MODEL THEORY

MWF at 3:00 | | Ward Henson
443 Altgeld | Math 571 (G1) | Spring 2008

This course gives an introduction to the methods of model theory for first order logic. Model theory is the branch of mathematical logic that deals with mathematical structures and the formal languages they interpret. First order logic is the most important formal language and its model theory has many connections to the main body of mathematics.

The central object of study in model theory is the collection of *definable* sets on a structure $\mathcal{M}$. Let M be the underlying set of $\mathcal{M}$. A set $A \subseteq M^m$ is *definable* (without parameters) in $\mathcal{M}$ if there is a formula $\varphi(x_1, \ldots, x_m)$ in the first order language associated to $\mathcal{M}$ such that

$$A = \{(a_1, \ldots, a_m) \in M^m \mid \varphi \text{ is true of } (a_1, \ldots, a_m) \text{ in } \mathcal{M}\}.$$

A function is definable if its graph is a definable set. We will develop a number of tools for analyzing the definable sets and functions on a structure $\mathcal{M}$, including the method of quantifier elimination. A fundamental idea involves passing to elementary extensions of $\mathcal{M}$ (for example, to ultrapowers of $\mathcal{M}$). This enriches the underlying set M without changing the structure of the category of definable sets.

As a focus for developing "pure" model theory, we will prove Morley's Theorem: if T is a complete theory in a countable language and T is κ-categorical for some uncountable κ, then T is κ-categorical for *all* uncountable κ. In developing the tools needed to prove this theorem we will introduce *stability*, one of the key concepts of modern model theory.

This course will treat many applications and examples in order to show how model theory can be useful in mathematics. For example, we will treat the model theory of the field of real numbers (real closed fields) and will show how this can be used to obtain the solution to Hilbert's 17th Problem. (Theorem: a rational function over $\mathbb{R}$ is positive semi-definite iff it is a sum of squares.) Our treatment of real closed fields will allow us to show that the definable sets in the field $\mathbb{R}$ (equipped with names for its elements) are exactly the semi-algebraic sets.

Prerequisites: Familiarity with basic syntax and semantics of first order logic. (For example, the first half of Math 570 gives adequate preparation for Math 571.)

Texts and References: There is no required text; a set of lecture notes will be made available by the instructor. Recent books that can be used as references include *Model Theory: An Introduction* by David Marker and *A Course in Model Theory* by Bruno Poizat, both published by Springer-Verlag.

Required Work: Students will write solutions to problems assigned during the course. There will be six problem sets during the semester.