Some exercises for sections 10 and 11
Also see all the exercises that are implicit in the lecture notes, especially the “Facts”.

10.1. Let $L$ be the language of pure equality and let $T$ be the theory in $L$ of all infinite sets. From Example 3.16 we know that $T$ admits QE and is complete.
  • Show that $T$ is strongly minimal.
  • Explain the meaning of the dimension of a given model of $T$, in the sense of Section 10.

10.2. Let $L$ be the language whose nonlogical symbols consist of a unary function symbol $F$. Let $T$ be the theory in $L$ of the class of all $L$-structures $(A, f)$ in which $f$ is a bijection from $A$ onto itself and $f$ has no finite cycles. From Problem 2.2 we know that $T$ admits QE and is complete. Note that $(\mathbb{Z}, S)$ is a model of $T$, where $S(a) = a + 1$ for all $a \in \mathbb{Z}$; therefore $T = \text{Th}(\mathbb{Z}, S)$.
  • Show that $T$ is strongly minimal.
  • Explain the meaning of the dimension of a given model of $T$, in the sense of Section 10.

10.3. Let $K$ be a field and let $L$ be the language of vector spaces over $K$. Let $T$ be the theory in $L$ of all infinite vector spaces over $K$. (See Exercises 3.6, 5.4, and 9.3.)
  • Show that $T$ is strongly minimal.
It follows that Section 10 applies to infinite $K$-vector spaces. Exercise 9.3 shows that algebraic closure in the sense of model theory and linear span in the sense of linear algebra are identical, when applied to subsets of a fixed infinite vector space over $K$.
  • Let $V, W$ be infinite $K$-vector spaces and let $X \subseteq V, Y \subseteq W$ be $K$-linear subspaces. Suppose $F : X \to Y$ is a $K$-linear isomorphism. Show that $F$ is an elementary map in the sense of the $L$-structures $V, W$. (Note that if $K$ is a finite field, and $X, Y$ are finitely generated, then they are not models of $T$.)
  • If $V$ is an infinite $K$-vector space and $X \subseteq V$ is a $K$-linear subspace, show that the model theoretic dimension of $X$ in the sense of algebraic closure in $V$ does not depend on $V$. Show that this dimension is the same as the dimension of $X$ in the sense of linear algebra.
  • Check that Theorem 10.3 implies all of the standard facts about linearly independent sets, spanning sets, and bases, for arbitrary vector spaces over $K$. 
10.4. Let $T$ be a strongly minimal $L$-theory and let $\kappa$ be an infinite cardinal. Let $\mathcal{A}$ be an infinite model of $T$.

$\bullet$ Show that $\mathcal{A}$ is $\kappa$-saturated iff the dimension of $\mathcal{A}$ in the sense of Section 10 is $\geq \kappa$.

10.5. Let $L$ be the language whose only nonlogical symbol is a binary predicate symbol $<$. Let $\mathcal{A}$ be any infinite linear ordering, considered as an $L$-structure.

$\bullet$ Show that $\text{Th}(\mathcal{A})$ is not strongly minimal.

11.1. Let $K$ be a countable ordered field, considered as an $L_{or}$-structure, and let $T = \text{Th}(K)$. Show that there exists a 1-type $p \in S_1(T)$ that is not realized in $K$. Therefore, no countable ordered field is $\omega$-saturated.

11.2. Let $R$ be an ordered field. Let $x$ be a transcendental element over $R$ and consider the field $R(x)$ of rational functions in $x$ with coefficients in $R$.

$\bullet$ Show that there is linear ordering $<$ on $R(x)$ that makes $R(x)$ into an ordered field, such that $r < x$ for all $r \in R$.

$\bullet$ Show that this ordering is unique.

$\bullet$ Show how to embed the ordered field $R(x)$ with this ordering into a suitable ultrapower of $R$.

$\bullet$ Describe all the embeddings of the field $R(x)$ into an ultrapower of $R$. (Each one induces a field ordering on $R(x)$.)

11.3. Use the preceding Exercise and results in Section 11 to show that the theory $RCOF$ is not $\kappa$-categorical for any infinite cardinal $\kappa$. (For example, construct models of $RCOF$ of cardinality $\kappa$, such that one has an ordering of cofinality $\omega$ and the other has an ordering of uncountable cofinality.)