For this subsection we fix a compact Hausdorff space \((T, \tau)\), and a metric space \((M, \rho)\), both in \(U(X)\).

Let \(C(T, M)\) denote the space of continuous functions from \(T\) into \(M\). We give \(C(T, M)\) the topology of uniform convergence; this is the topology defined by the supremum metric, which is defined for \(f, g \in C(T, M)\) by

\[
d(f, g) = \sup\{\rho(f(t), g(t)) | t \in T\}.
\]

Note that because \((T, \tau)\) is compact and \(f\) and \(g\) are continuous, the set \(\{\rho(f(t), g(t)) | t \in T\}\) is bounded in \(\mathbb{R}\).

5.16. Notation. If \(a, b \in ^*T\) we write \(a \approx b\) if \(\text{st}(a) = \text{st}(b)\). If \(\alpha, \beta \in ^*M\) we write \(\alpha \approx \beta\) if \(*\rho(\alpha, \beta) \approx 0\). Note that both of these relations are equivalence relations.

5.17. Definition. An internal function \(F: ^*T \to ^*M\) is said to be \(S\)-continuous if \(a \approx b \Rightarrow F(a) \approx F(b)\) holds for all \(a, b \in ^*T\).

5.18. Theorem. If \(F \in ^*C(T, M)\), then \(F\) is nearstandard if and only if \(F\) is \(S\)-continuous and \(F(*t) \in \text{ns}(^*M)\) for all \(t \in T\).

Proof. \((\Rightarrow)\) Assume \(F\) is nearstandard and let \(f \in C(T, M)\) be the standard part of \(F\). This means that \(F(a) \approx *f(a)\) holds for all \(a \in ^*T\).

First we prove \(F\) is \(S\)-continuous. Take any \(a, b \in ^*T\) such that \(a \approx b\), and thus we have \(t \in T\) that is the common \(\tau\)-standard part of \(a\) and \(b\). Since \(f\) is continuous at \(t\) we have \(F(a) \approx *f(a) \approx *f(t) \approx *f(b) \approx F(b)\).

Second, we note that \(F(*t)\) is \(\rho\)-nearstandard in \(^*M\) for every \(t \in T\), since we have \(F(*t) \approx *f(*t) = *\left(\text{st}(F(*t))\right)\).

\((\Leftarrow)\) Assume \(F: ^*T \to ^*M\) is \(S\)-continuous and that \(F(*t) \in \text{ns}(^*M)\) for all \(t \in T\). Define \(f: T \to M\) by \(f(t) = \text{st}(F(*t))\) for \(t \in T\). We will show that \(f\) is in \(C(M, T)\) and that it is the standard part of \(F\).

First we show that \(f\) is continuous. Let \(t \in T\) and \(\epsilon > 0\). Because \(F\) is \(S\)-continuous, \(*\rho(F(a), F(*t))\) is infinitesimal for every \(a \in \mu(t)\). Thus we have

\[
\bigcap\{*O \mid t \in O \text{ and } O \text{ is } \tau\text{-open}\} \subseteq \{a \in ^*T \mid *\rho(F(a), F(*t)) < \epsilon\}.
\]

Recall that we are assuming \(\text{card}(\tau)\)-saturation. Therefore we may apply Proposition 4.16 to show that there exist \(\tau\)-open neighborhoods \(O_1, \ldots, O_n\).
of $t$ such that $^*\rho(F(a), F(*t)) < \epsilon$ holds for every $a \in \ast O_1 \cap \cdots \cap \ast O_n$. Let $O$ be the open neighborhood $O_1 \cap \cdots \cap O_n$ of $t$ and consider any $s \in O$. We have
\[
\rho(f(s), f(t)) = ^*\rho(^*f(s), ^*f(t)) \\
\leq ^*\rho(^*f(s), F(^*s)) + ^*\rho(F(^*s), F(*t)) + ^*\rho(F(*t), ^*f(t)) \\
< \epsilon + ^*\rho(^*f(s), F(^*s)) + ^*\rho(F(*t), ^*f(t)).
\]
From the definition of $f$ we conclude $\rho(f(s), f(t)) \leq \epsilon$ for every $s \in O$. Since $t$ and $\epsilon$ were arbitrary, this shows that $f$ is continuous on $T$.

Finally we must show that $f$ is the standard part of $F$. Take any $a \in ^*T$ and let $t = \text{st}(a)$. Then using the definition of $f$ and the fact that $F$ is $S$-continuous and that $f$ is continuous, we have $F(a) \approx F(*t) \approx ^*f(t) \approx ^*f(a)$, which means that $^*\rho(F(a), ^*f(a))$ is infinitesimal. Since $d(F, ^*f)$ is the internal supremum of $^*\rho(F(a), ^*f(a))$ as $a$ ranges over $^*T$, we have that $d(F, ^*f)$ is also infinitesimal, as needed. $\square$

5.19. Definition. A subset $\mathcal{F}$ of $C(T, M)$ is equicontinuous if for every $t \in T$ and $\epsilon > 0$ there is an open neighborhood $O$ of $t$ such that $\rho(f(s), f(t)) < \epsilon$ holds for all $s \in O$.

5.20. Lemma. If $\mathcal{F} \subseteq C(T, M)$, then $\mathcal{F}$ is equicontinuous if and only if every $F \in \ast \mathcal{F}$ is $S$-continuous.

Proof. $(\Rightarrow)$ Assume $\mathcal{F} \subseteq C(T, M)$ is equicontinuous and suppose $F \in \ast C(T, M)$. Fix $a \approx b$ in $^*T$ and let $\epsilon > 0$ be standard. We may take $t \in T$ to be the common standard part of $a, b$. Let $O$ be an open neighborhood of $t$ that witnesses the equicontinuity of $\mathcal{F}$, in the sense that $\rho(f(s), f(t)) < \epsilon/2$ holds for every $f \in \mathcal{F}$ and every $s \in O$. Since $a, b \in \mu(t) \subseteq \ast O$, we have by transfer that $^*\rho(F(a), F(*t)) < \epsilon/2$ and $^*\rho(F(b), F(*t)) < \epsilon/2$. The triangle inequality for $^*\rho$ yields $^*\rho(F(a), F(b)) < \epsilon$. Since $\epsilon > 0$ was arbitrary, we conclude $F(a) \approx F(b)$, as desired.

$(\Leftarrow)$ Assume every $F \in \ast \mathcal{F}$ is $S$-continuous. Fix $t \in T$ and a standard $\epsilon > 0$. Consider the internal set
\[
A = \{a \in ^*T \mid ^*\rho(F(a), F(*t)) < \epsilon \text{ for all } F \in \ast \mathcal{F}\}.
\]
Our assumption yields that $\mu(t) \subseteq A$. By Proposition 4.16, which applies since we are assuming $\text{card}(\tau)^+\text{-saturation}$ and $\mu(t)$ is equal to the intersection of at most $\text{card}(\tau)$ many standard sets, there exist open sets $O_1, \ldots, O_n$
such that \( t \in O_1 \cap \cdots \cap O_n \) and \( \ast O_1 \cap \cdots \cap \ast O_n \subseteq A \). By transfer we have that \( \rho(f(s), f(t)) < \epsilon \) holds for every \( f \in \mathcal{F} \) and \( s \in O_1 \cap \cdots \cap O_n \). Since \( \epsilon > 0 \) was arbitrary, this proves \( \mathcal{F} \) is equicontinuous. \( \square \)

Using the nonstandard analysis results above, we give a quick proof of the following standard theorem:

5.21. **Theorem** (Ascoli’s Theorem). For \( \mathcal{F} \subseteq C(T, M) \) the following are equivalent:

1. \( \mathcal{F} \) is relatively compact in \( C(T, M) \).
2. \( \mathcal{F} \) is equicontinuous and \( \{ f(t) \mid f \in \mathcal{F} \} \) is relatively compact in \( M \) for all \( t \in T \).

**Proof.** Since metric spaces are regular, Proposition 5.14 applies and shows that condition (1) is equivalent to

\( (1') \ast \mathcal{F} \subseteq \text{ns}(\ast C(T, M)) \).

For each \( t \in T \) we let \( \mathcal{F}_t = \{ f(t) \mid f \in \mathcal{F} \} \). The Standard Definition Principle yields \( \ast \mathcal{F}_t = \{ F(\ast t) \mid F \in \ast \mathcal{F} \} \) for each \( t \in T \).

By Theorem 5.18 we have that \( (1') \) holds if and only if every \( F \in \ast \mathcal{F} \) is \( S \)-continuous and \( \ast \mathcal{F}_t \subseteq \text{ns}(\ast M) \) for all \( t \in T \). The first part of this condition is equivalent to \( \mathcal{F} \) being equicontinuous, by the previous Lemma. The second part is equivalent to \( \mathcal{F}_t \) being relatively compact in \( M \) for each \( t \in T \), using Proposition 5.14 in \( M \). Hence \( (1') \) is equivalent to (2). \( \square \)

We finish this subsection with another application of the nonstandard analysis characterization of compactness in spaces of continuous functions. In it we take \( T \) to be \([0, 1]\) and \( M \) to be \( \mathbb{R} \), and we write \( C[0, 1] \) for \( C([0, 1], \mathbb{R}) \).

Fix a continuous function \( K : [0, 1]^2 \to \mathbb{R} \). Consider the linear operator \( \Phi : C[0, 1] \to C[0, 1] \) defined by

\[
\Phi(f)(t) = \int_0^1 K(t, s)f(s)ds.
\]

5.22. **Theorem.** The operator \( \Phi \) on \( C(T, M) \) is compact; that is, the set

\[
R = \{ \Phi(f) \mid f \in C[0, 1] \text{ and } ||f(t)|| \leq 1 \}
\]

is relatively compact in \( C[0, 1] \).
Proof. Since $K$ is continuous on $[0, 1]^2$, which is compact, there exists $C > 0$ such that $|K(s, t)| \leq C$ for each $s, t \in [0, 1]$. Also, $K$ is uniformly continuous on $[0, 1]^2$, which yields that $^*K(a, b) \approx ^*K(a', b')$ in $^*R$ whenever $a \approx a'$ and $b \approx b'$ in $^*[0, 1]$.

Using Proposition 5.14 in the regular space $C[0, 1]$, it suffices to show $^*R \subseteq \text{ns}(^*C[0, 1])$. By Theorem 5.18 it therefore suffices to show that for each $t \in T$ the set $\{G(t) \mid G \in ^*R\}$ is contained in $\text{ns}(^*R) = \text{fin}(^*R)$ and that each $G \in ^*R$ is $S$-continuous.

Let $G \in ^*R$. There is a $^*$continuous $F : ^*[0, 1] \to ^*R$ satisfying $|F(u)| \leq 1$ for all $u \in ^*[0, 1]$, such that $G = ^*\Phi(F)$.

First we show that $G(t) \in ^*[-C, C] \subseteq \text{fin}(^*R)$ for all $t \in [0, 1]$. For $t \in [0, 1]$ we have

$$|G(t)| = \left| \int_0^1 ^*K(t, v)F(v)dv \right| \leq \int_0^1 |^*K(t, v)||F(v)||dv \leq \int_0^1 Cdv = C.$$ 

It remains to show that $G$ is $S$-continuous. Let $a \approx b$ in $^*[0, 1]$. Since $K$ is uniformly continuous,

$$\{^*K(a, v) - ^*K(b, v) : v \in ^*[0, 1]\} \subseteq \mu(0).$$

By underspill there exists a positive infinitesimal $\delta$, such that $|^*K(a, v) - ^*K(b, v)| \leq \delta$ for all $v \in ^*[0, 1]$. Then we have

$$|G(a) - G(b)| = \left| \int_0^1 ^*K(a, v)F(v)dv - \int_0^1 ^*K(b, v)F(v)dv \right|$$

$$= \left| \int_0^1 [^*K(a, v) - ^*K(b, v)] F(v)dv \right|$$

$$\leq \int_0^1 |^*K(a, v) - ^*K(b, v)||F(v)||dv$$

$$\leq \int_0^1 \delta dv = \delta \approx 0.$$ 

This finishes the proof. $\square$