Uncountable Categoricity for Structures Based on Banach Spaces

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There are three sub-themes in this talk (under the overall topic of uncountable categoricity for metric structures):

- Toward characterizations of uncountable categoricity for metric structures.
- Toward structure theory for uncountably categorical metric structures, which seems to require finding analogues of strongly minimal sets.
- Enlarge the collection of uncountably categorical metric structures. Use them as guides toward what could be true in general.

All three projects are very much works in progress.

But first: A skeptical question:

Why study uncountably categorical structures?

After all, there are few of them and the mathematically interesting ones can be studied well without using model theory.

A possible answer:

It's for the tools that you have to develop

to completely understand uncountably categorical metric structures, and the later applications of those tools.

Let T be an arbitrary theory of metric structures, in a countable signature.

Definition (κ-categoricity; $\kappa > \omega$)

T is κ-categorical if T has exactly one model of density $= \kappa$.

Morley's Theorem is known to hold for metric structures:

Theorem

Let κ, λ be uncountable cardinals. Then T is κ-categorical if and only if T is λ-categorical.

This was proved independently by Itaï Ben Yaacov [JSL 2005, for CATs] and Alex Usvyatsov [in his PhD thesis, published with Shelah in Isr.J.Math 2011, for more general notions of categoricity].

- The proofs of Ben Yaacov and Shelah-Usvyatsov have a skeletal structure roughly similar to Morley's argument. In particular, they show κ-categorical $(\kappa > \omega)$ implies ω-stable. This yields tools that are critical for obtaining the result.
- At the moment there is nothing close to a characterization of uncountably categorical theories in the general metric setting.
- However, if one restricts attention to structures based on Banach spaces, the problem of understanding uncountable categoricity seems more tractable at the moment.

For comparison and reference, here is a characterization and structural result due to Baldwin and Lachlan in classical model theory:

Theorem

Let T be a complete theory in a countable language. The following are equivalent:
(1) T is uncountably categorical.
(2) There exists a strongly minimal set D definable over the prime model of T, such that each uncountable model M of T is minimal and prime over D^M.

Prime: every elementary map from D^M into a model N of T extends to an elementary embedding of M into N.
Minimal: There is no proper elementary submodel of M that contains D^M.

We take the signature for unit balls of Banach spaces (over $\mathbb{R}$) to be $\mathcal{L}_b = \{0, c_{r,s}, \| \; \|\}$ where r, s range over rational scalars such that $|r| + |s| \leq 1$. This is a countable continuous signature.

- 0 is a constant symbol.
- $c_{r,s}$ are binary function symbols.
  - Interpret $c_{r,s}(x, y)$ as $rx + sy$.
- $\| \; \|$ is a unary predicate symbol.
- the functions and predicate are 1-Lipschitz in each variable.

Notation

We let T_b denote a theory axiomatizing the class of all (unit balls of) Banach spaces.

In what follows, we always have:

- $\mathcal{L}$ is a countable continuous signature expanding $\mathcal{L}_b$.
- T is a consistent $\mathcal{L}$-theory extending T_b.

So models of T are unit balls of Banach spaces equipped with extra predicates and operations (whose operations map unit balls to unit balls). For example:

- Banach lattices
- C^*-algebras
- Operator spaces

- The most fundamental uncountably categorical theory of Banach structures is the theory IHS of infinite dimensional Hilbert spaces.
- IHS has quantifier elimination.
- $\mathrm{acl}(A) = \mathrm{dcl}(A) =$ closed linear span of A.
- The independence relation for *IHS* is what you would expect, based on orthgonal projection.
- *IHS* is κ-categorical for all infinite κ.

- As model theory in functional analysis developed, it was observed by the speaker that all known examples of uncountably categorical Banach structures are very closely related to Hilbert spaces. (However, it was also observed that very few such structures were known.)
- Shelah and Usvyatsov treated this as a conjecture, and proved the following remarkable result, which describes uncountably categorical Banach structures using Hilbert spaces as the foundation.

Theorem (Shelah,Usvyatsov; arXiv:1402.6513; Adv. in Math.)

Assume T is an uncountably categorical theory of Banach space structures in a countable language. Then there is a separable model $\mathcal{M}_0$ of T and a 1-type $p_0(x)$ over $\mathcal{M}_0$ with the following properties:
(1) Every Morley sequence in p_0 is isometric to the standard orthonormal basis of a Hilbert space.
(2) Every non-separable model $\mathcal{M}$ of T is prime over a Morley sequence $(a_\alpha \mid \alpha < \lambda)$ in p_0, where λ is the density of $\mathcal{M}$.
(3) The type $p_0(x)$ is minimal wide.

Wide types

Let $\pi(x)$ be a partial 1-type over A. With no loss of generality assume $\pi(x)$ contains the condition "$\|x\| = 1$".

Definition

$\pi(x)$ is wide if there exists $\mathcal{M} \models T$ containing A such that $\pi(\mathcal{M})$ contains the unit sphere of an infinite dimensional subspace of $\mathcal{M}$.

Definition

A complete 1-type $p(x)$ over A is minimal wide if $p(x)$ has exactly one extension to a wide global type.

For proving the existence of wide complete 1-types, Milman's far-reaching extension of Dvoretsky's Theorem is used.

Setting: $f\colon S(X) \to \mathbb{R}$ is a uniformly continuous function.

($S(X)$ is the unit sphere of the Banach space X)

Definition

The Hilbert spectrum of f, denoted $\gamma_H(f)$, is the set of $r \in \mathbb{R}$ such that for all $\epsilon > 0$ and all $k \geq 1$ there exists a k-dimensional subspace F of X such that

(1) F is $(1+\epsilon)$-isomorphic to Hilbert space and

(2)for all $x \in S(F)$, we have $|f(x) - r| \leq \epsilon$.

Theorem (Dvoretsky-Milman)

For any infinite dimensional Banach space X and any uniformly continuous function $f: S(X) \to \mathbb{R}$,

$$\gamma_H(f) \neq \varnothing.$$

Corollary (1-step extension of a wide type)

If $\pi(x)$ is a wide partial 1-type over A, and $\varphi(x)$ is any formula over A, then there exists $r \in [0,1]$ such that the partial type $\pi(x) \cup [\varphi(x) = r]$ is wide.

Things to do in pure model theory of Banach structures:

- Improve the Shelah-Usvyatsov theorem. In particular, consider the Hilbert space $H =$ closed linear span of the Morley sequence in $p_0(x)$, and try to:
  - understand how definable H can be in $\mathcal{M}$;
  - explain how $\mathcal{M}$ sits over H in clear functional analysis language.
  - understand the "induced structure" on H.
- The ultimate goal is to obtain

  an exact characterization and structural description

  of uncountably categorical Banach structures.

More things to do:

- Find uncountably categorical Banach structures in whatever corner of functional analysis you find yourself. (Or prove there aren't any there.)
- Isolate model theoretic properties of Hilbert space that are sufficient to imply uncountable categoricity for any Banach structure that satisfies them. (How exactly does Hilbert space deserve to be considered analogous to a strongly minimal set?)

Some cautionary examples:

- No infinite dimensional C^*-algebra is uncountably categorical. (Indeed, they are not even ω-stable.)
- Fix $1 \leq p < \infty, p \neq 2$. No infinite dimensional L_p-space is uncountably categorical, neither as Banach space nor as Banach lattice (even though these theories are all ω-stable).
- No L_2-space is uncountably categorical as a Banach lattice.

Conjecture: No infinite dimensional Banach lattice is uncountably categorical. (This would imply all of the above examples.)

The Jordan-von Neumann (JvN) constant

The JvN constant $a(X)$ of a Banach space X is the smallest $C \geq 1$ such that for all $x, y \in X$

$$(2/C)(\|x\|^2 + \|y\|^2) \leq \|x+y\|^2 + \|x-y\|^2 \leq (2C)(\|x\|^2 + \|y\|^2).$$

Hence $a(X) = 1$ iff X satisfies the parallelogram equality iff X is linearly isometric to a Hilbert space. More generally, the distance between $a(X)$ and 1 measures the extent to which X fails to satisfy the parallelogram identity. Note that $a(X)$ depends only on the 2-dimensional subspaces of X.

Joint with Yves Raynaud; arxiv 1606.03122 (= Comment. Math. 2016) and work in progress

For the next slides we consider the ℓ_2-sum $X = \oplus_2(E_n : n \in \mathbb{N})$, where each E_n is finite dimensional with norm $\| \cdot \|_n$.

Here X consists of all $x = (x_n \mid n \in \mathbb{N})$ with $x_n \in E_n$ for all n and

$$\Theta(x) := \sum_n \|x_n\|_n^2 < \infty.$$

The norm $\| \cdot \|$ on X is given by $\|x\| = \sqrt{\Theta(x)}$.

Theorem (main result)

If $a(E_n) \to 1$ as $n \to \infty$, then X is uncountably categorical.

Indeed, there is a subspace $\mathcal{B}(X)$ of X such that every Banach space elementarily equivalent to X is of the form $\mathcal{B}(X) \oplus_2 H$ where H is a Hilbert space.

Further, if $\mathcal{B}(X)$ is infinite dimensional, then $\mathcal{B}(X) \oplus_2 H$ is ee to X for every Hilbert space H. If X is infinite dimensional but $\mathcal{B}(X)$ is finite dimensional, then H must be infinite dimensional.

Proposition (step 1)

If $a(E_n) \to 1$ as $n \to \infty$, then
(Ult) every ultrapower of X is of the form $DX \oplus_2 H$, where H is a Hilbert space and D is the diagonal embedding of X into the ultrapower.

(This is in Theorem 3.1 of our 2016 paper.)

Remark

If a Banach space Y has the form $E \oplus_2 F$, then each factor determines the other uniquely. For example, F is the set of all $y \in Y$ such that for all $e \in E$, we have $\|e + y\|^2 = \|e - y\|^2$.

Definition

$F \subseteq Y$ is a 2-summand of Y if there exists $E \subseteq Y$ such that $Y = E \oplus_2 F$.
$\mathcal{H}(Y)$ is the set of $y \in Y$ such that $\langle y \rangle$ is a 2-summand of Y.

Proposition

Let Y be any Banach space. Then $\mathcal{H}(Y)$ is a closed linear subspace of Y, it is isometrically a Hilbert space, and it is a 2-summand of Y.

We let $\mathcal{B}(Y)$ denote the unique subspace of Y for which $Y = \mathcal{B}(Y) \oplus_2 \mathcal{H}(Y)$. Note that $\mathcal{B}(Y)$ has no 1-dimensional 2-summand.

Note that every automorphism J of a Banach space Y satisfies $J(\mathcal{H}(Y)) = \mathcal{H}(Y)$ and $J(\mathcal{B}(Y)) = \mathcal{B}(Y)$.

Proposition

Suppose $Y = \oplus_2(Y_i \mid i \in I)$. Then $\mathcal{H}(Y) = \oplus_2(\mathcal{H}(Y_i) \mid i \in I)$, and $\mathcal{B}(Y) = \oplus_2(\mathcal{B}(Y_i) \mid i \in I)$.

Recall $X = \oplus_2(E_n : n \in \mathbb{N})$, where each E_n is finite dimensional. To avoid complete triviality, assume X is infinite dimensional.

By the preceding result $\mathcal{B}(X) = \oplus_2(\mathcal{B}(E_n) \mid n \in \mathbb{N})$ and we know $X = \mathcal{B}(X) \oplus_2 \mathcal{H}(X)$.

We focus on the interesting case, in which $\mathcal{B}(X)$ is infinite dimensional.

Step 1 of our analysis applies to $\mathcal{B}(X)$: if $a(E_n) \to 1$ as $n \to \infty$, then every ultrapower of $\mathcal{B}(X)$ is of the form $D\mathcal{B}(X) \oplus_2 H$ for some Hilbert space H, where D is the diagonal embedding of $\mathcal{B}(X)$ into its ultrapower.

Corollary

(Aut) Every automorphism J of $D\mathcal{B}(X) \oplus_2 H$ satisfies $J(D\mathcal{B}(X)) = D\mathcal{B}(X)$ and $J(H) = H$.

Uncountable categoricity then follows from the following result, proved using model theory, applied to $Y = \mathcal{B}(X)$.

Theorem

Let Y be an infinite dimensional separable Banach space such that

(Ult) Every ultrapower $Y_{\mathcal{U}}$ of Y is the ℓ_2 sum $DY \oplus_2 H$ for some Hilbert space H, where D is the diagonal embedding of Y into $Y_{\mathcal{U}}$, and
(Aut) Every automorphism J of an ultrapower $Y_{\mathcal{U}}$ represented as $DY \oplus_2 H$ as in (Ult), satisfies $J(DY) = DY$ and $J(H) = H$.

Then the elementary class of Y consists exactly of all ℓ_2 direct sums $Y \oplus_2 H$ where H is an arbitrary Hilbert space.