Uncountable Categoricity for Structures Based on Banach Spaces

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There are three sub-themes in this talk (under the overall topic of uncountable categoricity for metric structures):

- Toward characterizations of uncountable categoricity for metric structures.
- Toward structure theory for uncountably categorical metric structures, which seems to require finding analogues of strongly minimal sets.
- Enlarge the collection of uncountably categorical metric structures. Use them as guides toward what could be true in general.

All three projects are very much works in progress.

But first: A skeptical question:

Why study uncountably categorical structures?

After all, there are few of them and the mathematically interesting ones can be studied well without using model theory.

A possible answer:
The model theoretic tools developed to understand uncountable categoricity proved very useful for a large number of important applications having little or nothing to do with categoricity. The original project was focused, and the tools needed to complete it became generally useful, after they were developed further.

Let T be an arbitrary theory of metric structures, in a countable signature.

Definition (κ-categoricity; $\kappa > \omega$)

T is κ-categorical if T has exactly one model of density $= \kappa$.

Morley's Theorem is known to hold for metric structures:

Theorem

Let κ, λ be uncountable cardinals. Then T is κ-categorical if and only if T is λ-categorical.

This was proved independently by Itaï Ben Yaacov [JSL 2005, for CATs] and Alex Usvyatsov [in his PhD thesis, published with Shelah in Isr.J.Math 2011, for more general notions of categoricity].

- The proofs of Ben Yaacov and Shelah-Usvyatsov have a skeletal structure roughly similar to Morley's argument. In particular, they show κ-categorical ($\kappa > \omega$) implies ω-stable. This yields tools that are critical for obtaining the result.
- At the moment there is nothing close to a characterization of uncountably categorical theories in the general metric setting.
- However, if one restricts attention to structures based on Banach spaces, the problem of understanding uncountable categoricity seems more tractable at the moment.

For comparison and reference, here is a characterization and structural result due to Baldwin and Lachlan in classical model theory:

Theorem

Let T be a complete theory in a countable language. The following are equivalent:
(1) T is uncountably categorical.
(2) There exists a strongly minimal set D definable over the prime model of T, such that each uncountable model M of T is minimal and prime over D^M.

Prime: every elementary map from D^M into a model N of T extends to an elementary embedding of M into N.
Minimal: There is no proper elementary submodel of M that contains D^M.

We take the signature for unit balls of Banach spaces to be $\mathcal{L}_b = \{0, c_{r,s}, \| \ \|\}$ where r, s range over rational scalars such that $|r| + |s| \leq 1$. This is a countable continuous signature.

- 0 is a constant symbol.
- $c_{r,s}$ are binary function symbols.
  - Interpret $c_{r,s}(x, y)$ as $rx + sy$.
- $\| \ \|$ is a unary predicate symbol.
- the functions and predicate are 1-Lipschitz in each variable.

Notation

We let T_b denote a theory axiomatizing the class of all (unit balls of) Banach spaces.

In what follows, we always have:

- $\mathcal{L}$ is a countable continuous signature expanding $\mathcal{L}_b$.
- T is a consistent $\mathcal{L}$-theory extending T_b.

So models of T are unit balls of Banach spaces equipped with extra predicates and operations (whose operations map unit balls to unit balls). For example:

- Banach lattices
- C^*-algebras
- Operator spaces

- The most fundamental uncountably categorical theory of Banach structures is the theory IHS of infinite dimensional Hilbert spaces.
- IHS has quantifier elimination.
- $\mathrm{acl}(A) = \mathrm{dcl}(A) =$ closed linear span of A.
- The independence relation for *IHS* is what you would expect, based on orthgonal projection.
- *IHS* is κ-categorical for all infinite κ.

- As model theory in functional analysis developed, it was observed by the speaker that all known examples of uncountably categorical Banach structures are very closely related to Hilbert spaces. (However, it was also observed that very few such structures were known.)
- Shelah and Usvyatsov treated this as a conjecture, and proved the following remarkable result, which describes uncountably categorical Banach structures using Hilbert spaces as the foundation.

Theorem (Shelah,Usvyatsov; arXiv:1402.6513)

Assume T is an uncountably categorical theory of Banach space structures in a countable language. Then there is a separable model $\mathcal{M}_0$ of T and a 1-type $p_0(x)$ over $\mathcal{M}_0$ with the following properties:
(1) Every Morley sequence in p_0 is isometric to the standard orthonormal basis of a Hilbert space.
(2) Every non-separable model $\mathcal{M}$ of T is prime over a Morley sequence $(a_\alpha \mid \alpha < \lambda)$ in p_0, where λ is the density of $\mathcal{M}$.
(3) The type $p_0(x)$ is minimal wide.

Wide types

Let $\pi(x)$ be a partial 1-type over A. With no loss of generality assume $\pi(x)$ contains the condition "$\|x\| = 1$".

Definition

$\pi(x)$ is wide if there is exists $\mathcal{M} \models T$ containing A such that $\pi(\mathcal{M})$ contains the unit sphere of an infinite dimensional subspace of $\mathcal{M}$.

Definition

A complete 1-type $p(x)$ over A is minimal wide if $p(x)$ has exactly one extension to a wide global type.

For proving the existence of wide complete 1-types, Milman's far-reaching extension of Dvoretsky's Theorem is used.
Setting: X is an infinite dimensional Banach space, $S(X)$ is the unit sphere of X, and $f\colon S(X) \to \mathbb{R}$ is a uniformly continuous function.

Definition

The Hilbert spectrum of f, denoted $\gamma_H(f)$, is the set of $r \in \mathbb{R}$ such that for all $\epsilon > 0$ and all $k \geq 1$ there exists a k-dimensional subspace F of X such that
(1) F is $(1+\epsilon)$-isomorphic to Hilbert space and
(2)for all $x \in S(F)$, we have $|f(x) - r| \leq \epsilon$.

Theorem (Dvoretsky-Milman)

For any infinite dimensional Banach space X and any uniformly continuous function $f\colon S(X) \to \mathbb{R}$,

$$\gamma_H(f) \neq \varnothing.$$

Corollary

If $\pi(x)$ is a wide partial 1-type over A, and $\varphi(x)$ is any formula over A, then there exists $r \in [0,1]$ such that the partial type $\pi(x) \cup [\varphi(x) = r]$ is wide.

Things to do in pure model theory of Banach structures:

- Improve the Shelah-Usvyatsov theorem. In particular, consider the Hilbert space $H =$ closed linear span of the Morley sequence in $p_0(x)$, and try to:
  - understand how definable H can be in $\mathcal{M}$;
  - explain how $\mathcal{M}$ sits over H in clear functional analysis language.
  - understand the "induced structure" on H.
- Prove the converse.

More things to do:

- Find uncountably categorical Banach structures in whatever corner of functional analysis you find yourself. (Or prove there aren't any there.)
- Isolate model theoretic properties of Hilbert space that are sufficient to imply uncountable categoricity for any Banach structure that satisfies them. (How exactly does Hilbert space deserve to be considered analogous to a strongly minimal set?)

Some cautionary examples:

- No infinite dimensional C^*-algebra is uncountably categorical. (Indeed, they are not even ω-stable.)
- Fix $1 \leq p < \infty, p \neq 2$. No infinite dimensional L_p-space is uncountably categorical, neither as Banach space nor as Banach lattice (even though these theories are all ω-stable).
- No L_2-space is uncountably categorical as a Banach lattice.

Conjecture: No infinite dimensional Banach lattice is uncountably categorical.

Joint work with Yves Raynaud; arxiv 1606.03122

We consider separable, infinite dimensional Banach spaces X that are asymptotically Hilbertian in the sense that *every ultrapower of X is (isometrically) of the form $X \oplus H$ for some Hilbert space H*. Here X is identified with its canonical copy in the ultrapower (image of X under the diagonal embedding).

Such X are necessarily reflexive, and hence there is a canonical projection P from each ultrapower of X onto X.

For our examples, the kernel of P is a Hilbert space, and the structure of the decomposition $X \oplus \ker(P)$ is uniquely determined by the density of $\ker(P)$. (For the moment we avoid the issue of describing $\oplus$ precisely.)

Let X be as above.

We try to show that the models of $\mathrm{Th}(X)$ are exactly the Banach spaces of the form $Y \oplus H$, where Y is the prime model of $\mathrm{Th}(X)$ and H is any Hilbert space. If so, then $\mathrm{Th}(X)$ would be uncountably categorical. (For our examples, this is exactly what we prove.)

At the moment we work with spaces satisfying an additional strong automorphism condition on ultrapowers:

(Aut) For any ultrapower of X, represented as $X \oplus H$, where H is the kernel of the canonical projection P onto X,

- The automorphisms of $X \oplus H$ are exactly the maps $Q_X \oplus Q_H$, where Q_X is an automorphism of X and Q_H is an automorphism of H.

Such Banach spaces X are uncountably categorical:

Theorem

Suppose X is a separable, infinite dimensional, asymptotically Hilbertian Banach space satisfying (Aut). Then $\mathrm{Th}(X)$ *is uncountably categorical. Indeed, the models of* $\mathrm{Th}(X)$ *are exactly the Banach spaces of the form $X \oplus H$ where H is a Hilbert space.*

Note that in the situations covered by this Theorem, X is the prime model of its theory. Indeed, X is the algebraic closure of $\varnothing$, and hence every model $X \oplus H$ of $\mathrm{Th}(X)$ is the algebraic closure of its Hilbert part H.

Modular Banach spaces

A convex modular on a linear space X is a convex function $\Theta\colon X \to \mathbb{R}^+$ satisfying: $\Theta(-x) = \Theta(x)$ and $(\Theta(x) = 0 \Leftrightarrow x = 0)$. An associated norm on X is then defined by the formula:

$$\|x\| = r \Leftrightarrow \Theta(\tfrac{1}{r}x) = 1.$$

When $(X_i, \Theta_i \mid i \in I)$ is a family of modular spaces, their modular direct sum consists of the linear space of families $x = (x_i \mid i \in I)$ such that $x_i \in X_i$ for all $i \in I$ and

$$\Theta(x) := \sum_{i \in I} \Theta_i(x_i) < \infty$$

on which Θ is a convex modular.

We always assume Θ satisfies the Δ_2 condition with constant C for all the modular spaces we consider; this means

$$\Theta(2x) \leq C \cdot \Theta(x) \quad \text{for all } x$$

When C is fixed, this yields a signature with respect to which all such modular spaces are pre-structures, allowing us to apply the ultraproduct construction and other model thoretic tools to them.

Note that when X is a normed space and $p \in [1, \infty)$, the function $\Theta(x) = \|x\|^p$ is a convex modular on X, and the norm defined from Θ is equal to the original one. This Θ satisfies the Δ_2 condition with constant 2^p.

The Jordan-von Neumann constant of a normed space

The Jordan-von Neumann constant $a(X)$ of a normed space X is

$$a(X) := \frac{1}{2}\sup\{\|x+y\|^2 + \|x-y\|^2 : x, y \in X, \|x\|^2 + \|y\|^2 = 1\}.$$

By setting $u = x+y$, $v = x-y$ one shows

$$a(X) = 2\sup\{\|x\|^2 + \|y\|^2 : x, y \in X, \|x+y\|^2 + \|x-y\|^2 = 1\}.$$

Equivalently: $a(X)$ is the smallest $C \geq 1$ such that for all $x, y \in X$

$$(2/C)(\|x\|^2 + \|y\|^2) \leq \|x+y\|^2 + \|x-y\|^2 \leq (2C)(\|x\|^2 + \|y\|^2).$$

Hence $a(X) = 1$ iff X is linearly isometric to a Hilbert space. More generally, the distance between $a(X)$ and 1 measures the extent to which X fails to satisfy the parallelogram identity.

Nakano direct sums

Theorem

Let $(E_n, \|\cdot\|_n \mid n \in \mathbb{N})$ be any family of nonzero finite dimensional Banach spaces such that $a(E_n) \to 1$ as $n \to \infty$. Let $(p_n \mid n \in \mathbb{N})$ be a sequence from $[1, \infty)$ such that $p_n \to 2$ as $n \to \infty$ and $p_n \neq 2$ for all n. Let X consist of the sequences $x = (x_n \mid n \in \mathbb{N})$ such that $x_n \in E_n$ for all n and define a Δ_2 convex modular on X by

$$\Theta(x) := \sum_n \|x_n\|_n^{p_n} < \infty.$$

Then X, equipped with the norm associated to Θ, is an uncountably categorical Banach space. Indeed, the Banach space models of $\mathrm{Th}(X)$ are exactly the modular direct sums $X \oplus_m H$ where H is a Hilbert space. (The modular on H is $x \mapsto \|x\|^2$.)

Some Hilbert direct sum examples

Now consider $X = \oplus_2(E_n : n \in \mathbb{N})$, where each E_n is finite dimensional with norm $\| \cdot \|_n$. Here X consists of the families $x = (x_n \mid n \in \mathbb{N})$ such that $x_n \in E_n$ for all n and

$$\Theta(x) := \sum_n \|x_n\|_n^2 < \infty$$

and we define the norm $\| \cdot \|$ on X by $\|x\| = \sqrt{\Theta(x)}$. (This is, of course, the familiar ℓ_2 direct sum, described in the modular framework.)

We show that X is an uncountably categorical Banach space in the following concrete examples.

In the first two of these examples, we assume that $p_n \to 2$ (this ensures that $a(E_n) \to 1$) and that $p_n \neq 2$ for all n.

- $X = \oplus_2(\ell_{p_n}^{d_n} : n \in \mathbb{N})$;
- $X = \oplus_2(S_{p_n}^{d_n} : n \in \mathbb{N})$;
  here S_p^d is the Schatten class consisting of $d \times d$ matrices with complex entries and using the exponent p trace norm.

- $X = \oplus_2(E_n : n \in \mathbb{N})$ where each E_n is polyhedral of finite dimension ≥ 2 and $a(E_n) \to 1$.

In each case above, the theory of X is uncountably categorical and X is its prime model; indeed, the models of the theory of X are exactly the spaces of the form $X \oplus_2 H$ where H is any Hilbert space.