

Ultraproducts as a tool in the model theory of metric structures

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Preliminary comments

- Ultraproducts play a more significant role in model theory of metric structures than in classical model theory of discrete, algebraic structures.
- In the metric setting, ultraproducts do more than proving compactness.
- They provide an important experimental tool for studying metric structures model theoretically.
- Especially toward **verifying axiomatizability** of classes of structures and clarifying **what is definable**.

Early uses of ultraproducts

- hints in work of Skolem (1934) and von Neumann (1942).
- a metric ultraproduct occurs in work of Wright, a student of Kaplansky (1954; Ann. of Math.)
- for discrete structures, the work of Łoś was published in 1955; Robinson's nonstandard analysis in 1961.
- ultraproducts of some operator algebras in Sakurai (1962), McDuff (1969), Connes (1974),
- Banach space ultraproducts in Krivine (1967), nonstandard hulls in Luxemburg (1968); used extensively by Lindenstrauss, Pisier, Johnson, . . . and studied by Stern, Heinrich,
- metric geometry: (ultralimits) Gromov (1981), (asymptotic cones) van den Dries–Wilkie (1984).

A crash course in continuous logic

- **Signatures \mathcal{L}** specify *moduli of uniform continuity* for their function symbols F and predicate symbols P .
- **\mathcal{L} -structures**: complete metric spaces equipped with constants, functions, and predicates that interpret the symbols of \mathcal{L} .
- **predicates** are $[0, 1]$ -valued; **predicates and functions** must satisfy the given moduli.
- **\mathcal{L} -terms**: built as usual from constants and function symbols.
- **Atomic \mathcal{L} -formulas**: of the form $P(t_1, \dots, t_n)$ and $d(t_1, t_2)$.
- **\mathcal{L} -formulas**: obtained by closing under continuous **connectives** $u : [0, 1]^n \rightarrow [0, 1]$ and **quantifiers** sup and inf.
- A **theory T** in \mathcal{L} is a set of **axioms**, which are expressions of the form “ $\sigma = 0$ ” where σ is an \mathcal{L} -sentence (formula with no free variables). We identify T with the set of sentences σ .

Ultraproducts defined

- $(M_i \mid i \in I)$ are \mathcal{L} -structures, \mathcal{U} an ultrafilter on I .
- We let $N_0 = \prod_{i \in I} M_i$ as a set; its members are $(a) = (a_i \mid i \in I)$, $a_i \in M_i$.
- We interpret the symbols on N_0 :

$$\begin{aligned} F^{N_0}((a_i \mid i \in I), \dots) &= (F^{M_i}(a_i, \dots) \mid i \in I) \in N_0 \\ P^{N_0}((a_i \mid i \in I), \dots) &= \lim_{i \rightarrow \mathcal{U}} P^{M_i}(a_i, \dots) \in [0, 1] \end{aligned}$$

- d^{N_0} is only a pseudometric; $E((a), (b)) :\Leftrightarrow d^{N_0}((a), (b)) = 0$ is an equivalence relation and the quotient N_0/E has canonical metric and interpretations of the symbols of \mathcal{L} .
- We define the **ultraproduct** to be N_0/E (which is complete) and denote it by $N := \prod_{i \in I} M_i / \mathcal{U}$.
- The image of $(a) \in N_0$ in N is denoted $(a)_{\mathcal{U}}$; note that

$$(a)_{\mathcal{U}} = (b)_{\mathcal{U}} \Leftrightarrow 0 = \lim_{i \rightarrow \mathcal{U}} d(a_i, b_i) \quad \left[= d^{N_0}((a), (b)) \right].$$

Properties of ultraproducts

- **Łoś's Theorem:** for every \mathcal{L} -formula $\varphi(x, y, \dots)$ and elements $(a)_{\mathcal{U}}, (b)_{\mathcal{U}}, \dots \in N = \prod M_i / \mathcal{U}$:

$$\varphi^N((a)_{\mathcal{U}}, (b)_{\mathcal{U}}, \dots) = \lim_{i \rightarrow \mathcal{U}} \varphi^{M_i}(a_i, b_i, \dots).$$

- **Corollary** The Compactness Theorem for continuous logic of metric structures.

Elementary

Let M, N be \mathcal{L} -structures. Let $A \subseteq M, B \subseteq N$ and $f: A \rightarrow B$.

Definition

(a) M and N are **elementarily equivalent** if $\sigma^M = \sigma^N$ for all \mathcal{L} -sentences σ . We write $M \equiv N$.

(b) The map f is **elementary** if

$$(M, \langle a \mid a \in A \rangle) \equiv (N, \langle f(a) \mid a \in A \rangle).$$

Examples

Any isomorphism from M onto N is elementary.

The diagonal map $D_M: M \rightarrow M_{\mathcal{U}}$ is elementary.

Any restriction of an elementary map is elementary

The composition of elementary maps is elementary.

The inverse of an elementary map is elementary.

Let M, N be \mathcal{L} -structures. Let $A \subseteq M, B \subseteq N$ and $f: A \rightarrow B$.

- **Keisler-Shelah Theorem** M and N are elementarily equivalent if and only if there is an ultrafilter \mathcal{U} such that $M_{\mathcal{U}} \cong N_{\mathcal{U}}$.
- **Corollary** The map f is elementary iff there is an ultrafilter \mathcal{U} and an isomorphism J from $M_{\mathcal{U}}$ onto $N_{\mathcal{U}}$ such that
$$J \circ D_M = D_N \circ f \quad \text{on } A.$$

Definition

A class \mathcal{C} of \mathcal{L} -structures is **axiomatizable** if there is an \mathcal{L} -theory T such that \mathcal{C} is the class of all models of T ; that is,

$$\mathcal{C} = \{M \mid M \text{ is an } \mathcal{L}\text{-structure and } \sigma^M = 0 \text{ for all } \sigma \in T\}.$$

Theorem

For any class \mathcal{C} of \mathcal{L} -structures, the following are equivalent:

- (a) \mathcal{C} is axiomatizable.*
- (b) \mathcal{C} is closed under isomorphisms, ultraproducts, and ultraroots.*

M is an **ultraroot** of N if there is an ultrafilter \mathcal{U} such that N is isomorphic to $M_{\mathcal{U}}$.

T -formulas

Let T be an \mathcal{L} -theory and $x = x_1, \dots, x_n$ a tuple of distinct variables.

- A **T -formula** $\varphi(x)$ is a sequence $(\varphi_n(x))$ of \mathcal{L} -formulas such that $\varphi_n^M(a)$ converges as $n \rightarrow \infty$, uniformly for all models M of T and all $a \in M^x$.
- The **interpretation** φ^M of a T -formula $\varphi(x) = (\varphi_n(x))$ in a model M of T is defined by
$$\varphi^M(a) := \lim_n \varphi_n^M(a) \text{ for all } a \in M^x.$$
- We refer to the interpretations of T -formulas in the models of T (as above) as **T -predicates**.
- Example: for any \mathcal{L} -formulas $(\theta_n(x))$, the weighted sum $\sum_n 2^{-n} \theta_n^M(x)$ is a T -predicate.

Uniform assignments of sets to models of a theory

Let T be an \mathcal{L} -theory; fix a tuple of distinct variables $x = x_1, \dots, x_n$.

We will denote by \mathbb{X} any operation that assigns a subset $\mathbb{X}(M) \subseteq M^x$ to each model M of T . We refer to such an \mathbb{X} as a **uniform assignment of sets to models of T** .

For example, if $\varphi(x)$ is a T -formula, we consider \mathbb{X}_φ defined by

$$\mathbb{X}_\varphi(M) := \{a \in M^x \mid \varphi^M(a) = 0\}.$$

Such an operation will be referred to as a **T -zeroset** in the variables x .

We note the following properties of T -zerosets $\mathbb{X} = \mathbb{X}_\varphi$:

(iso) If M, N are models of T and J is an isomorphism of M onto N , then $J(\mathbb{X}(M)) = \mathbb{X}(N)$.

(diag) If M is a model of T and \mathcal{U} is an ultrafilter, with $D: M \rightarrow M_{\mathcal{U}}$ the diagonal embedding, then

$$D(\mathbb{X}(M)) = D(M^x) \cap \mathbb{X}(M_{\mathcal{U}}).$$

These two conditions yield that if M, N are models of T and J is any elementary embedding from M into N , then

$$J(\mathbb{X}(M)) = J(M^x) \cap \mathbb{X}(N).$$

Fact: a uniform assignment \mathbb{X} satisfies (iso) and (diag) if and only if there is a set $\tau(\mathbb{X}) \subseteq S_x(T)$ such that for any model M of T

$$\mathbb{X}(M) = \{a \in M^x \mid \text{tp}_M(a) \in \tau(\mathbb{X})\}.$$

Note: Properties of \mathbb{X} that we discuss later correspond to properties of $\tau(\mathbb{X})$ in the topometric space $S_x(T)$.

Let T be an \mathcal{L} -theory and let $x = x_1, \dots, x_n$ be a tuple of distinct variables. Let \mathbb{X} be a uniform assignment of sets to models of T .

Theorem

The following are equivalent:

(1) *There exists a family $(\varphi_j(x) \mid j \in J)$ of T -formulas such that*

$$\mathbb{X}(M) = \bigcap (\mathbb{X}_{\varphi_j}(M) \mid j \in J)$$

for all models M of T .

(2) \mathbb{X} *satisfies (iso) and (diag), and the* **containment**

$$\prod \mathbb{X}(M_i)/\mathcal{U} \subseteq \mathbb{X}(\prod M_i/\mathcal{U})$$

holds for every ultraproduct $\prod M_i/\mathcal{U}$ of models of T .

(3) \mathbb{X} *satisfies (iso) and (diag), and $\tau(\mathbb{X})$ is closed in $S_x(T)$.*

Characterization of T -zerosets

Let T be an \mathcal{L} -theory and let $x = x_1, \dots, x_n$ be a tuple of distinct variables. Let \mathbb{X} be a uniform assignment of sets to models of T .

Corollary

The following are equivalent:

(1) *There exists a T -formula $\varphi(x)$ such that $\mathbb{X} = \mathbb{X}_\varphi$.*

(2) *(containment condition)*

\mathbb{X} satisfies (iso) and (diag), and the containment

$$\prod \mathbb{X}(M_i)/\mathcal{U} \subseteq \mathbb{X}(\prod M_i/\mathcal{U})$$

holds for every ultraproduct $\prod M_i/\mathcal{U}$ of models of T ,

and $\tau(\mathbb{X})$ is a G_δ in $S_x(T)$.

(3) *\mathbb{X} satisfies (iso) and (diag), and $\tau(\mathbb{X})$ is a closed G_δ in $S_x(T)$.*

Note: if the signature \mathcal{L} is countable, then the G_δ condition in (2) and (3) is automatically true, because the topology on $S_x(T)$ is metrizable.

T -Definable sets

Let T be an \mathcal{L} -theory, and let $x = x_1, \dots, x_n$ be a tuple of distinct variables. Let \mathbb{X} be a uniform assignment of sets to models of T .

Definition

We say \mathbb{X} is **T -definable** if the class of T -predicates is closed under the quantifiers $\sup_{x \in \mathbb{X}(M)}$ and $\inf_{x \in \mathbb{X}(M)}$.

For a T -definable assignment \mathbb{X} , the central idea is that we have a natural concept of **induced model theoretic structure** on $\mathbb{X}(M)$ that is obtained uniformly from \mathcal{L} -formulas applied to M . Among other things, this can help giving explicit axioms for T .

Note: If \mathbb{X} is T -definable, and we define $P^M(a) := \text{dist}(a, \mathbb{X}(M))$, then there is a T -formula $\varphi(x)$ such that $P^M(a) = \varphi^M(a)$, for all models M of T and all $a \in M^x$. In particular, $\mathbb{X} = \mathbb{X}_\varphi$ is a zeroset.

Characterizations of T -definable sets

Let T be an \mathcal{L} -theory, and let $x = x_1, \dots, x_n$ be a tuple of distinct variables. Let \mathbb{X} be a uniform assignment of sets to models of T .

Theorem

The following are equivalent:

- (1) \mathbb{X} is T -definable.
- (2) *(equality condition)*

\mathbb{X} satisfies (iso), and the equality

$$\prod \mathbb{X}(M_i)/\mathcal{U} = \mathbb{X}(\prod M_i/\mathcal{U})$$

holds for every ultraproduct $\prod M_i/\mathcal{U}$ of models of T .

(3) *The predicate $P(x)$ defined by $P^M(a) := \text{dist}(a, \mathbb{X}(M))$ for all models M of T and all $a \in M^x$ is a T -predicate.*

(4) \mathbb{X} satisfies (iso) and (diag) and $\text{dist}(\cdot, \tau(\mathbb{X})) : S_x(T) \rightarrow [0, 1]$ is continuous.

Note that (2) omits the condition (diag) and the statement that $\tau(\mathbb{X})$ is a G_δ in $S_x(T)$; they follow from what is stated in (2).

Let T be an \mathcal{L} -theory, and let $x = x_1, \dots, x_n$ be a tuple of distinct variables. Let \mathbb{X} be a uniform assignment of sets to models of T .

Corollary

The following are equivalent:

(1) \mathbb{X} is T -definable.

(2) \mathbb{X} is a T -zerset and **some** T -predicate $P(x)$ with \mathbb{X} as its zerset has the **almost-near property**;

namely, for all epsilon > 0 there exists $\delta > 0$ such that for all models M of T and all $a \in M^x$, we have

$$P^M(a) < \delta \text{ implies } \text{dist}(a, \mathbb{X}(M)) \leq \epsilon.$$

(3) \mathbb{X} is a T -zerset and **every** T -predicate $P(x)$ with \mathbb{X} as its zerset has the almost-near property.

A framework for applications

Frequently in applications we have a class \mathcal{C} of \mathcal{L} -structures (closed under isomorphism) that we want to investigate using model theory. We let $T = \text{Th}(\mathcal{C})$ and need to study the models of T .

Let $\mathcal{C}^* \subseteq \text{Mod}(T)$ be the class of all structures isomorphic to ultraproducts of families from \mathcal{C} . We suppose a good understanding of the structures in \mathcal{C}^* , but not much knowledge about the full class of models of T .

Facts

- \mathcal{C}^* is closed under ultraproducts.
- every model M of T is elementarily equivalent to a member of \mathcal{C}^* , and therefore $M_{\mathcal{U}}$ is in \mathcal{C}^* for some ultrafilter \mathcal{U} .
- whether or not (φ_n) defines a T -formula can be verified by considering only members of \mathcal{C} .

Now suppose we have a uniform assignment of sets \mathbb{X} that is only defined for models in \mathcal{C}^* .

Fact

If \mathbb{X} satisfies (iso) and (diag) on \mathcal{C}^ , then there is a set $\tau(\mathbb{X}) \subseteq S_x(T)$ such that*

$$\mathbb{X}(M) = \{a \in M^x \mid \text{tp}_M(a) \in \tau(\mathbb{X})\}$$

for all M in \mathcal{C}^ .*

Consequently, \mathbb{X} has a unique extension to $\text{Mod}(T)$ that satisfies (iso) and (diag) on all models of T .

Fact

Suppose \mathbb{X} is a uniform assignment of sets satisfying (iso) and (diag) on all models of T . Then, to verify the containment condition or the equality condition for \mathbb{X} on all models of T , it suffices to verify the condition on all models in \mathcal{C}^ .*

Example 1

Let \mathcal{C} be the class of pointed metric spaces (M, c) of diameter ≤ 1 in which the metric d is an ultrametric (i.e., d satisfies $d(x, z) \leq \max(d(x, y), d(y, z))$). Let T be the theory of \mathcal{C} .

Fix r in the interval $0 < r < 1$ and for every (M, c) in \mathcal{C} let $\mathbb{X}(M) = \{a \in M \mid d(c, a) \leq r\}$.

Obviously \mathbb{X} is a T -zeroset. However, \mathbb{X} is not T -definable.

Proof: Let (M, c) be a member of \mathcal{C} in which there exist elements (a_n) such that $\inf_n d(c, a_n) = r$ and $r < d(c, a_n)$ for all $n \in \mathbb{N}$. Let \mathcal{U} be a nonprincipal ultrafilter on \mathbb{N} . The element $a = (a_n)_{\mathcal{U}}$ of the ultrapower $(M, c)_{\mathcal{U}}$ satisfies $d(c, a) = r$. However, the ultrametric inequality prevents the existence of any sequence (b_n) from M that satisfies $a = (b_n)_{\mathcal{U}}$ and $d(c, b_n) \leq r$ for all $n \in \mathbb{N}$.

Example 2

Let \mathcal{L} be a countable signature and T an \mathcal{L} -theory. Recall that we say T is ω -categorical if T has a unique separable model, which is not compact.

Proposition

The following are equivalent:

- (1) T is ω -categorical.*
- (2) Every T -zeroset is T -definable.*

Proof.

For each model M_0 of T and $a_0 \in M_0^x$, let \mathbb{X} be defined on models M of T by

$$\mathbb{X}(M) := \{a \in M^x \mid (M, a) \equiv (M_0, a_0)\}.$$

Obviously \mathbb{X} is a T -zeroset. The omitting types theorem for continuous logic implies that \mathbb{X} is T -definable iff $\mathbb{X}(M) \neq \emptyset$ for all models M of T . The characterization of separable categoricity then implies the equivalence of (1) and (2). □

Example 3

Here we consider the theory Pr of probability measure algebras.

Given a model M of Pr we define

$$\mathbb{X}(M) = \{a \in M \mid a \text{ is an atom or } 0\}$$

and show that \mathbb{X} is Pr -definable.

Proof: We need to check that

$$\prod \mathbb{X}(M_i)/\mathcal{U} = \mathbb{X}(\prod M_i/\mathcal{U})$$

holds for every ultraproduct $\prod M_i/\mathcal{U}$ of models of Pr .

\subseteq : It is easy to show by contradiction that if e_i is an atom or 0 in M_i for all i , then $(e)_\mathcal{U}$ is an atom or 0 in $\prod M_i/\mathcal{U}$.

\supseteq : (next slide)

\supseteq : Suppose $(a)_{\mathcal{U}} \in M = \prod M_i/\mathcal{U}$ is an atom and let $r = \mu((a)_{\mathcal{U}}) > 0$. For each $i \in I$, let e_i be the atom of largest measure contained in a_i , or $e_i = 0$ if a_i is atomless. By the first part of the proof, either $(e)_{\mathcal{U}}$ is an atom in $\prod M_i/\mathcal{U}$ or it is 0.

We have $(e)_{\mathcal{U}} \leq (a)_{\mathcal{U}}$. Therefore, if $(e)_{\mathcal{U}}$ is an atom in $\prod M_i/\mathcal{U}$, we must have $(a)_{\mathcal{U}} = (e)_{\mathcal{U}} \in \prod \mathbb{X}(M_i)/\mathcal{U}$ as needed.

It remains to show that $(e)_{\mathcal{U}} = 0$ is impossible. Otherwise, the set $S = \{i \in I \mid \mu(e_i) \leq r/2\} \in \mathcal{U}$. For each $i \in S$ we may write $a_i = b_i \cup c_i$ where $b_i \cap c_i = 0$ and $\mu(c_i) \leq \mu(b_i) \leq \mu(c_i) + r/2$. (Put $e_i \leq b_i$ and alternate the other atoms $\leq a_i$ between b_i and c_i and split the atomless part of a_i in two equal parts.)

Hence $(a)_{\mathcal{U}} = (b)_{\mathcal{U}} \cup (c)_{\mathcal{U}}$ and $(b)_{\mathcal{U}} \cap (c)_{\mathcal{U}} = 0$, while $\mu((c)_{\mathcal{U}}) \leq \mu((b)_{\mathcal{U}}) \leq \mu((c)_{\mathcal{U}}) + r/2 = \mu((c)_{\mathcal{U}}) + \mu((a)_{\mathcal{U}})/2$, which contradicts the assumption that $(a)_{\mathcal{U}}$ is an atom. \square

Example 4

This example comes from work in progress with Yves Raynaud, in which we are constructing new families of uncountably categorical Banach spaces. These spaces are asymptotically hilbertian in a strong sense.

Definition

A Banach space E is **asymptotically 2-hilbertian** if every ultrapower of E has the form $E_{\mathcal{U}} = D(E) \oplus_2 H$ for some Hilbert space $H \subseteq E_{\mathcal{U}}$, where D is the diagonal embedding of E into $E_{\mathcal{U}}$.

Here we only have time to discuss how the Hilbert part of $E_{\mathcal{U}}$ can be identified.

Let T_b denote a theory that axiomatizes the class of all **unit balls** of Banach spaces over \mathbb{R} . For each such Banach space E , let $B(E)$ denote the unit ball of E , considered as a model of T_b .

For each $a \in B(E)$ we let $\langle a \rangle$ denote the linear subspace of E generated by a . We call $\langle a \rangle$ a **2-summand** of E if there is a subspace F of E such that $E = F \oplus_2 \langle a \rangle$; note that if this equation holds, then F is uniquely determined by a . Note further that if $E = F \oplus_2 H$ and H is a Hilbert space, then $\langle a \rangle$ is a 2-summand of E for every $a \in H$.

For each model $B(E)$ of T_b , we define

$$\mathbb{X}(B(E)) := \{a \in B(E) \mid \langle a \rangle \text{ is a 2-summand of } E\}.$$

Proposition

The set assignment \mathbb{X} is a T_b -zeroset but it is not T_b -definable.

To prove that \mathbb{X} is a T_b -zeroset, we verify the conditions in the ultraproduct characterization of zerosets. Condition (iso) is obvious. Condition (diag) follows from the easy stronger fact that if $a \in B(E_1)$ and $E_1 \subseteq E_2$, and if $E_2 = F \oplus_2 \langle a \rangle$, then $E_1 = (F \cap E_1) \oplus_2 \langle a \rangle$. The ultraproduct containment condition holds because the \oplus_2 direct sum operation commutes with the ultraproduct construction.

To prove that \mathbb{X} is not T_b -definable, consider $E = \oplus_2(E_n \mid n \in \mathbb{N})$, where each E_n is polyhedral and 2-dimensional, with the unit sphere of E_n approximating the unit circle more and more closely as $n \rightarrow \infty$. Then $\mathbb{X}(B(E)) = \{0\}$, whereas E is asymptotically 2-hilbertian, so $\mathbb{X}(B(E_U))$ will be very large. Thus the ultraproduct equality condition fails to hold.