Banach lattice methods for proving axiomatizability of Banach spaces

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(work in progress)
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- Banach spaces and Banach lattices over $\mathbb{R}$ are the only structures considered here.
- They are treated model-theoretically using continuous first order logic applied to appropriate structures on their unit balls.

Main Problem:

- *Given an axiomatizable class $\mathcal{C}$ of Banach lattices, when is the class $\mathcal{C}^B$ of their underlying Banach spaces axiomatizable?*
- Rationale: there were many classes of Banach lattices known to be axiomatizable, but not many such classes of Banach spaces. A systematic method was lacking.

- To directly apply ideas and tools from model theory to a class $\mathcal{S}$ of structures, you need to know that $\mathcal{S}$ is axiomatizable.
- Otherwise, you could do something like: replace $\mathcal{S}$ by the smallest axiomatizable class containing $\mathcal{S}$ (call it $\mathcal{S}^*$), and then
    - (1) apply model theory to $\mathcal{S}^*$
    - (2) understand the relation between $\mathcal{S}$ and $\mathcal{S}^*$.
- $\mathcal{S}^* = \mathrm{Mod}(\mathrm{Th}(\mathcal{S}))$.
- $\mathcal{S}^*$ is the class of all structures that are ultraroots of ultraproducts of a family from $\mathcal{S}$.

Some axiomatizable classes of Banach lattices

- L_p-spaces, $1 \leq p < \infty$, p fixed (old, easy).
  - their model theory studied by Berenstein, Ben Yaacov, and WH in Isr. J. Math, 2011.
- $C(K)$-spaces, M-spaces, some other classes of M-spaces (old, easy).
- Nakano spaces, with essential range $\subseteq [r, s]$, $1 \leq r \leq s < \infty$, r, s fixed (Poitevin, 2006; Ben Yaacov, 2009).
- BL_pL_q-spaces, $1 \leq p, q < \infty$ fixed (WH, Raynaud, 2007).
  - their model theory studied by WH and Raynaud in Positivity, 2007, and J. Logic and Analysis, 2011.

Some axiomatizable classes of Banach spaces

- L_p-spaces, $1 \leq p < \infty$, p fixed. (WH, Moore; $\mathcal{L}_{p,1+}$-spaces)
- L_1-preduals (WH,Moore; $\mathcal{L}_{\infty,1+}$-spaces)
- $C(K)$-spaces, other classes of L_1-preduals. (Heinrich, 1981)

Recent work:

- Nakano spaces, with essential range $\subseteq [r,s]$, $1 \leq r \leq s < \infty$, r, s fixed, BUT with added restrictions $2 < r$ or $s < 2$ (Raynaud, new, using disjointness methods described here).
- BL_pL_q-spaces, $1 \leq p, q < \infty$ fixed, BUT with added restrictions $2 < p, q$ or $1 < p, q < 2$ or ... (Raynaud, new, same methods).

A Banach lattice is an M-space if it satisfies the condition:

$$x, y \text{ disjoint} \quad \Rightarrow \quad \|x + y\| = \max(\|x\|, \|y\|)$$

This is a negative example for axiomatizability: The class of M-spaces is axiomatizable. However the class of Banach space reducts of M-spaces is *not* axiomatizable

Work of WH, Heinrich, and Moore in the 1980s gave a Banach space E elementarily equivalent to c_0 such that E does not have any compatible lattice structure at all.

Signature for Banach spaces over reals

We take the signature for unit balls of Banach spaces to be $\mathcal{L}_{sp} = \{0, c_{r,s}, \dot{2}, \| \ \|\}$ where r, s range over rational scalars such that $|r| + |s| \leq 1$. This is a countable signature.

- 0 is a constant symbol.
- $c_{r,s}$ are binary function symbols.
- $\dot{2}$ is a unary function symbol.
  - Interpret $c_{r,s}(x, y)$ as $rx + sy$.
  - Interpret $\dot{2}(x)$ as $2x$ if $\|x\| \leq \frac{1}{2}$, as $x/\|x\|$ otherwise.
- $\| \ \|$ is a unary predicate symbol.
- the functions and predicate are 1-Lipschitz (2-Lipschitz for $\dot{2}$) in each variable.

There is also a predicate symbol d for the underlying metric; it is treated the same as $=$ in classical signatures.

Signature for Banach lattices over reals

We take the signature for unit balls of Banach lattices to be $\mathcal{L}_{latt} = \mathcal{L}_{sp} \cup \{|\cdot|\}$.

- $|\cdot|$ is a unary function symbol, interpreted as the absolute value on the unit ball of any Banach lattice; it is 1-Lipschitz.
- Suitably scaled versions of the other Banach lattice operations can be explicitly defined by terms using $\|\cdot\|$ and the allowed linear operations. For example:

$$\tfrac{1}{2}(x \vee y) = \tfrac{1}{2}\left(\tfrac{1}{2}(x+y) + |\tfrac{1}{2}(x-y)|\right)$$

$$\tfrac{1}{2}(x \wedge y) = \tfrac{1}{2}\left(\tfrac{1}{2}(x+y) - |\tfrac{1}{2}(x-y)|\right)$$

$$x_+ = \tfrac{1}{2}(|x| + x) \qquad x_- = \tfrac{1}{2}(|x| - x)$$

The classes of (unit balls) of Banach spaces and of Banach lattices (as structures for the signatures $\mathcal{L}_{sp}$ and $\mathcal{L}_{latt}$, respectively) are axiomatizable, and the axioms can be taken to be universal. (For example, both classes are closed under ultraproducts and substructures.)

Notation

- T_{sp} is a universal $\mathcal{L}_{sp}$-theory axiomatizing the class of all (unit balls of) Banach spaces.
- T_{latt} is a universal $\mathcal{L}_{latt}$-theory axiomatizing the class of all (unit balls of) Banach lattices.

Examples of $\mathcal{L}_{latt}$-sentences:

- $\sigma := \sup_x \left| \|x\|^p - (\|x_+\|^p + \|x_-\|^p) \right|$
  - $T_{latt} \cup \{\sigma = 0\}$ axiomatizes L_p Banach lattices.
- $\psi := \sup_x \left| \|x\| - \max(\|x_+\|, \|x_-\|) \right|$
  - $T_{latt} \cup \{\psi = 0\}$ axiomatizes M-spaces.
- $\tau := \inf_x \max\left(\left| \|x\| - 1 \right|, \sup_y \| |y| - (|y| \wedge |x|) \| \right)$
  - $T_{latt} \cup \{\tau = 0\}$ axiomatizes Banach lattices with a strong order unit.

Note: some scaling is needed in σ and τ to make them syntactically correct.

Notation

- $X, Y, \ldots$ are Banach lattices.
- $E, F, \ldots$ are Banach spaces
- B_X, B_E are the unit balls.

Definition

$x, y \in X$ are disjoint if $|x| \wedge |y| = 0$; we write $x \perp y$.

$$D(X) = \{(u, v) \mid u, v \in B_X \text{ and } u \perp v\}$$

Definition

A class $\mathcal{C}$ of Banach lattices has property *DPA* if for every $X, Y \in \mathcal{C}$, every surjective linear isometry from X to Y is disjointness preserving. A Banach lattice X has *DPA* if the class $\{X\}$ has *DPA*.

Main Theorem (WH, strengthening a result of Raynaud)

Suppose $\mathcal{C}$ is an axiomatizable class of Banach lattices. If every member of $\mathcal{C}$ has DPA and is order continuous, then the class $\mathcal{C}^B$ of Banach space reducts of members of $\mathcal{C}$ is axiomatizable.

The proof uses results from Banach lattice theory (worked out by Raynaud) as well as definability results from model theory (noticed by WH). We discuss these next, in that order.

- A band in X is a linear subspace $I \subseteq X$ such that
  - $\{x \in X, y \in I, |x| \leq |y|\}$ imply $x \in I$. (I is an ideal.)
  - $S \subseteq I, x = \sup S$ imply $x \in I$.
- A band projection of X onto a band I is a projection P such that $0 \leq P(x) \leq x$ for all $x \geq 0$ in X. Such a projection is necessarily a lattice homomorphism, and $Id - P$ is a band projection onto the band orthogonal to I.
- A sign-change operator on X is an operator $U\colon X \to X$ for which there exists a band projection P for which $U = 2P - Id$.
- If U is a sign-change operator on X, then U is a surjective linear isometry and it is disjointness preserving.

Proposition (Raynaud)

Suppose $T: X \to Y$ *is a linear isometry (into) that preserves disjointness. Then the map* $|T|: X \to Y$ *defined by*

$$|T|(x) = |T(x_+)| - |T(x_-)|$$

is a linear lattice isometry. Further, if T *is surjective or if* Y *is order complete, then there is a sign-change operator* U *on* Y *such that* $T = U \circ |T|$*.*

Proposition (Raynaud)

A bounded linear operator $T: X \to Y$ preserves disjointness if and only if it preserves the lattice term $a(x,y) = (x_+ \wedge y_+) - (x_- \wedge y_-)$; that is, if we have $T(a(x,y)) = a(Tx, Ty)$ for all $x, y \in X$.

Proposition (Raynaud)

Assume X is order continuous. For any closed linear subspace E of X, the following are equivalent:
(1) There is a sign-change operator U on X such that $U(E)$ is a closed vector sublattice of X.
(2) The function $a(x,y)$ maps $E \times E$ into E.

Recall

Consider a theory T in continuous logic.

- *A definable predicate for T is a uniform limit of formulas. (Uniform over all models $\mathcal{M}$ of T and all elements of the model).*
- *Let $P(x)$ be a definable predicate for T. The zeroset of P, which is the set $ZP^{\mathcal{M}} := \{x \in M^n \mid P^{\mathcal{M}}(x) = 0\}$ in each model $\mathcal{M}$ of T, is a definable set for T if there is a definable predicate Q for T such that*

$$Q^{\mathcal{M}}(x) = dist(x, ZP^{\mathcal{M}})$$

for all $\mathcal{M} \models T$ and all $x \in M^n$. (On M^n we use the metric given by the maximum of coordinate distances.)

The set $D(X)$ of disjoint pairs in $(B_X)^2$ was defined to be the zeroset of the formula $\||x| \wedge |y|\|$. The next result says that $D(X)$ is a definable set in all Banach lattices (*i.e.*, in all models of T_{latt}).

Lemma

For any Banach lattice X, for any $x, y \in B_X$, we have

$$dist((x,y), D(X)) = \varphi^X_{disj}(x,y)$$

where $\varphi_{disj}(x,y)$ is the formula

$$\inf_w \inf_z \left(\||w| \wedge |z|\| \dotplus \max(\|x-w\|, \|y-z\|) \right).$$

Proof.

Given $x, y \in B_X$ the key is to construct $(u,v) \in D(X)$ such that $\max(\|x-u\|, \|y-v\|) \leq \||x| \wedge |y|\|$. □

Theorem

Let $\mathcal{C} = \mathrm{Mod}(T)$ *be an axiomatizable class of Banach lattices, and let* $T^B := \mathrm{Th}(\mathcal{C}^B)$*. The following conditions are equivalent:*
(1) $\mathcal{C}$ *has DPA.*
(2) There exists a definable predicate $P(x,y)$ *for the theory* T^B *such that for all* $X \in \mathcal{C}$ *and all* $x, y \in B_X$ *one has*

$$dist((x,y), D(X)) = P^X(x,y).$$

(3) There exists a definable predicate $Q(x,y,z)$ *for* T^B *such that for all* $X \in \mathcal{C}$ *and all* $x,y,z \in B_X$ *one has*

$$\|z - a_X(x,y)\| = Q^X(x,y,z).$$

Note: in (2) and (3) the predicates P and Q have to be uniform limits of formulas in the *Banach space language*.

Corollary (Key Lemma for the Main Theorem)

Let $\mathcal{C}$ be an axiomatizable class of Banach lattices such that every member of $\mathcal{C}$ has DPA. Suppose $X \in \mathcal{C}$ and E is a linear subspace of X. If E is an elementary Banach subspace of X, then the lattice function $a_X(x, y)$ maps $E \times E$ into E.

Proof. Let $\mathcal{C}_X \subseteq \mathcal{C}$ be the class of all Banach lattices Y that are elementarily equivalent to X. A small argument shows that the class $\mathcal{C}_X$ has *DPA*, so we can apply the previous Theorem to it. Let $Q(x, y, z)$ be the definable predicate for the class $\mathcal{C}_X$ that is described in statement (3) of that Theorem. Then Q^E is the restriction of Q^X to E^3. Moreover, for any $x, y \in B_E$ we have

$$\inf_{z \in B_E} Q^E(x, y, z) = \inf_{z \in B_X} Q^X(x, y, z) = \inf_{z \in B_X} \|z - a_X(x, y)\| = 0.$$

It follows that there is a sequence in E converging to $a_X(x, y)$. □

Remark

The class of $C(K)$ Banach lattices is axiomatizable, and has property *DPA* (by the Banach-Stone Theorem).

Of course there exist (many) $C(K)$ spaces which are not order continuous, so the Main Theorem does not apply to this case. Heinrich (1981) did show that the class of $C(K)$ Banach spaces is axiomatizable, but without giving explicit axioms.

Question

Give an explicit Banach space definable predicate whose value in each $C(K)$ space X is $\mathrm{dist}((x, y), D(X))$.
Give explicit axioms for the class of $C(K)$ Banach spaces.

Proposition

Let $\mathcal{C}$ be an axiomatizable class of Banach spaces with DPA. Then the following are equivalent, for any $X, Y \in \mathcal{C}$:
(1) X, Y are elementarily equivalent as Banach spaces.
(2) X, Y are elementarily equivalent as Banach lattices.

Proof.

Assume (1). Then there exist elementary extensions X', Y' of X, Y respectively, and a surjective linear isometry $J \colon X' \to Y'$. Since $\mathcal{C}$ has *DPA*, the map J is disjointness preserving. Therefore $|J|$ (defined above: $|J|(x) = |J(x_+)| - |J(x_-)|$) is a linear lattice isometry from X' onto Y'. Hence X', Y' are elementarily equivalent as Banach lattices, so the same is true of X, Y. □