Exercises on Elementary Inequalities

Problem 1. Verify (for all real numbers \(x, y\)) that \(|x| \geq x\), that \(|xy| = |x||y|\), and that \(|x|^2 = x^2\).

Problem 2. Give a precise definition of the maximum of two real numbers \(x\) and \(y\) (we denote this by max\((x, y)\)). The definition should be similar to the definition of \(|x|\) in the textbook. Prove the following (for all real numbers \(x, y, z\)):

\[
\max(x + z, y + z) = \max(x, y) + z.
\]

Problem 3. Prove (for all real numbers \(x, y\)) that

\[
\max(x, y) = \frac{|x - y| + x + y}{2}.
\]

Problem 4. Give a precise definition of the minimum of two real numbers \(x\) and \(y\) (we denote this by min\((x, y)\)). Prove the following (for all real numbers \(x, y\)):

\[
\min(x, y) = -\max(-x, -y).
\]

Use this to prove results for min\((x, y)\) analogous to what is given for max\((x, y)\) in Problems 2 and 3.

Problem 5. Define \(f(x)\) for all nonzero real numbers \(x\) by

\[
f(x) = x + \frac{1}{x}.
\]

Determine the possible values of \(f(x)\). For which real numbers \(y\) does the equation \(y = f(x)\) have a unique solution \(x\)? For which \(y\) does \(y = f(x)\) have 3 or more solutions? When \(y\) is such that \(y = f(x)\) has exactly two solutions, say \(x = a\) and \(x = b\), how are \(a\) and \(b\) related?

Problem 6. Prove the inequality \(2xy \leq x^2y^2 + 1\) for all real numbers \(x, y\). Determine the precise condition for equality.

Problem 7. Find the largest number \(\lambda\) such that

\[
\lambda xy \leq x^2y^2 + x^2 + y^2 + 1
\]

holds for all real numbers \(x, y\).

Problem 8. Determine the set of real numbers \(c\) such that

\[
x^4 + cx^2 + 1 \geq 0
\]

holds for all real numbers \(x\).