Packing of Steiner trees and S-connectors in graphs

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June 15, 2010

Abstract

Nash-Williams and Tutte independently characterized when a graph has k edge-disjoint spanning trees; a consequence is that $2k$-edge-connected graphs have k edge-disjoint spanning trees. Kriesell conjectured a more general statement: defining a set $S \subseteq V(G)$ to be *j-edge-connected in G* if S lies in a single component of any graph obtained by deleting fewer than j edges from G, he conjectured that if S is $2k$-edge-connected in G, then G has k edge-disjoint trees containing S. Lap-Chi Lau proved that the conclusion holds whenever S is $24k$-edge-connected in G.

We improve Lau's result by showing that it suffices for S to be $6.5k$-edge-connected in G. This and an analogous result for packing stronger objects called "S-connectors" follow from a common generalization of the Tree Packing Theorem and Hakimi's criterion for orientations with specified outdegrees. We prove the general theorem using submodular functions and the Matroid Union Theorem.

1 Introduction

In 1961, Nash-Williams [7] and Tutte [9] independently obtained a necessary and sufficient condition for a graph to have k edge-disjoint spanning trees. A consequence is that every $2k$-edge-connected graph has k edge-disjoint spanning trees. Kriesell [4] conjectured a generalization of this Tree Packing Theorem that seeks edge-disjoint trees containing only a specified subset S of the vertices. Finding the most such trees for given S is the *Steiner-Tree Packing Problem*. Lap-Chi Lau [6] gave a partial result toward Kriesell's Conjecture. In this paper, we use a stronger concept called *S-connector* to improve Lau's result.

We use "graph" in the general sense, allowing loops and multi-edges. In a graph G, let S be a set of distinguished vertices called *terminals*. An *S-Steiner-tree* or simply *S-tree* in

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G is a tree T contained in G such that $S \subseteq V(T)$. An *S-path* is a path in G with both ends in S. *Short-cutting* a u, v-path means replacing its edges with one edge uv. An *S-connector* in G is the union of a family of edge-disjoint S-paths such that short-cutting them yields a connected graph with vertex set S.

Given a graph G, a vertex set S is *connected in G* if S lies in a single component of G. A set S is *k-edge-connected in G* if S remains connected in every graph obtained by deleting fewer than k edges from G.

Conjecture 1.1 (Kriesell's Conjecture [4]). *If S is $2k$-edge-connected in G, then G contains k edge-disjoint S-trees.*

Known partial results toward Kriesell's Conjecture include the following.

Theorem 1.2 (Kriesell [4]). *If S is $2k$-edge-connected in G, and every vertex outside S has even degree, then G contains k edge-disjoint S-trees.*

Theorem 1.3 (Frank–Király–Kriesell [2]). *If S is $3k$-edge-connected in G, and $G - S$ has no edges, then G contains k edge-disjoint S-trees.*

Theorem 1.4 (Lau [6]). *If S is $24k$-edge-connected in G, then G has k edge-disjoint S-trees.*

We obtain the following improvements.

Theorem 1.5. *If S is $6.5k$-edge-connected in G, then G contains k edge-disjoint S-trees.*

Theorem 1.6. *If S is $10k$-edge-connected in G, then G contains k edge-disjoint S-connectors.*

An S-tree need not be an S-connector. For example, when $|S| \geq 3$, a star whose leaf set is S is an S-tree but not an S-connector. Thus stricter conditions may be needed to guarantee S-connectors. We pose an analogue for S-connectors of Kriesell's Conjecture.

Conjecture 1.7. *If S is $3k$-edge-connected in G, then G contains k edge-disjoint S-connectors.*

Our results will follow from a theorem that generalizes the Tree Packing Theorem of Nash-Williams and Tutte. Stating it requires some terminology and notation. For $S \subseteq V(G)$, write $\overline{S}$ for $V(G) - S$; following Lovász, write $\delta(S)$ for the number of edges having endpoints in S and $\overline{S}$. A partition $A_1, \ldots, A_l$ of a set containing S in $V(G)$ is an *S-partition* if each A_i intersects S. When P is an S-partition with blocks $A_1, \ldots, A_l$, let $B_P = V(G) - \bigcup_{i=1}^{l} A_i$. When discussing several S-partitions, let l_P denote the number of blocks in the S-partition P. Let $\mathcal{P}(S)$ be the set of all S-partitions of G.

Let $\mathbb{N}_0$ be the set of nonnegative integers. Given a graph G, an *S-parity function* is a function $g\colon V(G) \to \mathbb{N}_0$ such that $g(v) \equiv d_G(v) \pmod 2$ for all $v \in \overline{S}$ (there is no restriction on $g(v)$ for $v \in S$). For any vertex set A and function h, let $h(A) = \sum_{v\in A} h(v)$. Given an S-parity function g, let T_P be the set of vertices v in S such that $g(v) \neq 0$ and v is the only vertex of S in the block of P containing v.

In a graph G with terminal set S, a *(k, g)-family* is a set of $k + g(V(G))$ edge-disjoint subgraphs in which k are S-connectors and the others are paths of positive length that can be oriented so that all end in S and $g(v)$ of them start from v, for each vertex v in G. Our main result gives a necessary and sufficient condition for existence of a (k, g)-family in G.

Theorem 1.8. *Let S be a set of terminals in G. If g is an S-parity function of G, then G has a (k, g)-family if and only if $f_g(P) \geq 0$ for all $P \in \mathcal{P}(S)$, where f_g is defined by*

$$f_g(P) = \sum_{A_i\in P} \delta(A_i) - 2k(l-1) - g(B_P) - 2g(T_P). \tag{1}$$

We call the condition that $f_g(P) \geq 0$ for all $P \in \mathcal{P}(S)$ the *Strong Partition Condition (SPC)*. The notion of S-parity function enables us to generalize the problem of packing S-connectors in a way (existence of (k, g)-families) that permits a characterization of existence and facilitates the proof of our results about packing of S-trees and S-connectors.

Necessity of the Strong Partition Condition follows easily. A (k, g)-family consists of k S-connectors and $g(V(G))$ oriented paths, pairwise edge-disjoint. For any S-partition P, where $P = \{A_1, \ldots, A_l\}$, let $t = \sum_{A_i\in P} \delta(A_i)$. Each S-connector contributes at least $2(l_P - 1)$ to t. For each vertex v in B_P, the paths starting from v contribute at least $g(v)$ to t. For a vertex $v \in S$ that is the only vertex of S in its block, the oriented paths starting from v contribute at least $2g(v)$ to t. Thus $t \geq 2k(l_P - 1) + g(B_P) + 2g(T_P)$, so $f_g(P) \geq 0$. We will prove the converse: the Strong Partition Condition suffices for the existence of a (k, g)-family.

A notable application of Theorem 1.8 follows from an appropriate choice of g. Given a vertex set $A \in V(G)$, let $n_o(A)$ be the number of vertices of A having odd degree in G.

Theorem 1.9. *Let S be a set of terminals in a graph G. If each $P \in \mathcal{P}(S)$ satisfies $\sum_{A_i\in P} \delta(A_i) - 2k(l_P - 1) - n_o(B_P) \geq 0$, then G contains k edge-disjoint S-connectors.*

Proof. Define an S-parity function by $g(v) = 1$ when v is a vertex of $\overline{S}$ having odd degree in G and otherwise $g(v) = 0$. For $P \in \mathcal{P}(S)$, we have $B_P \subseteq \overline{S}$, and hence $g(B_P) = n_o(B_P)$. Also, $g(T_P) = 0$ for any S-partition P. Hence the left side of the assumed equality is $f_g(P)$. In other words, we have assumed the SPC to hold for this S-parity function. By Theorem 1.8, G has a (k, g)-family, and hence there are k edge-disjoint S-connectors. □

The condition in Theorem 1.9 is sufficient but not necessary, as seen by adding to such G a large component in which every vertex has odd degree. The case of Theorem 1.9 when no vertex of $\overline{S}$ has odd degree implies Theorem 1.2 in the same way that the Tree Packing Theorem implies that $2k$-edge-connected graphs have k edge-disjoint spanning trees; indeed, we obtain S-connecters instead of just S-trees, thereby strengthening Theorem 1.2. We mention two other special cases.

Theorem 1.10 (Nash-Williams [7], Tutte [9])**.** *A graph G contains k edge-disjoint spanning trees if and only if $\sum_{i=1}^{l}\delta(A_i) \geq 2k(l-1)$ for every partition $A_1, \ldots, A_l$ of $V(G)$.*

Proof. Set $S = V(G)$, and make g is identically 0. The S-partitions are the partitions of $V(G)$, and the terms in the SPC involving g are always 0. Hence the hypothesis here is precisely the SPC for this S and g, and the resulting S-connectors are the spanning trees. □

Theorem 1.11 (Hakimi [3])**.** *Given a graph G and a function $g\colon V(G) \to \mathbb{N}_0$, there is an orientation D of G such that $d_D^+(v) \geq g(v)$ for all $v \in V(G)$ if and only if for all $T \subseteq V(G)$ there are at least $g(T)$ edges incident to T.*

Proof. Set $S = V(G)$ and $k = 0$. Since $S = V(G)$, always $B_P = \varnothing$ for an S-partition P. Hence the only requirement imposed on $\sum_{i=1}^{l}\delta(A_i)$ is from the singleton blocks; the sum must be at least $2g(T_P)$. Indeed, the sum counts the edges leaving singleton blocks twice, and it counts nothing else if the remainder of $V(G)$ is in one block.

Hence Hakimi's condition implies the SPC, and by Theorem 1.8 a $(0, g)$-family exists. Since $S = V(G)$, the paths can be taken to be single edges, and orienting the $g(v)$ edges chosen for v outward from v yields the desired orientation of G (orient the non-chosen edges arbitrarily). □

The proof of Theorem 1.8 has many ingredients, including submodular functions on lattices and the Matroid Union Theorem. Proving Theorem 1.5 from Theorem 1.8 (Section 4) uses still more, including Mader's Splitting Lemma and Theorem 1.11, which we have just seen is a special case of Theorem 1.8.

2 S-partitions and submodularity of f_g

We begin by defining a partial order on $\mathcal{P}(S)$. For any S-parity function g, we will prove that f_g is submodular for special pairs in this poset (the poset is a lattice). In the next section, we combine the submodularity of f_g with the Matroid Union Theorem to prove the characterization of (k, g)-families stated in Theorem 1.8.

If $x \leq y$ in a poset $\mathcal{P}$, then x is a *lower bound* for y and y is an *upper bound* for x. If z is an upper bound for both x and y, and also $z \leq w$ for every common upper bound w of $\{x, y\}$, then z is the *least upper bound* of $\{x, y\}$. When such an element z exists, it is the *join* of x and y, written $x \vee y$. Similarly, the *meet* $x \wedge y$, if it exists, is the *greatest lower bound* of x and y. A *lattice* is a poset in which meets and joins exist for all pairs of elements; a finite lattice has a unique maximal element and a unique minimal element.

The partition lattice Π_G on $V(G)$ is the poset of all partitions of $V(G)$, ordered by refinement. That is, when Q and Q' are partitions of $V(G)$, we have $Q \leq Q'$ if for every block $A_i \in Q$, there is a block $A'_j \in Q'$ such that $A_i \subseteq A'_j$.

To define the order relation on $\mathcal{P}(S)$, we map an S-partition P to a partition Q_P of $V(G)$ by defining $Q_P = \{A_1, \ldots, A_l, \{b_1\}, \ldots, \{b_{|B_P|}\}\}$, where $P = \{A_1, \ldots, A_l\}$ and $B_P = \{b_1, \ldots, b_{|B_P|}\}$. This mapping is injective; it simply splits B_P into singleton sets. Let $\mathcal{Q}(S)$ be the range of the mapping. The order relation on $\mathcal{P}(S)$ puts $P \leq P'$ if and only if $Q_P \leq Q'_P$ in $\mathcal{Q}(\mathcal{S})$. This makes $\mathcal{P}(S)$ isomorphic to the subposet $\mathcal{Q}(S)$ of Π_G.

To show that meet and join are well defined on $\mathcal{P}(S)$, we must show that $\mathcal{Q}(S)$ is closed under meet and join as a subposet of Π_G. Let $\wedge_\Pi$ and $\vee_\Pi$ denote the meet and join operations in Π_G, and let l_Q be the number of blocks of a partition Q. We use the following well-known properties of the partition lattice; statement (2) is the *submodularity property* for the rank function of Π_G, since the rank of a partition Q in $\Pi(G)$ is $|V(G)| - l_Q$.

Proposition 2.1. *For partitions Q and Q' of $V(G)$,*
(1) $Q \wedge_\Pi Q' = \{A_i \cap A_j \colon\ A_i \in Q, A_j \in Q'\}$;
(2) $l_{Q \wedge_\Pi Q'} + l_{Q \vee_\Pi Q'} \geq l_Q + l_{Q'}$.

Let the symbols $\wedge$ and $\vee$ without subscripts denote the meet and join in $\mathcal{P}(S)$.

Proposition 2.2. *For $P, P' \in \mathcal{P}(S)$, the meet and join of P and P' are well defined and*
(1) $P \wedge P' = \{A_i \cap A'_j \colon\ A_i \in P, A'_j \in P', A_i \cap A'_j \cap S \neq \varnothing\}$;
(2) $Q_{P \vee P'} = Q_P \vee_\Pi Q_{P'}$;
(3) $B_{P \vee P'} = B_P \cap B_{P'}$.

Proof. (1) Let $\hat{P} = \{A_i \cap A'_j \colon\ A_i \in P, A'_j \in P', A_i \cap A'_j \cap S \neq \varnothing\}$. By definition, $\hat{P} \in \mathcal{P}(S)$ and $\hat{P} \leq P, P'$. Let P'' be any common lower bound of P and P'. For $A''_t \in P''$, there exist $A_i \in P$ and $A'_j \in P'$ such that $A''_t \subseteq A_i \cap A'_j$. Since $A'' \cap S \neq \varnothing$, we have $A_i \cap A'_j \in \hat{P}$. Hence $P'' \leq \hat{P}$, and we conclude that $\hat{P} = P \wedge P'$.

(2) Let $Q'' = Q_P \vee_\Pi Q_{P'}$. We first prove that $Q'' \in \mathcal{Q}(S)$. If not, then there exists $A_i \in Q''$ such that $A_i \cap S = \varnothing$ and $|A_i| \geq 2$. For $a \in A_i$, the block containing a in Q_P is

contained in A_i. Since $A_i \cap S = \varnothing$ and P is an S-partition, this block in Q_P is $\{a\}$. Similarly, $\{a\} \in Q_{P'}$. Now $\{a\}$ is a block in $Q_P \vee_\Pi Q_{P'}$, contradicting the hypothesis.

Since $Q'' \in \mathcal{Q}(S)$, also Q'' is the least upper bound in $\mathcal{Q}(S)$ for Q_P and $Q_{P'}$. Since $\mathcal{P}(S)$ and $\mathcal{Q}(S)$ are isomorphic, also $P \vee P'$ exists, and its image is $Q_P \vee_\Pi Q_{P'}$.

(3) This follows immediately from (2). $\square$

Two S-partitions $\{A_1, \ldots, A_l\}$ and $\{A'_1, \ldots, A'_l\}$ form a *good pair* if $A_i \cap A'_j \neq \varnothing$ implies $A_i \cap A'_j \cap S \neq \varnothing$. If S-partitions P and P' form a good pair, then $Q_{P \wedge P'} = Q_P \wedge_\Pi Q_{P'}$.

Proposition 2.3. *If P and P' form a good pair, then:*
(1) $B_{P \wedge P'} = B_P \cup B_{P'}$;
(2) $l_{P \wedge P'} + l_{P \vee P'} \geq l_P + l_{P'}$.

Proof. (1) Since they form a good pair, the expression for their meet simplifies to $P \wedge P' = \{A_i \cap A'_j \colon\ A_i \in P, A'_j \in P', A_i \cap A'_j \neq \varnothing\}$. An element lies in $B_{P \wedge P'}$ if and only if it is not in any $A_i \cap A'_j$, which puts it in B_P or $B_{P'}$.

(2) Since $l_P = l_{Q_P} - |B_P|$ for any S-partition P, and

$$|B_P| + |B_{P'}| = |B_P \cap B_{P'}| + |B_P \cup B_{P'}| = |B_{P \wedge P'}| + |B_{P \vee P'}|,$$

the claim follows from $l_{Q_P \wedge_\Pi Q_{P'}} + l_{Q_P \vee_\Pi Q_{P'}} \geq l_{Q_P} + l_{Q'_P}$ (Proposition 2.3(2)). $\square$

For two sets $A, B \subseteq V(G)$, write $[A, B]$ for the set of edges with endpoints in both A and B, and let $G[A]$ denote the subgraph induced by A. Given an S-partition P with blocks $A_1, \ldots, A_l$, group the edges of G into four classes:

Class 1: $e \in [A_i, A_j]$ for some i and j; Class 3: $e \in E(G[A_i])$ for some i;
Class 2: $e \in [A_i, B_P]$ for some i; Class 4: $e \in E(G[B_p])$.

Let

$$h_P(e) = \begin{cases} 2, & \text{if } e \text{ is Class 1;} \\ 1, & \text{if } e \text{ is Class 2;} \\ 0, & \text{if } e \text{ is Class 3 or Class 4.} \end{cases}$$

For any S-partition P, we have $\sum_{i=1}^{l} \delta(A_i) = \sum_{e \in E(G)} h_P(e)$.

Proposition 2.4. *If $S \subseteq V(G)$ and P and P' form a good pair in $\mathcal{P}(S)$, then*

$$h_{P \wedge P'}(e) + h_{P \vee P'}(e) \leq h_P(e) + h_{P'}(e)$$

for all e in $E(G)$. Also, if the endpoints of e lie in different blocks in both P and P', but in the same block in $P \vee P'$, then the two sides of the inequality differ by 2.

Proof. For $uv \in E(G)$, let $W = \{u, v\}$. Note that $h_P(uv) = 2 - |W \cap B_P| - 2t_P(uv)$, where $t_P(uv) = 1$ if W is contained in a single block in P, and otherwise $t_P(uv) = 0$. Since $B_{P\wedge P'} = B_P \cup B_{P'}$ and $B_{P\vee P'} = B_P \cap B_{P'}$, we have $|W \cap B_P| + |W \cap B_{P'}| = |W \cap B_{P\vee P'}| + |W \cap B_{P\wedge P'}|$. Therefore $h_{P\wedge P'}(uv) + h_{P\vee P'}(uv) \le h_P(uv) + h_{P'}(uv)$ if and only if $t_{P\wedge P'}(uv) + t_{P\vee P'}(uv) \ge t_P(uv) + t_{P'}(uv)$. This holds when P and P' form a good pair, since $\max\{t_P(uv), t_{P'}(uv) = 1$ implies $t_{P\vee P'}(uv) = 1$.

If u and v lie in different blocks in P and P' but in the same block in $P \vee P'$, then $t_{P\wedge P'}(uv) + t_{P\vee P'}(uv) = t_P(uv) + t_{P'}(uv) + 1$, and the difference between the two sides is 2. □

Lemma 2.5 (Submodularity Lemma)**.** *For $S \subseteq V(G)$, let g be a S-parity function. If P and P' form a good pair in $\mathcal{P}(S)$, then*

$$f_g(P \wedge P') + f_g(P \vee P') \le f_g(P) + f_g(P'). \tag{2}$$

Proof. From the definition of f_g and the observation that $\sum_{A_i \in P} \delta(A_i) = \sum_{e \in E(G)} h_P(e)$ when P is an S-partition, we have

$$f_g(P) = \sum_{e \in E(G)} h_P(e) - 2k(l_P - 1) - g(B_P) - 2g(T_P). \tag{3}$$

We consider the contributions of the various terms in turn. Proposition 2.4 yields

$$\sum_{e \in E(G)} [h_{P\wedge P'}(e) + h_{P\vee P'}(e)] \le \sum_{e \in E(G)} [h_P(e) + h_{P'}(e)].$$

By Proposition 2.3(2),

$$2k(l_{P\wedge P'} - 1) + 2k(l_{P\vee P'} - 1) \ge 2k(l_P - 1) + 2k(l_{P'} - 1).$$

Since $B_{P\wedge P'} = B_P \cup B_{P'}$ and $B_{P\vee P'} = B_P \cap B_{P'}$,

$$g(B_{P\wedge P'}) + g(B_{P\vee P'}) = g(B_P) + g(B_{P'}).$$

For the last term, recall the definition: $T_P = \{v \in S \colon |C_P(v) \cap S| = 1\}$, where $C_P(v)$ is the block containing v in P. If $v \in T_P \cup T_{P'}$, then $v \in T_{P\wedge P'}$; if $v \in T_P \cap T_{P'}$, then since P and P' form a good pair, $v \in T_{P\vee P'}$. Summing the contributions made by each vertex yields

$$g(T_{P\wedge P'}) + g(T_{P\vee P'}) \ge g(T_P) + g(T_{P'}).$$

Summing the inequalities for all four terms completes the proof of (2). □

In special circumstances, we will need a stronger inequality than the Submodularity Inequality, ensuring a difference of 4.

Lemma 2.6. *Let P and P' be S-partitions that form a good pair. Let uv be an edge such that u and v lie in different blocks in both P and P' but in the same block in $P \vee P'$. If $N_{G-uv}(v)$ intersects both $C_P(u)$ and $C_{P'}(u)$, then $f_g(P) + f_g(P') - f_g(P \wedge P') - f_g(P \vee P') \geq 4$.*

Proof. The expression for f_g in (3) has terms using h, l, and g. We showed above that the contribution to $f_g(P) + f_g(P') - f_g(P \wedge P') - f_g(P \vee P')$ from terms using g is nonnegative. Hence it suffices to gain 4 from those using h and l.

For each edge e, let $\hat{h}(e) = h_P(e) + h_{P'}(e) - h_{P\wedge P'}(e) - h_{P \vee P'}(e)$. Proposition 2.4 implies that always $\hat{h}(e) \geq 0$ and that the locations of u and v yield $\hat{h}(uv) \geq 2$. It suffices to find another edge e with $\hat{h}(e) \geq 2$ or gain 2 from the term involving l.

After deleting (one copy of) the edge vu, still v has a neighbor in each of $C_P(u)$ and $C_{P'}(u)$. Suppose that v still has a neighbor w in $C_P(u) - C_{P'}(v)$ or $C_{P'}(u) - C_P(v)$ (possibly $w = u$). In either case, w and v lie in different blocks in both P and P', and w and u lie in the same block of $P \vee P'$. By hypothesis, this block of $P \vee P'$ also contains v, so Proposition 2.4 applies to yield $\hat{h}(wv) \geq 2$, which suffices.

Therefore, we may assume that the given neighbors of v lie in $C_P(u) \cap C_{P'}(v)$ and $C_{P'}(u) \cap C_P(v)$. Since u and v lie in distinct blocks in both P and P', these neighbors of v are distinct (and different from u); let them be $w \in C_P(u) \cap C_{P'}(v)$ and $w' \in C_{P'}(u) \cap C_P(v)$.

Obtain P'' from P by splitting $C_P(v)$ into $C_P(v) - C_{P'}(u)$ and $C_P(v) \cap C_{P'}(u)$. Since P and P' form a good pair, P'' is an S-partition. Since all intersections of blocks in P'' and P' are intersections of blocks in P and P', also P'' and P' form a good pair, and $P'' \wedge P' = P \wedge P'$. Furthermore, $P'' \vee P' = P \vee P'$, since $C_{P'}(v)$, $C_P(u)$, and $C_{P'}(u)$ successively put the pairs $\{v, w\}$, $\{w, u\}$, and $\{u, w'\}$ into the same block of $P'' \vee P'$ (using $C_{P''}(u) = C_P(u)$).

Now, since $l_{P''\wedge P'} + l_{P''\vee P'} - l_{P''} - l_{P'} \geq 0$ (by Proposition 2.3(2)) and $l_{P''} = l_P + 1$, we obtain $l_{P\wedge P'} + l_{P\vee P'} - l_P - l_{P'} \geq 1$. Since it has the coefficient $2k$, the term using l now provides the additional contribution of 2 needed to complete the proof. $\square$

Proposition 2.7. *If P is an S-partition and g is an S-parity function, then $f_g(P)$ is even.*

Proof. For $A \subseteq V(G)$, recall that $n_o(A)$ is the number of vertices of A having odd degree in G. Using $B_P \subseteq \overline{S}$ and the definition of S-parity function,

$$\begin{aligned} f_g(P) &= \left(\textstyle\sum_{A_i \in P} \delta(A_i)\right) - 2k(l-1) - g(B_P) - 2g(T_P) \\ &\equiv \left[\textstyle\sum_{i=1}^{l} \left(\sum_{v \in A_i} d_G(v)\right) - 2|E(G[A_i])|\right] + n_o(B_P) \\ &\equiv \left[\textstyle\sum_{i=1}^{l} n_o(A_i)\right] + n_o(B_P) \;\equiv\; n_o(V(G)) \;\equiv\; 0 \pmod 2. \end{aligned}$$

$\square$

For $X \subseteq \overline{S}$ and $P = (A_1, \ldots, A_l)$, let $P - X = (A_1 - X, \ldots, A_l - X)$. Note that if P is an S-partition, then so is $P - X$.

Proposition 2.8. *If P is an S-partition and $X \subseteq A_i \cap \overline{S}$, where A_i is a block of P, then*

$$f_g(P) - f_g(P - X) \geq \big|[X, \overline{A_i}]\big| - \big|[X, A_i - X]\big|.$$

Proof. Since $f_g(P) = \sum_{i=1}^{l} \delta(A_i) - 2k(l_P - 1) - g(B_P) - 2g(T_P)$, we have

$$\begin{aligned} f_g(P) - f_g(P - X) &= \delta(A_i) - \delta(A_i - X) + g(X) \\ &\geq \delta(A_i) - \delta(A_i - X) \quad = \quad \big|[X, \overline{A_i}]\big| - \big|[X, A_i - X]\big| \end{aligned}$$ □

3 Existence of (k, g)-families

Let uv and vw be two edges of G. The *uv, vw-shortcut* of G is obtained by replacing uv and vw with uw. We add an extra copy of uw if u is already adjacent (or equal) to w. Fix an edge uv with $u \in S$. Let $N_G(x) = \{y \in V(G) \colon xy \in E(G)\}$. For $w \in N_G(v) - \{u\}$, let G_w denote the uv, vw-shortcut of G. For $x \in V(G)$ and an S-partition P of G, where $P = \{A_1, \ldots, A_l\}$, let $C_P(x)$ be the set in the list $A_1, \ldots, A_l, B_P$ that contains x.

In order to give an inductive proof of Theorem 1.8, we will prove that if uv is an edge in S with $u \in S$ and $v \notin S$, and G satisfies the Strong Partition Condition (SPC) for an S-parity function g such that $d_G(v) > g(v)$, then there exists $w \in N_G(v) - \{u\}$ such that G_w also satisfies the SPC. This is the main technical result of our paper.

If G satisfies the SPC but G_w does not, then there exists an S-partition P such that $f_g(P) \leq -2$ in G_w and $f_g(P) \geq 0$ in G, since $f_g(P)$ is always even (Proposition 2.7). The contributions to $f_g(P)$ for G and G_w can differ only in $\sum_{A_i \in P} \delta(A_i)$, which can decrease in moving from G to G_w only when $v \notin C_P(u)$ and $w \notin C_P(v)$. Under those conditions, the shortcut decreases $f_g(P)$ by 2 if $v \in B_P$ and $w \in C_P(u)$, and it decreases $f_g(P)$ by 2 or 4 if $v \notin B_P$ and $w \notin C_P(v)$, depending on whether or not $w \notin C_P(u)$ holds.

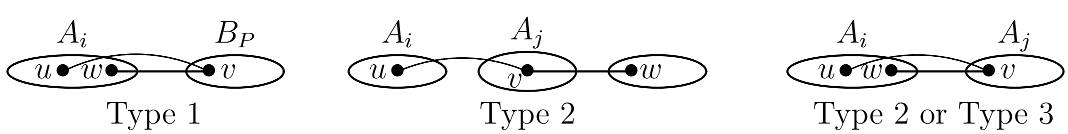

Figure 1: Dangerous locations for w

Whether this decrease is dangerous (making $f_g(P)$ negative for G_w) depends also on the value of $f_g(P)$ for G. In each of the three cases ("Types") listed below, the *dangerous set*

$D(P)$ is the location for w making $f_g(P)$ negative for G_w (grouping by the value of $f_g(P)$ causes the third sketch in Figure 1 to occur in two types). If v has a neighbor w outside the dangerous set, then G_w satisfies the SPC. To facilitate specification of $D(P)$, let $N'(v)$ denote the neighborhood of v after deleting the edge uv; thus $N'(v) = N_G(v) - \{u\}$ if uv has multiplicity 1, and otherwise $N'(v) = N_G(v)$. Let $D(P) = \varnothing$ when P is an S-partition such that no choice of w relative to uv can make $f_g(P)$ negative for G_w.

Type	$f_g(P)$	decrease	location of u, v	dangerous set $D(P)$
1	0	2	$v \in B_P$	$N'(v) \cap C_P(u)$
2	0	2 or 4	$v \notin B_P, u \notin C_P(v)$	$N'(v) - C_P(v)$
3	2	4	$v \notin B_P, u \notin C_P(v)$	$N'(v) \cap C_P(u)$

Fix u and v, and suppose the dangerous set $D(P)$ is nonempty for some S-partition P. Let $\mathcal{M}$ be the set of maximal S-partitions among those with maximal dangerous sets. That is, $P \in \mathcal{M}$ when there is no S-partition P' such that $D(P) \subsetneq D(P')$ or such that $D(P) = D(P')$ and $P < P'$. We will use the next lemma to obtain an S-partition whose dangerous set contains the dangerous sets for all S-partitions (relative to this fixed u and v).

Lemma 3.1. *If $P, P' \in \mathcal{M}$, then P and P' form a good pair.*

Proof. If P and P' do not form a good pair, then $\varnothing \neq A_i \cap A'_j \subseteq \overline{S}$ for some $A_i \in P$ and $A'_j \in P'$. Let $X = A_i \cap A'_j$; we have remarked that $P - X \in \mathcal{P}(S)$. We claim that $D(P) \subseteq D(P - X)$ or (symmetrically) $D(P') \subseteq D(P' - X)$, which contradicts $P, P' \in \mathcal{M}$.

If P is Type 3 and $P - X$ is Type 2, then $D(P) = N'(v) \cap C_P(u) \subseteq N'(v) - C_P(v) \subseteq N'(v) - C_{P-X}(v) = D(P - X)$, so the claim holds. If each of P and $P - X$ is Type 1 or 3, and $u \notin A_i$, then $X \subseteq B_{P-X}$ implies $D(P) = N'(v) \cap C_P(u) = N'(v) \cap C_{P-X}(u) = D(P - X)$; again the claim holds. We reduce most cases to these two.

By symmetry, we may assume $\big|[X, A_i - X]\big| \leq \big|[X, A'_j - X]\big|$. If $\big|[X, A_i - X]\big| < \delta(X)/2$, then $f_g(P) - f_g(P - X) \geq \big|[X, \overline{A_i}]\big| - \big|[X, A_i - X]\big| = \delta(X) - 2\big|[X, A_i - X)]\big| > 0$, by Proposition 2.8. Since G satisfies the SPC, $f_g(P - X) \geq 0$. Since $D(P) \neq \varnothing$, we have $f_g(P) \leq 2$. Therefore $f_g(P - X) = 0 = f_g(P) - 2$ and P is Type 3. If $v \notin X$, then $v \notin B_{P-X}$ and $P - X$ is Type 2. If $v \in X$, then $P - X$ is Type 1 (since $X \subseteq B_{P-X}$) and $u \notin A_i$ (since $v \notin C_P(u)$). We have proved $D(P) \subseteq D(P - X)$ in these cases.

Since $\big|[X, A_i - X]\big| + \big|[X, A'_j - X]\big| \leq \delta(X)$, we may henceforth assume $\big|[X, A_i - X]\big| = \big|[X, A'_j - X]\big| = \delta(X)/2$. Suppose first that $v \in X$. No matter what types P and P' are, $v \notin C_P(u) \cup C_{P'}(u)$, and hence $u \notin A_i \cup A'_j$. Since all edges leaving X go to $A_i - X$ or $A'_j - X$, we have $[X, \{u\}] = \varnothing$. This contradicts the existence of uv.

We may therefore assume $v \notin X$. Since $X \subseteq \overline{S}$, also $u \notin X$. Hence $u \notin A_i$ or $u \notin A'_j$, since $X = A_i \cap A'_j$. By symmetry, we may assume $u \notin A_i$, so $X \not\subseteq C_P(u)$. Proposition 2.8 and the hypothesis of Case 2 yield $f_g(P) - f_g(P - X) \geq \left|[X, \overline{A_i}]\right| - \left|[X, A_i - X]\right| = 0$.

If P is Type 3, then $f_g(P) = 2$ and $f_g(P - X) \in \{0, 2\}$. Since $v \notin X$ implies $v \notin B_{P-X}$, we conclude that $P - X$ is Type 2 if $f_g(P - X) = 0$ and Type 3 if $f_g(P - X) = 2$. If P is Type 1, then $f_g(P) = f_g(P - X) = 0$ and $v \in B_P \subseteq B_{P-X}$, so also $P - X$ is Type 1. Since $u \notin A_i$, we have proved $D(P) \subseteq D(P - X)$ in all these cases.

If P is Type 2, then $v \notin X$ implies that $P - X$ is Type 2, so $D(P) = N'(v) - C_P(v) \subseteq N'(v) - C_{P-X}(v) = D(P - X)$. □

We now obtain a single S-partition whose dangerous set contains all dangerous sets.

Lemma 3.2. *There exists an S-partition $\hat{P}$ such that $D(\hat{P}) = \bigcup_{P \in \mathcal{P}(S)} D(P)$.*

Proof. It suffices to show that the dangerous sets for all S-partitions in $\mathcal{M}$ are the same. Consider $P, P' \in \mathcal{M}$ with $D(P) \neq D(P')$. By Lemma 3.1, P and P' form a good pair. No matter what types P and P' are, we show in each case that $P \wedge P'$ is Type 2. This yields $D(P) \subseteq N'(v) - C_P(v) \subseteq N'(v) - C_{P \wedge P'}(v) = D(P \wedge P')$, which contradicts $P \in \mathcal{M}$.

Case 1: P and P' are Type 2. Here $f_g(P \wedge P') + f_g(P \vee P') \leq f_g(P) + f_g(P') = 0$, so $f_g(P \wedge P') = 0$. Also $v \notin B_P \cup B_{P'} = B_{P \wedge P'}$, so $P \wedge P'$ is Type 2.

Case 2: P or P' (not both) is Type 2. By symmetry, let P be Type 2. Hence $D(P) = N'(v) - C_P(v)$ and $D(P') = N'(v) \cap C_{P'}(u)$. If $C_P(v) \cap C_{P'}(u) = \varnothing$, then $D(P') \subseteq D(P)$, which contradicts $P' \in \mathcal{M}$. Thus $C_P(v) \cap C_{P'}(u) \neq \varnothing$, which puts u and v in the same block of $P \vee P'$. Applying Proposition 2.4 to the edge uv yields $f_g(P) + f_g(P') - f_g(P \wedge P') - f_g(P \vee P') \geq 2$. Since $f_g(P) = 0$, we have $f_g(P') = 2$ and $f_g(P \wedge P') = 0$. Now P' is Type 3, so $v \notin B_{P'}$. We conclude that $v \notin B_P \cup B_{P'} = B_{P \wedge P'}$, so $P \wedge P'$ is Type 2.

Case 3: neither P nor P' is Type 2. Since $D(P) \neq \varnothing$, we have $\left|[v, C_P(u)]\right| \geq 2$, and similarly for P'. If u and v lie in the same block of $P \vee P'$, then Lemma 2.6 and the SPC yield $f_g(P) + f_g(P') \geq f_g(P \wedge P') + f_g(P \vee P') + 4 \geq 4$. Since $D(P), D(P') \neq \varnothing$ requires $f(P), f(P') \leq 2$, we have $f_g(P) = f_g(P') = 2$, and P and P' are both Type 3. Therefore $v \notin B_P \cup B_{P'} = B_{P \wedge P'}$ and $f_g(P \wedge P') = f_g(P \vee P') = 0$, so $P \wedge P'$ is Type 2.

If u and v do not lie in the same block of $P \vee P'$, then $D(P \vee P') \supseteq N'(v) \cap C_{P \vee P'}(u) \supseteq (N'(v) \cap C_P(u)) \cup (N'(v) \cap C_P(u)) = D(P) \cup D(P')$. If $f_g(P \vee P') \leq 2$, then $P \vee P' \in \mathcal{M}$; with $D(P) \neq D(P')$, this contradicts $P, P' \in \mathcal{M}$. Therefore, $f_g(P \vee P') \geq 4$. Submodularity yields $f_g(P) + f_g(P') \geq 4$, so P and P' are Type 3. As in the preceding paragraph, we conclude that $P \wedge P'$ is Type 2. □

We now prove an analogue of Mader's Splitting Lemma (Lemma 4.3). Recall that $N'(v) = N_G(v) - \{u\}$ if uv has multiplicity 1, and otherwise $N'(v) = N_G(v)$.

Theorem 3.3. *If G satisfies the Strong Partition Condition and has an edge uv with $u \in S$, $v \notin S$, and $d_G(v) > g(v)$, then there is a vertex $w \in N'(v)$ such that G_w satisfies the SPC.*

Proof. By Lemma 3.2, there exists an S-partition $\hat{P}$ such that $D(\hat{P}) = \bigcup_{P \in \mathcal{P}(S)} D(P)$. If there is no such w, then $D(\hat{P}) = N'(v)$. Thus $\big|[v, C_{\hat{P}}(v)]\big| = 0$. Let P' be the S-partition obtained from $\hat{P}$ by moving v to $C_{\hat{P}}(u)$; note that $l_{P'} = l_{\hat{P}}$ and $T_{P'} = T_{\hat{P}}$.

Using the expression for f_g in (1), we have $f_g(\hat{P}) - f_g(P') = d_G(v) - g(v) > 0$ when $\hat{P}$ is Type 1, and $f_g(\hat{P}) - f_g(P') = 2\big|[v, C_{\hat{P}}(u)]\big| - 2\big|[v, C_{\hat{P}}(v)]\big| > 0$ when $\hat{P}$ is Type 2 or Type 3. Since $f_g(P') \geq 0$, this yields $f_g(\hat{P}) > 0$. Hence $\hat{P}$ is Type 3.

Since $N'(v) = D(\hat{P})$, now $N_G(v) \subseteq C_{\hat{P}}(u)$. Since g is an S-parity function, $v \notin S$, and $d_G(v) > g(v)$, we also have $\big|[v, C_{\hat{P}}(u)]\big| = d_G(v) \geq g(v) + 2 \geq 2$. Now $2 \geq f_g(\hat{P}) - f_g(P') = 2\big|[v, C_{\hat{P}}(u)]\big| \geq 4$, a contradiction. We conclude that the desired vertex w exists. □

We need one more result before proving Theorem 1.8. In fact, this theorem immediately yields the case $S = V(G)$ for Theorem 1.8. We use the Matroid Union Theorem. For a vertex $v \in V(G)$, let $E(v)$ denote the set of edges incident to v.

Theorem 3.4. *Let $S = V(G) = \{v_1, \ldots, v_n\}$. If the Strong Partition Condition holds for a function $g \colon V(G) \to \mathbb{N}_0$, then G contains edge-disjoint subgraphs $H_1, \ldots, H_{n+k}$ such that $d_{H_i}(v_i) = g(v_i)$ for $1 \leq i \leq n$ and $H_{n+1}, \ldots, H_{n+k}$ are spanning trees.*

Proof. For $1 \leq i \leq n$, let M_i be the matroid on $E(G)$ whose independent sets are $\{X \subseteq E(v_i) \colon |X| \leq g(v_i)\}$ (edges not incident to v_i are loops in M_i). Let $M_{n+1}, \ldots, M_{n+k}$ be copies of the cycle matroid of G. Let M be the union of the matroids $M_1, \ldots, M_{n+k}$ on $E(G)$; a subset of $E(G)$ is independent in M if and only if it partitions into sets $X_1, \ldots, X_{n+k}$ such that X_i is independent in M_i for each i. If M has an independent set of size $k(n-1) + g(V(G))$, then the resulting sets $X_1, \ldots, X_{n+k}$ are the edge sets of the desired subgraphs.

By the Matroid Union Theorem (Edmonds [1]), M is a matroid and the maximum size of an independent set in M is $\min_{X \subseteq E(G)} t(X)$, where $t(X) = |\overline{X}| + \sum_{i=1}^{n+k} r_i(X)$ and r_i is the rank function of M_i. Hence it suffices to show for each X that $t(X) \geq k(n-1) + g(V(G))$.

If $0 < r_i(X) < g(v_i)$, then deleting $X \cap E(v_i)$ from X shifts $r_i(X)$ from the term for M_i to the term for $\overline{X}$ without increasing other terms. Hence we may restrict our attention to sets X such that $r_i(X) \in \{0, g(v_i)\}$ for $1 \leq i \leq n$. Given such X, let P be the partition of $V(G)$ whose blocks are the vertex sets of the components of the spanning subgraph of G with edge set X. We express $t(X)$ in terms of P and then apply the SPC.

The set $\overline{X}$ consists of all edges joining blocks of P and possibly some edges within blocks of P. Hence $|\overline{X}| \geq \frac{1}{2}\sum_{A_i\in P}\delta(A_i)$.

A vertex v_i is a singleton block of P if and only if it has no incident edge in X. Thus $T_P = \{v_i\colon\ r_i(X) = 0\}$. With $r_i(X) \in \{0, g(V(G))\}$, we have $\sum_{i=1}^{n} r_i(X) = g(V(G)) - g(T_P)$. For $i > n$, the rank function of the cycle matroid yields $r_i(X) = n - l_P$.

By these computations, $2t(X) \geq \sum_{A_i\in P}\delta(A_i) - 2k(l_P - n) - 2g(T_P) + 2g(V(G))$. Thus $2t(X) \geq f_g(P) + 2k(n-1) + 2g(V(G))$. By the SPC, $f_g(P) \geq 0$, so the desired independent set and desired subgraphs exist. □

We can now prove our main result.

Theorem 1.8. *Let S be a set of terminals in G. If g is an S-parity function for G, then G has a (k, g)-family if and only if $f_g(P) \geq 0$ for all $P \in \mathcal{P}(S)$.*

Proof. We observed necessity in Section 1; here we prove sufficiency. We use induction on the number of vertices plus the number of edges, with trivial basis. By Theorem 3.4, the claim holds when $S = V(G)$, so we may assume $\overline{S} \neq \varnothing$. We will reduce the claim to a special case where Theorem 3.4 applies.

Let $R = \overline{S} \cap N(S)$. If $d(v) > g(v)$ for some $v \in R$, then $d(v) - g(v) \geq 2$, since g is an S-parity function. Since v has a neighbor $u \in S$, Theorem 3.3 guarantees a vertex $w \in N'(v)$ (for this choice of u) such that G_w satisfies the SPC. Since G_w is smaller than G, it has a (k, g)-family. If any of the resulting S-connectors or paths contain the edge uw that is not in G, then replacing that edge with the original uv and vw yields a (k, g)-family in G.

Hence we may assume $d_G(v) = g(v)$ for $v \in R$. We next reduce to the case $N(v) \subseteq S$ for all $v \in R$. Note that if $|S| = 1$ and the SPC holds, then the S-partition P whose only block is $V(G)$ requires $g(S) = 0$, since $0 \leq f_g(P) = -g(S)$. Now let P be the S-partition with $l_P = 1$ and $B_P = \overline{S}$. For any size of S, we have $f_g(P) = \left|[S, \overline{S}]\right| - g(\overline{S})$. By the SPC, $\left|[S, \overline{S}]\right| \geq g(\overline{S}) \geq \sum_{v\in R} d_G(v)$. However, $\left|[S, \overline{S}]\right| \leq \sum_{v\in R} d_G(v)$. We conclude that R is an independent set whose neighbors all lie in S and that $g(v) = 0$ for $v \in \overline{S} - R$.

We argue that in this remaining case $G[S]$ satisfies the SPC. Let $\hat{P}$ be an S-partition of $G[S]$; note that $B_{\hat{P}} = \varnothing$. Let P be the S-partition of G that is the same as $\hat{P}$ except $B_P = \overline{S}$. Note that $f_g(\hat{P}) - f_g(P) = g(B_P) - \left|[S, \overline{S}]\right|$. Since $g(B_P) = g(R)$, we have $f_g(\hat{P}) = f_g(P) \geq 0$.

Now that $G[S]$ satisfies the SPC, Theorem 3.4 yields $k + g(S)$ edge-disjoint subgraphs of $G[S]$ such that k are S-connectors in $G[S]$ and the others combine into disjoint sets of $g(v)$

edges at v for each $v \in S$. Since $g(v) = 0$ for $v \in \overline{S} - R$ and $g(v) = d_G(v)$ for $v \in R$, adding the edges from R to S as directed paths completes a (k, g)-family for G. □

4 Steiner tree packing

In this section we apply Theorem 1.8 to the problem of packing S-trees. Recall that $E(v)$ denotes the set of edges incident to a vertex v and that a vertex set S is j-edge-connected in a graph G when deleting any set of fewer than j edges leaves S in a single component. Our sufficient condition for k edge-disjoint S-trees uses the following theorem, which is the main technical result of this section and is proved using Theorem 1.8.

Theorem 4.1. *Fix $k \in \mathbb{N}$ and $\lambda k \in \mathbb{N}$ with $\lambda \geq 6.5$. Let S be a λk-edge-connected vertex set in a graph G, with $|S| \geq 3$. Given $v \in S$ with $d_G(v) = \lambda k$, let $E_0, \dots, E_k$ be a partition of $E(v)$, and let $N_i(v) = \{w \colon vw \in E_i\}$. If $|E_0| \geq k$, then G has edge-disjoint subgraphs $H_0, \dots, H_k$ such that*
(1) $E_i \subseteq E(H_i)$ for $0 \leq i \leq k$;
(2) $d_{H_0}(s) \geq k$ for all $s \in S$; and
(3) for $1 \leq i \leq k$, the vertex set $(S - \{v\}) \cup N_i(v)$ is connected in $H_i - v$.

We say that $H_0, \dots, H_k$ satisfying (1,2,3) in Theorem 4.1 *properly extend* $E_0, \dots, E_k$ or *form a proper extension* of $E_0, \dots, E_k$ in G. Note that by the meaning of "partition", each E_i is nonempty. Theorem 4.1 immediately yields Theorem 1.5.

Theorem 1.5 *If S is $6.5k$-edge-connected in G, then G contains k edge-disjoint S-trees.*

Proof. Form G' by adding a vertex v to G and adding $\lceil 6.5k \rceil$ edges joining v to S. Let $S' = S \cup \{v\}$; note that S' is $\lceil 6.5k \rceil$-edge-connected in G'. Partition the edges incident to v into $E_0, \dots, E_k$ with $|E_0| \geq k$. Applying Theorem 4.1 with G' and S' instead of G and S yields subgraphs $H_0, \dots, H_k$. By property (3) in the conclusion of Theorem 4.1, $H_1, \dots, H_k$ contain the desired S-trees. □

If Theorem 4.1 is not true, then there is a graph G_0 with fewest edges where S, v, λ, k and $E_0, \dots, E_k$ satisfy the hypotheses and yet no proper extension of $E_0, \dots, E_k$ exists. Among such structures, we choose one such that $V(G_0) - S$ is smallest. We describe this setting as "the counterexample G_0". In the next few lemmas, we obtain properties of such a counterexample. Minimality implies that G_0 is connected. Also, a λk-edge-connected set of size at least 2 cannot have a loop at a vertex of degree λk, so we may assume there is no loop at v when $|S| \geq 2$.

Lemma 4.2. *In the counterexample G_0, the set $\overline{S}$ of non-terminal vertices is independent.*

Proof. Suppose that e is an edge with endpoints in $\overline{S}$. If S is λk-edge-connected in $G_0 - e$, then by the minimality of G_0 there exist $H_0, \ldots, H_k$ that properly extend $E_0, \ldots, E_k$ in $G_0 - e$. These subgraphs also properly extend $E_0, \ldots, E_k$ in G_0.

Hence G_0 is a minimal graph in which S is λk-edge-connected. We can also discard isolated vertices, so we obtain an edge-cut F with the same properties as in the proof of Theorem 1.5. Define G', G'', S', S'' in the same way as before. Again the sets S' and S'' are λk-edge-connected in G' and G'', respectively. By symmetry, we may assume that the special vertex v in S lies in $V(G')$.

Since the endpoints of e are in $\overline{S}$, the cut F does not isolate a vertex, so both G' and G'' are smaller than G_0. Hence there exist $H_0', \ldots, H_k'$ that properly extend $E_0, \ldots, E_k$ in G'. Let $E_i'' = E(H_i') \cap F$ for $0 \le i \le k$. In G'', we obtain $H_0'', \ldots, H_k''$ that properly extend $E_0'', \ldots, E_k''$. For $0 \le i \le k$, let H_i be the subgraph of G with $E(H_i) = E(H_i') \cup E(H_i'')$. Now $H_0, \ldots, H_k$ properly extend $E_0, \ldots, E_k$ in G_0, a contradiction. □

For two vertices x and y of G, let $\kappa'(x, y; G)$ denote the *local edge-connectivity* of x and y in G, defined to be the minimum number of edges whose deletion leaves x and y not in the same component. We next state Mader's Splitting Lemma, a powerful tool for inductive arguments involving local edge-connectivity.

Theorem 4.3 (Mader's Splitting Lemma [8])**.** *Let x be a non-cut-vertex of G. If x has degree at least 2 (except when $d_G(x) = 3$ and x has three distinct neighbors), then there is a shortcut G' of G at x such that $\kappa'(u, v; G) = \kappa'(u, v; G')$ whenever $u, v \in V(G) - \{x\}$.*

From this we obtain another structural property of the counterexample G_0.

Lemma 4.4. *In the counterexample G_0, every vertex of $\overline{S}$ has degree 3, with three distinct neighbors in S.*

Proof. To facilitate the application of Mader's Lemma, we first show that G_0 has no cut-vertex. Suppose that x is a cut-vertex of G_0. If some component H of $G_0 - x$ contains no vertex of S, then S remains λk-edge-connected in the smaller subgraph $G_0 - V(H)$, and the proper extension of $E_0, \ldots, E_k$ in that subgraph is valid also in the full graph.

If S has a vertex in each component of $G_0 - x$, and S remains connected in a subgraph obtained by deleting fewer than λk edges leaves from G_0, then $S \cup \{x\}$ also remains connected in that subgraph, since every path connecting vertices from distinct components of $G_0 - x$ contains x. Hence $S \cup \{x\}$ is λk-edge-connected in G_0. This larger terminal set omits fewer vertices, so the choice of G_0 implies that with this terminal set there is a proper extension of $E_0, \ldots, E_k$ in G_0. That extension is also a proper extension of $E_0, \ldots, E_k$ using the original terminal set S.

We may therefore assume that G_0 is 2-connected. If a subgraph or a shortcut of G_0 has an proper extension of $E_0, \ldots, E_k$, then so does G_0. Minimality thus implies that in every subgraph or shortcut of G_0, the set S is not λk-edge-connected. Consider $u \in \overline{S}$. If $d_{G_0}(u) = 1$, then S is λk-edge-connected in $G_0 - \{u\}$. In all other cases except when $d_{G_0}(u) = 3$ and u has three distinct neighbors, Mader's Splitting Lemma implies that S is λk-edge-connected in some shortcut of G_0 at u. □

Within $V(G_0)$, pick a vertex u_i from $N_i(v)$ for $1 \le i \le k$. These vertices need not be distinct and may lie in S. Let $U = \{u_1, \ldots, u_k\}$, $S' = S - \{v\}$, $N'_i = N_i(v) - u_i - S'$ and $X = \bigcup_{i=1}^{l} N'_i$. Let M be the maximal bipartite subgraph of G_0 with partite sets X and S'.

Lemma 4.5. *In the counterexample G_0, there exists a subgraph M' of M such that:*
(1) $d_{M'}(x) = 1$ for all $x \in X$; and
(2) $d_{M'}(s) \ge \lfloor d_M(s)/2 \rfloor$ for all $s \in S'$.

Proof. Since all of S is among the set deleted from $N(v)$ to form X, we have $X \subseteq \overline{S}$. Since v has been deleted and all other vertices of S remain in S', every vertex in X two distinct neighbors in M. Let H be the graph with vertex set S' obtained from M by performing a shortcut at each vertex of X; each vertex of S' has the same degree in M and H.

As in any graph, each set $A \subseteq V(H)$ is incident to at least $\frac{1}{2}\sum_{s \in A} d_H(s)$ edges in H. Therefore, by Hakimi's Theorem (Theorem 1.11), there is an orientation D of H in which every vertex $s \in S'$ has outdegree at least $\lfloor d_M(s)/2 \rfloor$. For each $x \in X$, put the edge sx into M', where s is the tail of the edge in D corresponding to x. Now $d_{M'}(x) = 1$ for $x \in X$ and $d_{M'}(s) = d_D^+(s) \ge \lfloor d_M(s)/2 \rfloor$ for all $s \in S'$. □

In the counterexample G_0, let $G' = G_0 - v - X$. Using S' as the set of terminals, we will consider a special S'-parity function g, defined by

$$g(u) = \begin{cases} 0, & u \in \overline{S'}, d_{G'}(u) \text{ is even;} \\ 1, & u \in \overline{S'}, d_{G'}(u) \text{ is odd;} \\ \max\{k - d_{M'}(u) - \big|[u,v]\big|, 0\}, & u \in S'. \end{cases} \tag{4}$$

We will prove that G' has a (k, g)-family for the terminal set S' and this S'-parity function g. Because that proof is somewhat lengthy, we first motivate it by showing how to use it to complete the proof of Theorem 4.1.

Lemma 4.6. *Consider the counterexample G_0. If the derived graph G' has a (k, g)-family for the S'-parity function g defined by (4), then G_0 has a proper extension of $E_0, \ldots, E_k$ (implying that G_0 is not a counterexample and Theorem 4.1 holds).*

Proof. For $E_0, \ldots, E_k$ satisfying the hypotheses of Theorem 4.1, we will obtain a proper extension in G_0. Given a (k, g)-family in G', let $H'_1, \ldots, H'_k$ be the S'-connectors, and let $P_1, \ldots, P_{g(V(G'))}$ be the oriented paths. We may assume that $H'_1, \ldots, H'_k$ are minimal S'-connectors; short-cutting the paths turns each H'_i into a tree T'_i with vertex set S'.

When $u_i \notin S'$, define a *marked* edge e_i with endpoints in S' by short-cutting the two edges incident to u_i in G'. We will modify $T'_1, \ldots, T'_k$ so that e_i occurs in no tree other than the ith one. The union of $T'_1, \ldots, T'_k$ and the (at most k) marked edges is a graph with vertex set S'. We can permute the indices so that only the ith tree contains e_i unless some tree in the list, call it T, contains at least two marked edges; let e be one of them. Since the k trees are edge-disjoint, some other tree in the list, call it T', has no marked edge.

We apply a standard edge-switching argument. Adding e to T' completes a unique cycle via a path in T' containing an edge e' joining the two components of $T - e$. Replacing T and T' with $T - e + e'$ and $T' - e' + e$ yields a new set of k edge-disjoint spanning trees in which fewer trees contain more than one marked edge. The edge switch corresponds in G' to switching paths in the S'-connectors. Repeat the argument to eliminate trees containing more than one marked edge, and then index the resulting S'-connectors as $H'_1, \ldots, H'_k$ so that each vertex $u_i \in U - S'$ occurs in none of these other than H'_i.

If $u_i \notin S'$ and u_i precedes s on some path in the (k, g)-family, then let $C_i = \{u_i s\}$. If $u_i \notin S'$ and there is no such path, then let $C_i = E_{G'}(u_i)$. If $u_i \in S'$, then $C_i = \varnothing$. Let $C = \bigcup_{i=1}^{k} C_i$. Let B_i be the set of edges in $E(M) - E(M')$ incident to N'_i. Let H'_0 be the spanning subgraph of G' with edge set $\bigcup_{j=1}^{g(V(G))} E(P_j)$.

To construct the proper extension of $E_0, \ldots, E_k$, let H_0 be the spanning subgraph of G with edge set $E_0 \cup E(M') \cup (E(H'_0) - C)$, and for $1 \le i \le k$ let H_i be the spanning subgraph of G with edge set $E_i \cup E(H'_i) \cup B_i \cup C_i$. By explicit construction, (1) in Theorem 4.1 holds.

For (3), note that since H'_i is an S'-connector in G', all of $S - \{v\}$ is connected in $H_i - v$. If $x \in N'_i$, then x has two neighbors in S' in the bipartite graph M; one incident edge is in H_0 and the other is in H_i, connecting x to $S - \{v\}$ in $H_i - v$. Finally, if $u_i \notin S'$, then u_i is not in M but is in G'. By the switching argument given above, if either edge incident to u_i in G' is in $\bigcup_{j=1}^{k} H'_j$, then it is in H_i, which suffices. Otherwise, we have put at least one incident edge into C_i, which connects u_i to S' in H_i.

For (2), we check that H_0 gains enough edges at each vertex of S'. For $s \in S'$, in $H'_0 - C$

there are at least $g(s)$ edges incident to s, since the paths in the (k, g)-family guaranteed to start from s do not use edges of C at s. Adding $[s, v]$ and the edges of M' yields $d_{H_0}(s) \geq g(s) + \big|[s, v]\big| + d_{M'}(s) \geq k$. Also $d_{H_0}(v) \geq k$, since $|E_0| \geq k$. $\square$

By Lemma 4.6, the next lemma (occupying the remainder of this section) completes the proof of Theorem 4.1 and hence also Theorem 1.5. In the last case here we use $\lambda \geq 6.5$.

Lemma 4.7. *In the counterexample G_0, the derived graph G' has a (k, g)-family for the S'-parity function g defined by (4).*

Proof. If G' has no (k, g)-family, then by Theorem 1.8 the SPC fails, and there is an S'-partition P of G' such that

$$0 > f_g(P) = \sum_{A_i \in P} \delta(A_i) - 2k(l_P - 1) - g(B_P) - 2g(T_P).$$

By Lemma 4.4, every vertex of $\overline{S}$ has degree 3 in G_0, with three distinct neighbors in S; this permits some reductions in the structure of P. First, if $w \in A_i - S'$, and w has no neighbor in A_i, then w has a neighbor in some block A_j other than A_i, and moving w to that block produces an S'-partition of G' with $f_g(P') < f_g(P)$. Hence we may assume that every vertex outside S' in a block of P has a neighbor in that block.

Next, the definition of g yields $g(B_P) = n_o(B_P)$. Every vertex $w \in B_P$ has degree 2 or 3 in G', with its neighbors being distinct vertices of S'. If $d_{G'}(w) = 2$, or if $d_{G'}(w) = 3$ and w has two neighbors in one block of P, then form a new S'-partition P' by moving w into a block containing at least half the neighbors of w. Only the terms $\sum_{A_i \in P} \delta(A_i) - n_o(B_P)$ can change, and in all cases $f_g(P') \leq f_g(P) < 0$. By iterating this operation, we may thus assume that every vertex in B_P has neighbors in three different blocks of P, and $g(B_P) = |B_P|$.

Vertices of X are not in $V(G')$, but in G_0 they have exactly two neighbors in S'. For $x \in X$ and $j \in \{1, 2\}$, put $x \in X_j$ when $N(x) \cap V(G')$ intersects exactly j blocks in P; thus X_1 and X_2 partition X. Add each vertex of X_1 to the block of P containing its neighbors, forming $A'_1, \ldots, A'_l$ from $A_1, \ldots, A_l$; we study P in G' by studying these related blocks in G_0. Since S is λk-edge-connected in G_0, its subset S' is also λk-edge-connected in G_0, and hence $\delta_{G_0}(A'_i) \geq \lambda k$ for $1 \leq i \leq l$. Since each vertex of X_2 has v as a neighbor along with two neighbors in S', we have $\sum_{i=1}^{l} \big|[A'_i, X_2 \cup \{v\}]\big| = d(v) + |X_2|$. Now

$$\begin{aligned} \sum_{i=1}^{l} \delta_{G'}(A_i) &= \sum_{i=1}^{l} \Big(\delta_{G_0}(A'_i) - \big|[A'_i, X_2 \cup \{v\}]\big| \Big) \\ &\geq \lambda k l - d(v) - |X_2| \;=\; \lambda k(l-1) - |X_2|. \qquad (5) \end{aligned}$$

Also $3|B_P| = \big|[B_P, \overline{B_P}]\big| \le \sum_{i=1}^{l} \delta_{G'}(A_i)$, so $\sum_{i=1}^{l} \delta_{G'}(A_i) - g(B_P) \ge \frac{2}{3}\sum_{i=1}^{l} \delta_{G'}(A_i)$. Thus

$$\sum_{i=1}^{l} \delta_{G'}(A_i) - g(B_P) - 2k(l-1) \ge \frac{2}{3}[(\lambda-3)k(l-1) - |X_2|]. \tag{6}$$

After dividing the needed inequality by 2/3, it thus suffices to prove

$$(\lambda-3)k(l-1) - |X_2| - 3g(T_P) \ge 0. \tag{7}$$

Let L denote the left side of (7). Our last preliminary computation bounds $|X_2|$. Using $|X| \le \big|[\{v\}, N(v)]\big| \le \lambda k$ and $\big|[\{v\}, N_0(v) \cup U]\big| \ge 2k$, we have

$$|X_2| \le |X| \le (\lambda-2)k, \tag{8}$$

Case 1: $l = 1$. Since $|S| \ge 3$, we have $T_P = \varnothing$. Since vertices of B_P must have neighbors in three blocks, $B_P = \varnothing$. Hence also $\delta(A_1) = 0$, so $f_g(P) = 0$.

Case 2: $l = 2$ and $T_P = \varnothing$. Again $B_P = \varnothing$, so we may use (5) and (8) instead of (7) to obtain $f_g(P) \ge \lambda k(l-1) - (\lambda-2)k - 2k(l-1) = (\lambda-2)k(l-2) = 0$.

Case 3: $l = 3$ and $|T_P| = 1$. Let s be the one vertex of T_P, and index the blocks of P so that $s \in A_1$. Focusing on A_1, we compute

$$\begin{aligned}
\sum_{i=1}^{3} \delta_{G'}(A_i) - |B_P| &\ge 2\big|[A_1, A_2 \cup A_3]\big| + 3|B_P| - |B_P| \\
&= 2\delta_{G'}(A_1) \quad = \quad 2\big(\delta_G(A_1) - \big|[s, X_2 \cup \{v\}]\big|\big).
\end{aligned}$$

Since $s \in T_P$ requires $g(s) > 0$ by the definition of T_P, we have $g(s) = k - d_{M'}(s) - \big|[s, v]\big|$. By Lemma 4.5, $2d_{M'}(s) \ge d_M(s) - 1$. Using this and (8) along the way, we compute

$$\begin{aligned}
f_g(P) &= \sum_{i=1}^{3} \delta_{G'}(A_i) - |B_P| - 4k - 2g(s) \\
&\ge 2\delta_G(A_1) - 2\big|[s, X_2 \cup \{v\}]\big| - 6k + d_M(s) - 1 + 2\big|[s, v]\big| \\
&\ge 2\lambda k - (\lambda-2)k - 6k - \big|[s, X_2]\big| + d_M(s) - 1 \\
&\ge (\lambda-4)k - 1 \quad \ge \quad 0.
\end{aligned}$$

Case 4: $|T_P| < l - 2$, or $|T_P| = l - 2 \ge 2$. Using (8) and $\lambda \ge 6.5$ and $g(T_P) \le k|T_P|$,

$$L/k \ge (\lambda-3)(l-2) - 1 - 3|T_P| \ge -1 + .5(l-2) + 3(l-2-|T_P|) \ge 0.$$

Case 5: $|T_P| = l-1 \geq 1$. Each $x \in X_2$ has neighbors in S in two blocks of P; hence x has at least one neighbor in T_P, so $\sum_{u\in T_P} d_M(u) \geq |X_2|$. Also, for $u \in T_P$, we have $g(u) \leq k - d_{M'}(u)$. Since $\lambda k \in \mathbb{N}$ and $\lambda \geq 6.5$, we have $\lambda k \geq 6k+1$, so

$$\begin{aligned} L &= (\lambda-3)k(l-1) - |X_2| - 3g(T_P) \\ &\geq (\lambda-3)k(l-1) - |X_2| - 3k(l-1) + 3\sum_{u\in T_P} d_{M'}(u) \\ &\geq (l-1) + \sum_{u\in T_P} d_{M'}(u) + \sum_{u\in T_P}(d_M(u)-1) - |X_2| \quad \geq \quad 0. \end{aligned}$$

Case 6: $|T_P| = l \geq 2$. Here $T_P = S'$, and each block of P contains exactly one vertex of S', so $X_1 = \varnothing$ and $X = X_2$. Also, $d_{M'}(T_P) = d_{M'}(S') = d_{M'}(X) = |X|$. Hence $g(T_P) = kl - |X| - \big|[v,S']\big|$.

We now need to strengthen the lower bound on $\sum_{i=1}^{l} \delta_{G'}(A_i)$ and upper bound on $|B_P|$ used in (5). Let $W = \{w \in \overline{S}\colon vw \in E_0\}$. Note that $\big|[v,W]\big| = |W|$, since $W \subseteq \overline{S}$. If $w \in W \cap A_i$, then w is adjacent to the vertex of S' in A_i (by our initial reduction of P) and to a vertex of S' in another block A_j (by Lemma 4.4). Hence $\delta_{G'}(A_i) = \delta_{G'}(A_i - W)$. Since $\big|[S',X]\big| = 2|X|$, and $X \subseteq N(v)$, and S' is λk-edge-connected in G_0, we have

$$\begin{aligned} \sum_{i=1}^{l} \delta_{G'}(A_i) &= \sum_{i=1}^{l}\Big(\delta_{G_0}(A_i - W) - \big|[A_i - W, X\cup\{v\}]\big|\Big) \\ &\geq \lambda kl - d(v) - |X| + |W| \quad = \quad \lambda k(l-1) - |X| + |W|. \end{aligned}$$

Each vertex of B_P supplies three of the edges leaving blocks of P, but not the edges leaving blocks of P to or from vertices of W; hence $3|B_P| \leq \left(\sum_{i=1}^{l}\delta_{G'}(A_i)\right) - 2|W|$. Now

$$\begin{aligned} f_g(P) &= \sum_{i=1}^{l} \delta_{G'}(A_i) - |B_P| - 2k(l-1) - 2g(T_P) \\ &\geq \frac{2}{3}(\lambda k(l-1) - |X|) + \frac{4}{3}|W| - 2k(l-1) - 2kl + 2|X| + 2\big|[v,S']\big| \\ &= \left(\frac{2}{3}\lambda - 4\right)k(l-1) + \frac{4}{3}|W| + 2\big|[v,S']\big| + \frac{4}{3}|X| - 2k \\ &\geq \frac{1}{3}k(l-1) + \frac{4}{3}k - 2k. \end{aligned}$$

In the last step, we used $|W| + \big|[v,S']\big| \geq |E_0| \geq k$, along with $\lambda \geq 6.5$ and $|X| \geq 0$. The final expression is nonnegative when $l \geq 3$.

This leaves the case $|T_P| = l = 2$. As in Case 2, $B_P = \varnothing$, and we have

$$\begin{aligned} f_g(P) &= \sum_{i=1}^{2} \delta_{G'}(A_i) - 2k - 2g(T_P) \\ &\geq \lambda k - |X| + |W| - 2k - 4k + 2|X| + 2\big|[v, S']\big| \quad > \quad 0. \end{aligned}$$

□

5 S-connector packing

To prove Theorem 1.6, we prove a theorem for S-connectors analogous to Theorem 4.1. Note that Theorem 5.1 immediately yields Theorem 1.6 in the way that Theorem 4.1 yields Theorem 1.5, by applying it to a graph obtained from the given graph by adding one vertex.

Theorem 5.1. *Fix $k \in \mathbb{N}$ and $\lambda k \in \mathbb{N}$ such that $\lambda \geq 10$. Consider $S \subseteq V(G)$ and $v \in S$ such that S is λk-edge-connected in G and $d_G(v) = \lambda k$. If $E_0, \ldots, E_k$ is a partition of $E(v)$ such that $|E_0| \geq 2k$, then there exist edge-disjoint subgraphs $H_0, \ldots, H_k$ such that*
(1) $E_i \subseteq E(H_i)$;
(2) $d_{H_0}(s) \geq 2k$ for any $s \in S$;
(3) For $i > 0$, H_i is the union of an S-connecting family of edge-disjoint paths; and
(4) For $i > 0$, deleting from the family of paths forming H_i the paths that use edges of E_i leaves an $(S - v)$-connecting family.

The proof of Theorem 5.1 is very similar to the proof of Theorem 4.1; we describe the differences without repeating the entire argument.

Again we use the language of proper extension and consider a minimal counterexample G_0. The argument of Lemmas 4.2 and 4.4 is the same to show that the non-terminal vertices in G_0 form an independent set in which every vertex has degree 3, with three distinct neighbors in S. We again define an auxiliary bipartite graph M, but without choosing special vertices $u_1, \ldots, u_k$. That is, we simply have $S' = S - \{v\}$, $N'_i = N_i(v) - S'$, and $X = \bigcup_{i=1}^{l} N'_i$. Again M is the maximal bipartite subgraph of G_0 with partite sets X and S', and the argument of Lemma 4.5 is the same to obtain the subgraph M' such that $d_{M'}(x) = 1$ for $x \in X$ and $d_{M'}(s) \geq \lfloor d_M(s)/2 \rfloor$ for $s \in S$.

Again we define the derived graph G' by $G' = G_0 - v - X$. However, the S'-parity function we define on G' is different from before; when $u \in S'$ we replace k with $2k$:

$$g(u) = \begin{cases} 0, & u \in \overline{S'}, d_{G'}(u) \text{ is even;} \\ 1, & u \in \overline{S'}, d_{G'}(u) \text{ is odd;} \\ \max\{2k - d_{M'}(u) - \big|[u, v]\big|, 0\}, & u \in S'. \end{cases} \tag{9}$$

Again we reduce the problem to showing that G' has a (k,g)-family for S' and this g, by proving as in Lemma 4.6 that $E_0,\dots,E_k$ extend in G_0 as specified in Theorem 5.1 when G' has a (k,g)-family with g as in (9). This time the reduction is easier, because the absence of chosen vertices $u_1,\dots,u_k$ eliminates the need to choose S'-connectors that avoid them. We present the simpler argument.

Lemma 5.2. *Consider the counterexample G_0. If the derived graph G' has a (k,g)-family for the S'-parity function g defined by (9), then G_0 has a proper extension of $E_0,\dots,E_k$ (implying that G_0 is not a counterexample and Theorem 5.1 holds).*

Proof. For $E_0,\dots,E_k$ satisfying the hypotheses of Theorem 5.1, we will obtain a proper extension in G_0. Given a (k,g)-family in G', let $H_1',\dots,H_k'$ be the S'-connectors, and let $P_1,\dots,P_{g(V(G'))}$ be the oriented paths.

Let B_i be the set of edges in $E(M)-E(M')$ incident to N_i'. Let H_0' be the spanning subgraph of G' with edge set $\bigcup_{j=1}^{g(V(G))} E(P_j)$.

To construct the proper extension of $E_0,\dots,E_k$, let H_0 be the spanning subgraph of G with edge set $E_0\cup E(M')\cup E(H_0')$, and for $1\le i\le k$ let H_i be the spanning subgraph of G with edge set $E_i\cup E(H_i')\cup B_i$. By explicit construction, (1) in Theorem 4.1 holds.

For (3), note for $1\le i\le k$ that $E_i\cup B_i$ is a nonempty set of paths that join v to vertices of S'. We do not require H_0 to be an S-connector.

For (4), when we delete the paths formed by $E_i\cup B_i$, we return to H_i', which is an S'-connector in G' and hence is an $(S'-\{v\})$-connector in $G-v$.

For (2), we check that H_0 gains enough edges at each vertex of S'. For $s\in S'$, in H_0' there are at least $g(s)$ edges incident to s, provided explicitly by the paths in the (k,g)-family. Adding $[s,v]$ and the edges of M' yields $d_{H_0}(s)\ge g(s)+\big|[s,v]\big|+d_{M'}(s)\ge 2k$. Also $d_{H_0}(v)\ge 2k$, since $|E_0|\ge 2k$. □

Finally, we prove the analogue of Lemma 4.7.

Lemma 5.3. *In the counterexample G_0, the derived graph G' has a (k,g)-family for the S'-parity function g defined by (9).*

Proof. If G' has no (k,g)-family, then by Theorem 1.8 the SPC fails, and there is an S'-partition P of G' such that

$$0>f_g(P)=\sum_{A_i\in P}\delta(A_i)-2k(l_P-1)-g(B_P)-2g(T_P).$$

Arguing as in Lemma 4.7, we may assume that every vertex of $\overline{S}$ has degree 3 in G_0, that every vertex outside S' in a block of P has a neighbor in that block, that every vertex in B_P has neighbors in three different blocks of P, and that $g(B_P) = |B_P|$. Similarly, vertices of X have exactly two neighbors in S'. Again let X_2 be the subset of X whose vertices having neighbors in distinct blocks of P. Arguing exactly as in Lemma 4.7 yields (5), (6), (7), (8), except that now we use $\big|[\{v\}, N_0(v)]\big| = E_0 \geq 2k$ instead of $\big|[\{v\}, N_0(v) \cup U]\big| \geq 2k$, since there is no U and instead we increased the requirement on $|E_0|$ to $2k$.

There remain only the computations in the Cases. Cases 1 and 2 are unchanged.

Case 3: $l = 3$ and $|T_P| = 1$. Again with $T_P = \{s\}$ and $s \in A_1$, the argument is the same, except that $g(s)$ is larger: $g(s) = 2k - d_{M'}(s) - \big|[s, v]\big|$. The effect of this change is that $-4k - 2g(s) = -8k + d_M(s) - 1 + 2\big|[s, v]\big|$. Since $\lambda \geq 8$, the resulting computation is $f_g(P) \geq 2k + (\lambda - 8)k - 1 \geq 0$.

Case 4: $|T_P| < l - 2$, or $|T_P| = l - 2 \geq 2$. Using (8) and $\lambda \geq 9.5$ and $g(T_P) \leq 2k|T_P|$,

$$L/k \geq (\lambda - 3)(l - 2) - 1 - 6|T_P| \geq -1 + .5(l - 2) + 3(l - 2 - |T_P|) \geq 0.$$

Case 5: $|T_P| = l - 1 \geq 1$. With $g(u) \leq 2k - d_{M'}(u)$ and $\lambda k \geq 10k \geq 9k + 1$, the computation becomes

$$\begin{aligned} L &= (\lambda - 3)k(l - 1) - |X_2| - 3g(T_P) \\ &\geq (\lambda - 3)k(l - 1) - |X_2| - 6k(l - 1) + 3\sum_{u \in T_P} d_{M'}(u) \\ &\geq (l - 1) + \sum_{u \in T_P}(d_{M'}(u) - 1) + \sum_{u \in T_P} d_M(u) - |X_2| \quad \geq \quad 0. \end{aligned}$$

Case 6: $|T_P| = l \geq 2$. Reasoning exactly as for Case 6 in Lemma 4.7, the computation begins with $\sum_{i=1}^{l} \delta_{G'}(A_i) \geq \lambda k(l - 1) - |X| + |W|$ and $3|B_P| \leq \big(\sum_{i=1}^{l} \delta_{G'}(A_i)\big) - 2|W|$, and it ends with

$$\begin{aligned} f_g(P) &= \sum_{i=1}^{l} \delta_{G'}(A_i) - |B_P| - 2k(l - 1) - 2g(T_P) \\ &\geq \frac{2}{3}(\lambda k(l - 1) - |X|) + \frac{4}{3}|W| - 2k(l - 1) - 4kl + 2|X| + 2\big|[v, S']\big| \\ &= \left(\frac{2}{3}\lambda - 6\right) k(l - 1) + \frac{4}{3}|W| + 2\big|[v, S']\big| + \frac{4}{3}|X| - 4k \\ &\geq \frac{2}{3}k(l - 1) + \frac{8}{3}k - 4k. \end{aligned}$$

In the last step, we used $|W| + \big|[v, S']\big| \geq |E_0| \geq 2k$, along with $\lambda \geq 10$ and $|X| \geq 0$. The final expression is nonnegative when $l \geq 3$.

This leaves the case $|T_P| = l = 2$. As in Case 2, $B_P = \varnothing$, and $\lambda \geq 10$ is enough to give

$$\begin{aligned} f_g(P) &= \sum_{i=1}^{2} \delta_{G'}(A_i) - 2k - 2g(T_P) \\ &\geq \lambda k - |X| + |W| - 2k - 8k + 2|X| + 2\big|[v, S']\big| \quad \geq \quad 0. \end{aligned}$$

□

Acknowledgment

We thank Chandra Chekuri and Lap-Chi Lau for helpful discussions about this problem.

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