Banach lattice methods for proving axiomatizability of Banach spaces

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(work in progress)
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Banach spaces and Banach lattices over $\mathbb{R}$ are the only structures considered here.

They are treated model-theoretically using continuous first order logic applied to appropriate structures on their unit balls.

**Main Problem:**

*Given an axiomatizable class $C$ of Banach lattices, when is the class $C^B$ of their underlying Banach spaces axiomatizable?*

Rationale: there were many classes of Banach lattices known to be axiomatizable, but not many such classes of Banach spaces. A systematic method was lacking.
To directly apply ideas and tools from model theory to a class $S$ of structures, you need to know that $S$ is axiomatizable.

Otherwise, you could do something like: replace $S$ by the smallest axiomatizable class containing $S$ (call it $S^*$), and then

1. apply model theory to $S^*$
2. understand the relation between $S$ and $S^*$.

$S^* = \text{Mod}(\text{Th}(S))$.

$S^*$ is the class of all structures that are ultraroots of ultraproducts of a family from $S$. 
Some axiomatizable classes of Banach lattices

- $L_p$-spaces, $1 \leq p < \infty$, $p$ fixed (old, easy).

- $C(K)$-spaces, M-spaces, some other classes of M-spaces (old, easy).

- Nakano spaces, with essential range $\subseteq [r, s]$, $1 \leq r \leq s < \infty$, $r, s$ fixed (Poitevin, 2006; Ben Yaacov, 2009).

- $BL_p L_q$-spaces, $1 \leq p, q < \infty$ fixed (WH, Raynaud, 2007).
Some axiomatizable classes of Banach spaces

- $L_p$-spaces, $1 \leq p < \infty$, $p$ fixed. (WH, Moore; $L_{p,1^+}$-spaces)
- $L_1$-preduals (WH, Moore; $L_{\infty,1^+}$-spaces)
- $C(K)$-spaces, other classes of $L_1$-preduals. (Heinrich, 1981)

Recent work:
- Nakano spaces, with essential range $\subseteq [r, s]$, $1 \leq r \leq s < \infty$, $r, s$ fixed, BUT with added restrictions $2 < r$ or $s < 2$ (Raynaud, new, using disjointness methods described here).
- $BL_pL_q$-spaces, $1 \leq p, q < \infty$ fixed, BUT with added restrictions $2 < p, q$ or $1 < p, q < 2$ or ... (Raynaud, new, same methods).
A Banach lattice is an **M-space** if it satisfies the condition:

\[ x, y \text{ disjoint} \implies \|x + y\| = \max(\|x\|, \|y\|) \]

This is a **negative example** for axiomatizability: The class of M-spaces is axiomatizable. However the class of Banach space reducts of M-spaces is *not* axiomatizable.

Work of WH, Heinrich, and Moore in the 1980s gave a Banach space \( E \) elementarily equivalent to \( c_0 \) such that \( E \) does not have any compatible lattice structure at all.
Signature for Banach spaces over reals

We take the signature for unit balls of Banach spaces to be
\( \mathcal{L}_{sp} = \{0, c_{r,s}, \hat{2}, \parallel \parallel \} \) where \( r, s \) range over rational scalars such that \( |r| + |s| \leq 1 \). This is a countable signature.

- 0 is a constant symbol.
- \( c_{r,s} \) are binary function symbols.
- \( \hat{2} \) is a unary function symbol.
  - Interpret \( c_{r,s}(x, y) \) as \( rx + sy \).
  - Interpret \( \hat{2}(x) \) as \( 2x \) if \( \|x\| \leq \frac{1}{2} \), as \( x/\|x\| \) otherwise.
- \( \parallel \parallel \) is a unary predicate symbol.
  - the functions and predicate are 1-Lipschitz (2-Lipschitz for \( \hat{2} \)) in each variable.

There is also a predicate symbol \( d \) for the underlying metric; it is treated the same as \( = \) in classical signatures.
We take the signature for unit balls of Banach lattices to be $\mathcal{L}_{latt} = \mathcal{L}_{sp} \cup \{|\cdot|\}$.

- $|\cdot|$ is a unary function symbol, interpreted as the absolute value on the unit ball of any Banach lattice; it is 1-Lipschitz.
- Suitably scaled versions of the other Banach lattice operations can be explicitly defined by terms using $\|\cdot\|$ and the allowed linear operations. For example:

$$\frac{1}{2}(x \vee y) = \frac{1}{2}\left(\frac{1}{2}(x + y) + \frac{1}{2}(x - y)\right)$$

$$\frac{1}{2}(x \wedge y) = \frac{1}{2}\left(\frac{1}{2}(x + y) - \frac{1}{2}(x - y)\right)$$

$$x_+ = \frac{1}{2}(|x| + x) \quad x_- = \frac{1}{2}(|x| - x)$$
The classes of (unit balls) of Banach spaces and of Banach lattices (as structures for the signatures $\mathcal{L}_{sp}$ and $\mathcal{L}_{latt}$, respectively) are axiomatizable, and the axioms can be taken to be universal. (For example, both classes are closed under ultraproducts and substructures.)

**Notation**

- $T_{sp}$ is a universal $\mathcal{L}_{sp}$-theory axiomatizing the class of all (unit balls of) Banach spaces.
- $T_{latt}$ is a universal $\mathcal{L}_{latt}$-theory axiomatizing the class of all (unit balls of) Banach lattices.
Examples of $L_{latt}$-sentences:

- $\sigma := \sup_x \|x\|^p - (\|x_+\|^p + \|x_-\|^p)$
- $T_{latt} \cup \{\sigma = 0\}$ axiomatizes $L_p$ Banach lattices.

- $\psi := \sup_x \|x\| - \max(\|x_+\|, \|x_-\|)$
- $T_{latt} \cup \{\psi = 0\}$ axiomatizes M-spaces.

- $\tau := \inf_x \max (\|x\| - 1, \sup_y \|y\| - (|y| \wedge |x|))$
- $T_{latt} \cup \{\tau = 0\}$ axiomatizes Banach lattices with a strong order unit.

Note: some scaling is needed in $\sigma$ and $\tau$ to make them syntactically correct.
X, Y, . . . are Banach lattices.
E, F, . . . are Banach spaces
B_X, B_E are the unit balls.

Definition

x, y ∈ X are disjoint if |x| ∧ |y| = 0; we write x ⊥ y.

D(X) = {(u, v) | u, v ∈ B_X and u ⊥ v}
A class $C$ of Banach lattices has property $DPA$ if for every $X, Y \in C$, every surjective linear isometry from $X$ to $Y$ is disjointness preserving. A Banach lattice $X$ has $DPA$ if the class $\{X\}$ has $DPA$.

Main Theorem (WH, strengthening a result of Raynaud)

Suppose $C$ is an axiomatizable class of Banach lattices. If every member of $C$ has $DPA$ and is order continuous, then the class $C^B$ of Banach space reducts of members of $C$ is axiomatizable.

The proof uses results from Banach lattice theory (worked out by Raynaud) as well as definability results from model theory (noticed by WH). We discuss these next, in that order.
A band in $X$ is a linear subspace $I \subseteq X$ such that
- $\{x \in X, y \in I, |x| \leq |y|\}$ imply $x \in I$. ($I$ is an ideal.)
- $S \subseteq I, x = \sup S$ imply $x \in I$.

A band projection of $X$ onto a band $I$ is a projection $P$ such that $0 \leq P(x) \leq x$ for all $x \geq 0$ in $X$. Such a projection is necessarily a lattice homomorphism, and $Id - P$ is a band projection onto the band orthogonal to $I$.

A sign-change operator on $X$ is an operator $U : X \to X$ for which there exists a band projection $P$ for which $U = 2P - Id$.

If $U$ is a sign-change operator on $X$, then $U$ is a surjective linear isometry and it is disjointness preserving.
Proposition (Raynaud)

Suppose $T : X \to Y$ is a linear isometry (into) that preserves disjointness. Then the map $|T| : X \to Y$ defined by

$$|T|(x) = |T(x_+)| - |T(x_-)|$$

is a linear lattice isometry. Further, if $T$ is surjective or if $Y$ is order complete, then there is a sign-change operator $U$ on $Y$ such that $T = U \circ |T|$. 
**Proposition (Raynaud)**

A bounded linear operator $T : X \rightarrow Y$ preserves disjointness if and only if it preserves the lattice term $a(x, y) = (x_+ \land y_+) - (x_- \land y_-)$; that is, if we have $T(a(x, y)) = a(Tx, Ty)$ for all $x, y \in X$.

**Proposition (Raynaud)**

Assume $X$ is order continuous. For any closed linear subspace $E$ of $X$, the following are equivalent:
1. There is a sign-change operator $U$ on $X$ such that $U(E)$ is a closed vector sublattice of $X$.
2. The function $a(x, y)$ maps $E \times E$ into $E$. 
Recall

Consider a theory $T$ in continuous logic.

- A **definable predicate** for $T$ is a uniform limit of formulas. (Uniform over all models $\mathcal{M}$ of $T$ and all elements of the model).

- Let $P(x)$ be a definable predicate for $T$. The zero set of $P$, which is the set $ZP^\mathcal{M} := \{x \in M^n \mid P^\mathcal{M}(x) = 0\}$ in each model $\mathcal{M}$ of $T$, is a **definable set** for $T$ if there is a definable predicate $Q$ for $T$ such that

  $$Q^\mathcal{M}(x) = \text{dist}(x, ZP^\mathcal{M})$$

  for all $\mathcal{M} \models T$ and all $x \in M^n$. (On $M^n$ we use the metric given by the maximum of coordinate distances.)
The set $D(X)$ of disjoint pairs in $(B_X)^2$ was defined to be the zeroset of the formula $|||x| \land |y||$. The next result says that $D(X)$ is a definable set in all Banach lattices (i.e., in all models of $T_{latt}$).

**Lemma**

For any Banach lattice $X$, for any $x, y \in B_X$, we have

$$\text{dist}((x, y), D(X)) = \varphi_{\text{disj}}^X(x, y)$$

where $\varphi_{\text{disj}}(x, y)$ is the formula

$$\inf_w \inf_z (||w| \land |z|| + \max(||x - w||, ||y - z||)).$$

**Proof.**

Given $x, y \in B_X$ the key is to construct $(u, v) \in D(X)$ such that

$$\max(||x - u||, ||y - v||) \leq |||x| \land |y||.$$

\[\square\]
Let $\mathcal{C} = \text{Mod}(T)$ be an axiomatizable class of Banach lattices, and let $T^B := \text{Th}(\mathcal{C}^B)$. The following conditions are equivalent:

(1) $\mathcal{C}$ has DPA.

(2) There exists a definable predicate $P(x, y)$ for the theory $T^B$ such that for all $X \in \mathcal{C}$ and all $x, y \in B_X$ one has

$$\text{dist}((x, y), D(X)) = P^X(x, y).$$

(3) There exists a definable predicate $Q(x, y, z)$ for $T^B$ such that for all $X \in \mathcal{C}$ and all $x, y, z \in B_X$ one has

$$\|z - a_X(x, y)\| = Q^X(x, y, z).$$

Note: in (2) and (3) the predicates $P$ and $Q$ have to be uniform limits of formulas in the Banach space language.
Corollary (Key Lemma for the Main Theorem)

Let $C$ be an axiomatizable class of Banach lattices such that every member of $C$ has DPA. Suppose $X \in C$ and $E$ is a linear subspace of $X$. If $E$ is an elementary Banach subspace of $X$, then the lattice function $a_X(x, y)$ maps $E \times E$ into $E$.

Proof. Let $C_X \subseteq C$ be the class of all Banach lattices $Y$ that are elementarily equivalent to $X$. A small argument shows that the class $C_X$ has DPA, so we can apply the previous Theorem to it. Let $Q(x, y, z)$ be the definable predicate for the class $C_X$ that is described in statement (3) of that Theorem. Then $Q^E$ is the restriction of $Q^X$ to $E^3$. Moreover, for any $x, y \in B_E$ we have

$$\inf_{z \in B_E} Q^E(x, y, z) = \inf_{z \in B_X} Q^X(x, y, z) = \inf_{z \in B_X} \|z - a_X(x, y)\| = 0.$$ 

It follows that there is a sequence in $E$ converging to $a_X(x, y)$. 
Remark

The class of $C(K)$ Banach lattices is axiomatizable, and has property $DPA$ (by the Banach-Stone Theorem).

Of course there exist (many) $C(K)$ spaces which are not order continuous, so the Main Theorem does not apply to this case. Heinrich (1981) did show that the class of $C(K)$ Banach spaces is axiomatizable, but without giving explicit axioms.

Question

Give an explicit Banach space definable predicate whose value in each $C(K)$ space $X$ is $\text{dist}((x, y), D(X))$.

Give explicit axioms for the class of $C(K)$ Banach spaces.
Proposition

Let $C$ be an axiomatizable class of Banach spaces with DPA. Then the following are equivalent, for any $X, Y \in C$:

1. $X, Y$ are elementarily equivalent as Banach spaces.
2. $X, Y$ are elementarily equivalent as Banach lattices.

Proof.

Assume (1). Then there exist elementary extensions $X', Y'$ of $X, Y$ respectively, and a surjective linear isometry $J: X' \to Y'$. Since $C$ has DPA, the map $J$ is disjointness preserving. Therefore $|J|$ (defined above: $|J|(x) = |J(x_+) - J(x_-)|$) is a linear lattice isometry from $X'$ onto $Y'$. Hence $X', Y'$ are elementarily equivalent as Banach lattices, so the same is true of $X, Y$. 

\[\square\]