On the model theory of group actions on probability measure algebras

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The continuous signatures we use

- $L^{pr}$: signature for probability measure algebras:
  - Boolean operations: $\cup$, $\cap$, $\cdot^c$
  - Constants: 0, 1
  - Measure: $\mu$ (a unary predicate; values in $[0, 1]$)
  - Metric: $d$ (a binary predicate, values in $[0, 1]$)
  - Specify suitable moduli of uniform continuity; for example $|\mu(x) - \mu(y)| \leq d(x, y)$.

- $L^F$: signature for measure algebras with automorphisms
  - $F$ is a set of unary function symbols to be added to $L^{pr}$
  - For each $f \in F$, require $d(f(x), f(y)) \leq d(x, y)$. 
Let \((X, \mathcal{B}, \mu)\) be a probability space; i.e., \(X\) is a nonempty set, \(\mathcal{B}\) is a \(\sigma\)-complete boolean algebra of subsets of \(X\) that contains \(\emptyset\) and \(X\), and \(\mu: \mathcal{B} \to [0, 1]\) is a \(\sigma\)-additive probability measure on \(\mathcal{B}\).

The measure algebra of \((X, \mathcal{B}, \mu)\) is the \(L^\text{Pr}\)-structure \((\mathcal{B}/\mathcal{I}, \mu, d)\).

Here \(\mathcal{I} = \{a \in \mathcal{B} \mid \mu(a) = 0\}\) is the ideal of null sets,
\[
\mu([a]_{\mathcal{I}}) := \mu(a) \text{ for all } a \in \mathcal{B}, \text{ and }
\]
\[
d([a]_{\mathcal{I}}, [b]_{\mathcal{I}}) := \mu(a \triangle b) \text{ for all } a, b \in \mathcal{B}.
\]

We axiomatize the class of probability measure algebras using continuous logic.
Pr: $L^{pr}$-axioms for probability measure algebras

- (Boolean algebra axioms)
  \[
  \sup_{x_1} \ldots \sup_{x_n} d(t_1(x), t_2(x)) = 0
  \]
  for various boolean terms $t_1, t_2$.

- (Measure axioms)
  \[
  \mu(0) = 0 \text{ and } \mu(1) = 1 \text{ and }
  \sup_x \sup_y \left| \mu(x \cup y) + \mu(x \cap y) - \mu(x) - \mu(y) \right| = 0.
  \]

- (Connection between $d$ and $\mu$)
  \[
  \sup_x \sup_y \left| d(x, y) - \mu(x \Delta y) \right| = 0.
  \]

Here $x \Delta y$ denotes the symmetric difference:
\[
x \Delta y = (x \cap y^c) \cup (y \cap x^c).
\]
\( APr \): atomless probability measure algebras

- \( APr = Pr \) together with:

\[
\sup_x \inf_y |\mu(x \cap y) - \mu(x \cap y^c)| = 0.
\]

- Note that \( Pr \) is an \( \forall \) theory and \( APr \) is an \( \forall \exists \) theory.
The models of Pr are exactly the measure algebras of probability spaces.

The models of APr are exactly the measure algebras of atomless probability spaces.

Therefore \((APr)_\forall = Pr\).
Theorem

- The theory $\text{APr}$ admits quantifier elimination and is complete.
- Hence $\text{APr}$ is the model companion of $\text{Pr}$.
- Every model of $\text{APr}$ is $\omega$-saturated.
- Equivalently, if $M \models \text{APr}$ and $a \in M^n$, then $\text{Th}(M, a)$ is separably categorical.

Corollary

For every $L^{pr}$-formula $\theta(x_1, \ldots, x_n)$, the zero set of $\theta^M$ is a definable set $\subseteq M^n$ (uniformly) in all models $M$ of $\text{APr}$. 
Let $M = (\mathcal{B}, \mu, d) \models Pr$ and $C \subseteq M$.

- Let $C^#$ denote the subalgebra of $\mathcal{B}$ generated by $C$.
- Let $\langle C \rangle$ denote the $\sigma$-subalgebra of $\mathcal{B}$ generated by $C$.
- Note that $\langle C \rangle$ is the closure of $C^#$ in the metric space $(\mathcal{B}, d)$.
- Also, if $C$ is finite, then $C^#$ is also finite and $\langle C \rangle = C^#$. 
Again let $M = (\mathcal{B}, \mu, d) \models Pr$.

- For $a \in M^n$, let $a^*$ denote the $2^n$-tuple of elements of the form $a_1^{e_1} \cap \cdots \cap a_n^{e_n}$, where $e_1, \ldots, e_n$ is any sequence from $\{-1, +1\}$, with the coordinates of $a^*$ written in lexicographic order of $(e_1, \ldots, e_n)$. (Here $a_i^{+1} := a_i$ and $a_i^{-1} := a_i^c$.) Note that $a^*$ is a partition of 1 in $\mathcal{B}$, and that each $a_i$ is the union of those coordinates of $a^*$ with $e_i = 1$.

- We define $x^*$ in the same way when $x = x_1, \ldots, x_n$ is a tuple of distinct variables.
Proposition

Let $M = (B, \mu, d) \models APr$. Consider $a, b \in M^m$ and $c \in M^n$. The following are equivalent:

1. $\text{tp}(a/c) = \text{tp}(b/c)$;
2. $\text{tp}(a^*/c^*) = \text{tp}(b^*/c^*)$;
3. For each $i = 1, \ldots, 2^m$ and $j = 1, \ldots, 2^n$, we have
   \[
   \mu(a_i^* \cap c_j^*) = \mu(b_i^* \cap c_j^*);
   \]
4. For each $i = 1, \ldots, 2^m$, we have $\text{tp}(a_i^*/c) = \text{tp}(b_i^*/c)$;

From (3) we get an $L^{pr}$-formula $\varphi(x, y, z)$ such that
\[
\{(a, b, c) \in M^{2m+n} \mid \text{tp}(a/c) = \text{tp}(b/c)\}
\]
is the zeroset of $\varphi^M$ in all models $M$ of $APr$.

Hence this is a definable set, uniformly in $M$. 
Proposition

Let $M = (B, \mu, d) \models APr$. Consider $a, b \in M^m$ and $c \in M^n$, with each of $a, b, c$ being a partition of 1. Then

$$d\left( \text{tp}(a/c), \text{tp}(b/c) \right) = \max_i \sum_j |\mu(a_i \cap c_j) - \mu(b_i \cap c_j)|.$$ 

Corollary

Let $M = (B, \mu, d) \models APr$. Consider $a, b \in M^m$ and $C \subseteq M$, with each of $a, b$ being a partition of 1. Then

$$d\left( \text{tp}(a/C), \text{tp}(b/C) \right) = \max_i \| \mathbb{P}(a_i \mid \langle C \rangle) - \mathbb{P}(b_i \mid \langle C \rangle) \|_1.$$
Stability and non-dividing in $APr$

It follows from the preceding result that $APr$ is $\omega$-stable; that is, for each countable set $C \subseteq M \models APr$, the type spaces $S_n(C)$ are separable. Hence there is a well-behaved concept of non-dividing for types in $APr$, which can be described as follows:

**Proposition**

Let $D \subseteq C \subseteq M = (\mathcal{B}, \mu, d) \models APr$ and $a \in M^n$. Then the following are equivalent:

1. $tp(a/C)$ does not divide over $D$.
2. $tp(a^*/C)$ does not divide over $D$.
3. For every $j = 1, \ldots, 2^n$, we have
   $$\mathbb{P}(a_j^* | \langle C \rangle) = \mathbb{P}(a_j^* | \langle D \rangle).$$

It follows that every type $p$ in $S_n(D)$ is stationary; i.e., $p$ has a unique non-dividing extension to a type over $C$. 


A calculation based on preceding results yields:

**Corollary**

Suppose \( M = (\mathcal{B}, \mu, d) \models APr \), and that \( u \in M^r, v \in M^s \), and \( w \in M^t \) are each partitions of \( 1 \). Suppose \( q \in S_r(vw) \) is the unique non-dividing extension of \( tp(u/w) \). Then

\[
d(tp(u/vw), q) = \max \sum_i \left| \mu(u_i \cap v_j \cap w_k) - \frac{\mu(u_i \cap w_k)\mu(v_j \cap w_k)}{\mu(w_k)} \right|.
\]

This gives an \( L^{pr} \)-formula \( \psi(x, y, z) \) such that

\[
\{(a, c, d) \in M^{m+n+p} \mid tp(a/cd) \text{ is non-dividing over } d\} = \{(a, c, d) \in M^{m+n+p} \mid tp(a^*/c^*d^*) \text{ is non-dividing over } d^*\}
\]

is the zero set of \( \psi^M \) in all models \( M \) of \( APr \).

Hence this is a definable set, uniformly in \( M \).
Given a set $F$, recall that $L^F$ is the signature obtained by adding the elements of $F$ to $L^{pr}$ as unary function symbols.

Let $T^F$ denote the $L^F$-theory obtained by adding to $Pr$ conditions expressing that each $f \in F$ is interpreted by an automorphism of the underlying model of $Pr$.

The needed conditions express that $f$ preserves the boolean operations and the measure (hence also the metric) and that $f$ is surjective. This last requirement is expressed by

$$\sup_y \inf_x d(y, f(x)) = 0.$$
For a word $w = f_1 \cdots f_n$ on the alphabet $F$ and a variable $x$, we set

$$w(x) := f_1(\cdots(f_n(x))\cdots).$$

Further, if $x = x_1, \ldots, x_n$ is a sequence of variables, we set

$$w(x) := (w(x_1), \ldots, w(x_n)).$$

Suppose $G$ is a finite or countable discrete group. Let $T(G)$ denote the $L^G$-theory obtained by adding to $T^G$ the conditions

$$\sup_x d(v(x), w(x)) = 0$$

for every words $v, w$ on the set $G$ such that $v$ and $w$ represent the same element in the group $G$.

Obviously, models of $T(G)$ correspond exactly to actions of $G$ by automorphisms on probability measure algebras.
Theorem

- If $G$ is amenable, then $T(G)$ has a model companion $T^*(G)$.
- The theory $T^*(G)$ extends $APr$ and it admits quantifier elimination and is complete.
- Further, $T^*(G)$ is stable, with its non-dividing relation obtained by applying the non-dividing relation for $APr$ to the $T^*(G)$-definable closure of the sets involved.
- Hence, all types over sets are stationary.
The main ingredient of the proof is a generalization of Rokhlin’s Lemma (which applies to $\mathbb{Z}$-actions) to actions of amenable groups, due to Ornstein and Weiss. It allows us to give a reasonable description of the models of $T^*(G)$.

For example, $M \models T^*(\mathbb{Z})$ iff $1^M$ is an aperiodic automorphism of the reduct of $M$ to $L^{pr}$, which is a model of $APr$. 
Theorem

For each set $F$, the theory $T^F$ has a model companion, which has quantifier elimination and is complete.

To prove this result we need to axiomatize the class of existentially closed (e.c.) models of $T^F$. Recall that $M \models T^F$ is e.c. iff

$$(\inf_y \theta(a, y))^M = (\inf_y \theta(a, y))^N$$

whenever $\theta(x, y)$ is a quantifier-free $L^F$-formula, $N$ is a model of $T^F$ with $M \subseteq N$, and $a$ is a tuple from $M$.

Since $T^F$ is an inductive theory, we know that every model of $T^F$ is contained in an e.c. model.
To avoid notational complexity and make the key ideas more clear, we assume $F = \{ f, g \}$.

As a first step in the proof, one shows that a model $M$ of $T^F$ is e.c. iff $M$ satisfies $APr$ and the following statement for all finite tuples $a, b, c, d$ from $M$:

Assume

- $\text{tp}(b/f(d)) = \text{tp}(f(a)/f(d))$.
- $\text{tp}(c/g(d)) = \text{tp}(g(a)/g(d))$.
- $\text{tp}(a/df(d)g(d))$ is the non-dividing extension of $\text{tp}(a/d)$.
- $\text{tp}(b/df(d)g(d))$ is the non-dividing extension of $\text{tp}(b/d)$.
- $\text{tp}(c/df(d)g(d))$ is the non-dividing extension of $\text{tp}(c/d)$.

Then there exists a tuple $e$ from $M$ such that

$$\text{tp}(ef(e)g(e)/df(d)g(d)) = \text{tp}(abc/df(d)g(d)).$$

(Here “tp” and “non-dividing” are meant in the sense of $APr$.)
The proof of the preceding claim uses a stability argument that goes back to work of Kikyo and Pillay.

From our discussion of $APr$, we know that there are $L^{pr}$-formulas $\alpha$ and $\beta$ such that the assumptions are true of $a, b, c, d$ in $M$ iff
\[ \alpha^M(a, b, c, d, f(a), g(a), f(d), g(d)) = 0 \]
and the conclusion holds of $a, b, c, d, e$ in $M$ iff
\[ \beta^M(a, b, c, d, e, f(d), g(d), f(e), g(e)) = 0. \]
(Here $\alpha$ and $\beta$ depend on the lengths of the tuples $a, b, c, d, e$.)

AND we know that the zero sets of $\alpha$ and $\beta$ are definable sets in the theory $APr$. 
However, because we are working in continuous model theory, we cannot yet conclude that the desired implications are expressible by axioms. What happens is that the formulas $\alpha$ and $\beta$ can be chosen (without changing their zerosets) so that every e.c. model $M$ of $T^F$ satisfies the following stronger requirement (which is expressed by a condition in continuous logic):

$$\sup_{a,b,c,d} \left( \inf_e \beta(a, b, c, d, e, f(d), g(d), f(e), g(e)) \leq \alpha(a, b, c, d, f(a), g(a), f(d), g(d)) \right) = 0.$$ 

These conditions then axiomatize the e.c. models of $T^F$. 