Some exercises for sections 7 and 9

Also see all the exercises that are implicit in the lecture notes, especially the “Facts”.

7.1. Let $K$ be an algebraically closed field, considered as an $L_r$-structure; let $X$ be any subset of $K$ and let $k$ be the subfield of $K$ generated by $X$. Let $A$ denote the $L(X)$-structure $(K,a)_{a \in X}$.

- For any $a, b \in K$, show that $\text{tp}_A(a) = \text{tp}_A(b)$ iff either $a, b$ are both transcendental over $k$ or both $a, b$ are algebraic over $k$ and have the same minimal polynomial over $k$.

7.2. If $T$ is an $L$-theory, a model $A$ of $T$ is called existentially closed in $\text{Mod}(T)$ if it satisfies the following condition: whenever $A \subseteq B \models T$, $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a quantifier-free formula, and $a_1, \ldots, a_m \in A$, then $B \models \exists y_1 \ldots \exists y_n \varphi[a_1, \ldots, a_m]$ implies $A \models \exists y_1 \ldots \exists y_n \varphi[a_1, \ldots, a_m]$.

- Let $T$ be the theory of fields (in the language $L_r$). Show that a field $K$ is existentially closed in the class of all fields iff $K$ is algebraically closed.

9.1. Let $A$ be an infinite set, considered as a structure for the language of pure equality. For each $X \subseteq A$, show that $\text{acl}_A(X) = X$.

9.2. Let $A \models \text{DLO}$. For each $X \subseteq A$, show that $\text{acl}_A(X) = X$.

9.3. Let $K$ be a field and let $L$ be the language of vector spaces over $K$. (See Exercises 3.6 and 5.4.) For each infinite $K$-vector space $V$ (considered as an $L$-structure) and each $X \subseteq V$, show that $\text{acl}_V(X)$ is the $K$-linear subspace of $V$ spanned by $X$.

9.4. Consider the theory $T_{\text{dis}}$ of discrete linear orderings without endpoints. (See Example 5.6.) For $A \models T_{\text{dis}}$ and $X \subseteq A$, describe $\text{acl}_A(X)$. 