Model theory of $\mathbb{R}$-trees and of ultrametric spaces

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November, 2019

last revised 11/14/19
A metric space $(X, d)$ is an $\mathbb{R}$-tree if for every $a, b \in X$ there is a unique arc (denoted $[a, b]$) from $a$ to $b$ in $X$, and this arc is isometric to the interval $[0, d(a, b)]$.

Let $(X, d)$ be an $\mathbb{R}$-tree.

- A branch at $a$ is a connected component of $X \smallsetminus \{a\}$.
- The degree of $a$ is the number of branches at $a$.
- The height of a branch $\beta$ at $a$ is $\sup\{d(a, x) \mid x \in \beta\}$.
- $a$ is a branch point if its degree is $\geq 3$.
- $a$ is an endpoint if its degree is $\leq 1$. 
Let \((X, d)\) be a metric space.

- For \(a, b, x \in X\) the **Gromov product** is
  \[
  (a \cdot b)_x := \frac{1}{2}[d(a, x) + d(b, x) - d(a, b)].
  \]

- When \((X, d)\) is an \(\mathbb{R}\)-tree,
  \[
  (a \cdot b)_x = \text{dist}(x, [a, b]).
  \]

- \((X, d)\) is **0-hyperbolic** if for every \(a, b, c, x \in X\)
  \[
  \min[(a \cdot c)_x, (b \cdot c)_x] \leq (a \cdot b)_x.
  \]

- Every \(\mathbb{R}\)-tree is 0-hyperbolic. Indeed, for all points \(a, b, c\),
  \[
  [a, c] \cup [b, c] \supseteq [a, b].
  \]

- \((X, d)\) is **geodesic** if for every \(a, b \in X\) there is an isometric map \(\alpha\) from the interval \([0, d(a, b)]\) to \(X\) such that \(\alpha(0) = a\) and \(\alpha(d(a, b)) = b\).

- \((X, d)\) is an \(\mathbb{R}\)-tree if and only if it is
  geodesic and 0-hyperbolic.
Proposition

A complete metric space \((X, d)\) is an \(\mathbb{R}\)-tree iff it is 0-hyperbolic and has the approximate midpoint property, which says:

for all \(a, b \in X\) and all \(\epsilon > 0\) there exists \(c \in X\) such that both \(d(a, c)\) and \(d(b, c)\) are within \(\epsilon\) of \(\frac{1}{2}d(a, b)\).

Proof.

For each \(n \geq 1\), suppose \(c_n\) satisfies the displayed condition for \(\epsilon = \frac{1}{n}\). The 0-hyperbolic inequality can be used to show \(d(c_m, c_n) \leq \frac{1}{m} + \frac{1}{n}\). Therefore \((c_n)\) converges to a point \(c\) satisfying \(d(a, c) = d(b, c) = \frac{1}{2}d(a, b)\). Applying this in an inductive way, we get a function \(\mu\) from the set \(D\) of dyadic rationals in \([0, 1]\) to \(X\) such that \(\mu(0) = a, \mu(1) = b\), and for every \(s, t \in D\) we have \(d(\mu(s), \mu(t)) = |t - s|d(a, b)\). Then \(\mu\) extends to all of \([0, 1]\) by uniform continuity. The function \(f(t) = \mu(t/d(a, b))\) is isometric for \(t\) in the interval \([0, d(a, b)]\) and satisfies \(f(0) = a\) and \(f(d(a, b)) = b\). In other words, \((X, d)\) is geodesic. \(\square\)
Both the 0-hyperbolic condition and the approximate midpoint condition can obviously be expressed using continuous first order logic. This allows us to axiomatize many classes of $\mathbb{R}$-trees.

We focus on the class $\mathcal{C}_r$ of pointed $\mathbb{R}$-trees $(X, d, p)$ that have radius at most $r$, for a fixed $r > 0$.

**Theorem**

Let $\mathbb{RT}_r$ consist of continuous conditions that express the 0-hyperbolic and approximate midpoint conditions together with the condition $\sup_x d(p, x) \leq r$. Then

(a) The models of $\mathbb{RT}_r$ are exactly the members of $\mathcal{C}_r$ that are metrically complete.

(b) The class $\mathcal{C}_r$ is closed under completions, under unions of chains, and under ultraproducts.
The following fact about $\mathbb{R}$-trees is used in the reasoning behind several items that are stated later..

Let $(X, d)$ be an $\mathbb{R}$-tree.

**Proposition**

Suppose $E_1$ and $E_2$ are disjoint, closed, nonempty subtrees of $X$. Then there exist unique points $e_1 \in E_1$ and $e_2 \in E_2$ such that

$$d(e_1, e_2) = \text{dist}(E_1, E_2).$$

Moreover, for all $x_1 \in E_1$ and $x_2 \in E_2$, the geodesic segment $[x_1, x_2]$ contains $[e_1, e_2]$, so we have

$$d(x_1, x_2) = d(x_1, e_1) + d(e_1, e_2) + d(e_2, x_2).$$
Let $(X, d)$ be a metric space.

**Corollary**

(a) $(X, d)$ embeds into an $\mathbb{R}$-tree iff $(X, d)$ is 0-hyperbolic.

(b) Suppose for each $i = 1, 2$ that $(Y_i, d_i)$ is an $\mathbb{R}$-tree and $f_i : X \rightarrow Y_i$ is isometric.

For each $i$ let $Z_i$ be the smallest $\mathbb{R}$-tree contained in $Y_i$ and containing $f_i(X)$ (the subtree of $Y_i$ spanned by $f_i(X)$).

Then there exists a surjective isometry $g : Z_1 \rightarrow Z_2$ such that $f_2 = g \circ f_1$.

So, for every 0-hyperbolic metric space $(X, d)$ there is a unique $\mathbb{R}$-tree $(Y, d)$ of which $(X, d)$ is a subspace and for which $Y$ is spanned by $X$; that is, $Y$ is the union of the segments $[a, b]$, with $a, b \in Y$. 
Let \((X, d)\) be an \(\mathbb{R}\)-tree.

- For any \(a, b, c \in X\), there is a single point in \([a, b] \cap [a, c] \cap [b, c]\), which we denote \(Y(a, b, c)\).
- For any point \(x \in X\), we have
  \[
  d(x, Y(a, b, c)) = \max[(a \cdot b)_x, (a \cdot c)_x, (b \cdot c)_x].
  \]
- Therefore, the function \(Y(x, y, z)\) is a quantifier-free definable function on models of \(\mathbb{R}T_r\).

**Corollary**

Let \((X, d)\) be an \(\mathbb{R}\)-tree and let \((\overline{X}, d, p)\) be its completion. Then any point \(a \in \overline{X} \setminus X\) is the limit of a sequence \((a_n)\) from \(X\) such that for all \(n\) we have \(a_n \in [p, a_{n+1}] \subseteq [p, a]\).
Consequently, any such “new” point is an endpoint in \(\overline{X}\).
an $\mathbb{R}$-tree is **densely branching** if its set of branch points is dense.

in a densely branching $\mathbb{R}$-tree, for any distinct points $a, b$, the set of branch points that lie on the segment $[a, b]$ is dense in $[a, b]$.

in a separable densely branching $\mathbb{R}$-tree, the set of branch points that lie on $[a, b]$ is countable.
Let $(X, d, p)$ be an $\mathbb{R}$-tree of radius at most $r$ (i.e., it is a member of $C_r$).

- A point $a$ in $X$ is a rich branch point if there exist at least 3 branches at $a$ of height at least $r - d(p, a)$.
- $(X, d, p)$ is a richly branching $\mathbb{R}$-tree of radius $r$ if the set of rich branch points in $X$ is dense.
- If $(X, d, p)$ is a richly branching $\mathbb{R}$-tree of radius $r$, then so is the completion of $(X, d, p)$.

**Proposition**

Let $(X, d, p)$ be a richly branching $\mathbb{R}$-tree of radius $r$, and suppose $(X, d)$ is complete. For every $a \in X$ and every branch $\beta$ at $a$, there exists $b \in \beta$ such that $d(p, b) = r$. 
The theory $\mathbb{R}T_r$ has a model companion, which admits QE and is complete.

The model companion is axiomatized by adding to $\mathbb{R}T_r$ a single condition (to be given on the next slide) which is of the form $\sup_x \inf_{y_1} \inf_{y_2} \inf_{y_3} \eta(x, y_1, y_2, y_3) = 0$, where $\eta$ is quantifier-free. We denote this set of axioms by $\text{rb} \mathbb{R}T_r$.

The models of $\text{rb} \mathbb{R}T_r$, which must be the existentially closed models of $\mathbb{R}T_r$, are exactly the complete, richly branching $\mathbb{R}$-trees of radius $r$. 
We write $y$ for the sequence of variables $y_1, y_2, y_3$. Consider the following quantifier-free formulas $\alpha(x, y)$ and $\beta(x, y)$:

- $\alpha(x, y) := \max_{i=1,2,3}\{|d(x, y_i) - (r - d(p, x))|\}$;
- $\beta(x, y) := \max_{1 \leq i < j \leq 3}\{d(x, y_i) + d(x, y_j) - d(y_i, y_j)\}$.

Then the axioms of $\text{rbRT}_r$ consist of $\text{RT}_r$ together with the condition

$$\sup_x \inf_{y_1} \inf_{y_2} \inf_{y_3} \max[\alpha(x, y), \beta(x, y)] = 0.$$
Further properties of rbRTr:

- It is stable but not superstable. (we describe precisely)
- It has the maximum number of models of density $\kappa$, for every infinite cardinal $\kappa$.
- Its space of 2-types has density $2^{\omega}$ with respect to the induced metric. ($S_1(\text{rbRTr}) = [0, r]$)
- A model is $\kappa$-saturated iff every point has degree $\geq \kappa$. There exists a $\kappa$-saturated model of density $\kappa$ iff $\kappa^\omega = \kappa$.
- It has very few isolated types. In particular, it has no atomic model.
Introduction of the metric space $E_r(M)$

Consider $M = (M, d, p) \models \text{rbRT}_r$; so $M$ is complete.

- Let $E_r(M) := \{ a \in M \mid d(p, a) = r \}$.
- $E_r(M)$ is a definable set relative to the theory $\text{rbRT}_r$.
- $(E_r(M), d)$ satisfies the ultrametric inequality:
  \[ d(x, y) \leq \max[d(x, z), d(y, z)]. \]
- For each $a \in E_r(M)$, the set $D_a := \{ d(a, b) \mid b \in E_r(M) \}$ is dense in $[0, 2r]$.
- For $a, b, a', b' \in E_r(M)$ define
  \[ \rho((a, b), (a', b')) := d(Y(p, a, b), Y(p, a', b')). \]
  \[ \rho \] is a definable predicate relative to $\text{rbRT}_r$; it defines a pseudometric on the set of pairs from $E_r(M)$.
- The imaginary sort in $M^{eq}$ determined by $\rho$ is canonically isomorphic to $M$. 
So we let $\mathcal{T}_r$ be the theory of ultrametric spaces of diameter at most $2r$, and let $\mathcal{T}^*_r$ be the extension of $\mathcal{T}_r$ obtained by adding a condition that expresses the following statement: for every $x$ the distances $d(x, y)$ are dense in the interval $[0, 2r]$. As noted above,

if $M = (M, d, p) \models r\mathcal{B}\mathcal{R}\mathcal{T}_r$, then $(E_r(M), d) \models \mathcal{T}^*_r$.

**Theorem**

1. The theory $\mathcal{T}^*_r$ is the model companion of $\mathcal{T}_r$ and it has QE.
2. For every model $(E, d')$ of $\mathcal{T}^*_r$, there exists a model $M = (M, d, p)$ of $r\mathcal{B}\mathcal{R}\mathcal{T}_r$ such that $(E, d') = (E_r(M), d)$.
3. Indeed, the theories $r\mathcal{B}\mathcal{R}\mathcal{T}_r$ and $\mathcal{T}^*_r$ are bi-interpretable.