(x)) = 0$$

expresses the fact that τ is surjective (given that τ is isometric).

Of special interest are the $(\mathcal{M}, \tau) \models Pr_\tau$ that arise from an *aperiodic* automorphism S of an atomless probability space $(\Omega, \mathfrak{B}, \mu)$; this means that for each $n \in \mathbb{N}$ the set $\{\omega \in \Omega \mid S^n(\omega) = \omega\}$ has measure 0. This property can be axiomatized (over Pr_τ) by the conditions (for $n \geq 1$)

$$\inf_x \max \big(1/n \dot{-} \mu(x), \mu(x \cap \tau(x)), \ldots, \mu(x \cap \tau^{n-1}(x)) \big) = 0.$$

The theory consisting of Pr_τ and these conditions is: APr_A.

Theorem. (Berenstein and WH) The theory APr_A is the model companion of Pr_τ; it is complete, has quantifier elimination, and is stable. Further (Ben Yaacov), APr_A is not superstable; thus it is κ-stable iff $\kappa^\omega = \kappa$.

Model theory for metric structures, based on continuous logic

[BBHU] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, in Model Theory with Applications to Algebra and Analysis, Vol. II, eds. Z. Chatzidakis, D. Macpherson, A. Pillay, and A. Wilkie, Lecture Notes series of the London Mathematical Society, No. 350, Cambridge University Press, 2008, 315–427.
http://math.uiuc.edu/~henson/

[BU] Itaï Ben Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*, Transactions of the American Mathematical Society, to appear.
http://math.univ-lyon1.fr/~begnac/

Probability algebras

[Ben] Itaï Ben Yaacov, *Schroedinger's cat*, Israel Journal of Mathematics 153, 2006, 157–191.

[BH] Alexander Berenstein and C. Ward Henson, *Model theory of probability spaces with an automorphism*, in preparation.

[BB] Alexander Berenstein and Itaï Ben Yaacov, *On perturbations of Hilbert spaces and probability algebras with a generic automorphism*, Journal of Logic and Analysis 1:7, 2009, 1–8.