Decomposition of Sparse Graphs into Forests and a Graph with Bounded Degree

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Abstract

Say that a graph with maximum degree at most \(d\) is \(d\)-bounded. For \(d > k\), we prove a sharp sparseness condition for decomposability into \(k\) forests and a \(d\)-bounded graph. Consequences are that every graph with fractional arboricity at most \(k + \frac{d}{k+d+1}\) has such a decomposition, and (for \(k = 1\)) every graph with maximum average degree less than \(2 + \frac{2d}{d+2}\) decomposes into a forest and a \(d\)-bounded graph. When \(d = k+1\), and when \(k = 1\) and \(d \leq 6\), the \(d\)-bounded graph in the decomposition can also be required to be a forest. When \(k = 1\) and \(d \leq 2\), the \(d\)-bounded forest can also be required to have at most \(d\) edges in each component.

1 Introduction

A decomposition of a graph \(G\) consists of edge-disjoint subgraphs whose union is \(G\). The arboricity of a graph \(G\), written \(\Upsilon(G)\), is the minimum number of forests needed to decompose it. The famous Nash-Williams Arboricity Theorem is that a necessary and sufficient condition for \(\Upsilon(G) \leq k\) is that no subgraph \(H\) has more than \(k(|V(H)| - 1)\) edges. This is a sparseness condition. A slightly different sparseness condition places a bound on the average vertex degree in all subgraphs. The maximum average degree of a graph \(G\), denoted \(\text{Mad}(G)\),

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is $\max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$; it is the maximum over subgraphs $H$ of the average vertex degree in $H$. (Our model of “graph” allows multiedges but no loops.)

Many papers have obtained various types of decompositions from bounds on $\text{Mad}(G)$. Our results extend some of these and the Nash-Williams Theorem, which states that $\Upsilon(G) = \left\lceil \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1} \right\rceil$. We consider the fractional arboricity $\max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$, introduced by Payan [13]. For fractional arboricity we use the notation $\text{Arb}(G)$, by analogy with $\text{Mad}(G)$.

Three forests are needed to decompose a graph with fractional arboricity $2 + \epsilon$, but since this is just slightly above 2 one may hope that some restrictions can be placed on the third forest. Say that a graph is $d$-bounded if it has maximum degree at most $d$. Montassier et al. [11] posed the Nine Dragon Tree (NDT) Conjecture (honoring a famous tree in Kaohsiung, Taiwan that is far from acyclic): if $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$, then $G$ decomposes into $k+1$ forests with the last forest being $d$-bounded. They proved the cases $(k, d) = (1, 1)$ and $(k, d) = (1, 2)$, and they showed that no larger value of $\text{Arb}(G)$ is sufficient. In Sections 4–6 we will prove the cases $(1, d)$ for $d \leq 6$. In Section 3 we will prove the case $d = k + 1$.

Another stream of research considered decomposing a planar graph into a forest and a $d$-bounded graph, stimulated greatly by the seminal paper [8], which motivated the topic by its application to the concept of “game coloring number”. For a planar graph with girth $g$ to decompose into a forest and a matching, $g \geq 8$ suffices [11, 14] (earlier it was proved for $g \geq 11$ in [8], for $g \geq 10$ in [2], and for $g \geq 9$ in [6]). Also, the graph left by deleting the edges of a forest can be guaranteed to be 2-bounded when $g \geq 7$ [8] (improved to $g \geq 6$ in [9]) and 4-bounded when $g \geq 5$ [8]. Borodin, Ivanova, and Stechkin [3] disproved the conjecture from [8] that every planar graph $G$ decomposes into a forest and a $(\lceil \Delta(G)/2 \rceil + 1)$-bounded graph. In [4], there are sufficient conditions for a planar graph with triangles to decompose into a forest and a matching, and [5] shows that a planar graph without 4-cycles (3-cycles are allowed) decomposes into a forest and a 5-bounded graph.

Many results on planar graphs with restricted girth can be strengthened by showing that the conclusions hold when only corresponding bounds on $\text{Mad}(G)$ are assumed. If $G$ is a planar graph with girth $g$, then $G$ has at most $\frac{g}{2}(n-2)$ edges, by Euler’s Formula. This holds for all subgraphs, so girth $g$ implies $\text{Mad}(G) < \frac{2g}{g-2}$. Montassier et al. [10] initiated the problem of finding the best bound on $\text{Mad}(G)$ to guarantee decomposition into a forest and a $d$-bounded graph. They proved that $\text{Mad}(G) < 4 - \frac{8d+12}{d^2+6d+6}$ is sufficient and that $\text{Mad}(G) = 4 - \frac{4}{d+2}$ is not sufficient (seen by subdividing every edge of a $(2d+2)$-regular graph). Our general result in Theorem 1.1 completely solves this problem, implying that $\text{Mad}(G) < 4 - \frac{4}{d+2}$ suffices.

This result implies all the previous girth results yielding decompositions of planar graphs into a forest and a $d$-bounded graph. Girth 8, 6, and 5 imply that $\text{Mad}(G)$ is less than $8/3$, $3$, and $10/3$, respectively, which are precisely the bounds that by our result guarantee
decomposition into a forest and a graph with maximum degree at most 1, 2, or 4, respectively.

Other work brought the two problems that we have described closer together, requiring the leftover \(d\)-bounded graph to be a forest or considering the leftover after deleting more than one forest. Formally, a \((G, H)\)-decomposition of \(G\) decomposes it into a graph in \(G\) and a graph in \(H\); a graph having such a decomposition is \((G, H)\)-decomposable. Let \(F\) be the family of forests, let \(kdF\) be the family of edge-disjoint unions of (at most) \(k\) forests, let \(D_d\) be the family of \(d\)-bounded graphs, and let \(F_d\) be the family of \(d\)-bounded forests.

Examples of planar graphs with girth 7 having no \((F, F_2)\)-decomposition and examples with girth 5 having no \((F, F_2)\)-decomposition appear in [11, 9]. Gonçalves [7] proved the conjecture of Balogh et al. [1] that every planar graph is \((2F, F_4)\)-decomposable. He also proved that planar graphs with girth at least 6 are \((F, F_4)\)-decomposable and with girth at least 7 are \((F, F_2)\)-decomposable.

The NDT Conjecture is that \(\text{Arb}(G) \leq k + \frac{d}{k+d+2}\) guarantees a \((kF, F_d)\)-decomposition. The fractional arboricity of a planar graph can be arbitrarily close to 3, which is not small enough for the NDT Conjecture to guarantee \((2F, F_d)\)-decomposability for any constant \(d\). However, requiring girth at least 6 or 7 yields fractional arboricity less than 6/4 or 7/5, respectively, in which case the NDT Conjecture would guarantee \((F, F_4)\)- or \((F, F_2)\)-decompositions, respectively. Hence the NDT Conjecture implies the results of Gonçalves for \((F, F_d)\)-decomposition of planar graphs with large girth.

Our Theorem 1.1 holds whenever \(d > k\) but yields only a \((kF, D_d)\)-decomposition, weaker than the NDT Conjecture. To understand the relationship between the two problems and develop the statement of Theorem 1.1, we compare \(\text{Mad}(G)\) and \(\text{Arb}(G)\). Always \(\text{Mad}(G) < 2\text{Arb}(G)\), but the conditions \(\text{Mad}(G) < 2a\) and \(\text{Arb}(G) \leq a\) are not equivalent.

To compute \(\text{Arb}(G)\) or \(\text{Mad}(G)\), it suffices to perform the maximization only over induced subgraphs. Hence we restate the desired bounds in terms of integer computations and vertex subsets, writing \(|A|\) for the number of vertices in a vertex subset \(A\) and \(\|A\|\) for the number of edges in the subgraph of \(G\) induced by \(A\). Since \(k(k+d+1) + d = (k+1)(k+d)\), we have the following comparison, introducing an intermediate condition we call \((k, d)\)-sparse:

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equivalent constraint (when imposed for all (A \subseteq V(G)))</th>
</tr>
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<tbody>
<tr>
<td>(\text{Arb}(G) \leq k + \frac{d}{k+d+1})</td>
<td>((k+1)(k+d)\</td>
</tr>
<tr>
<td>(\text{Mad}(G) &lt; 2k + \frac{2d}{k+d+1})</td>
<td>((k+1)(k+d)\</td>
</tr>
<tr>
<td>((k, d))-sparse</td>
<td>((k+1)(k+d)\</td>
</tr>
</tbody>
</table>

Since \((k+1)(k+d) > k^2 \geq 1\), the condition on \(\text{Arb}(G)\) implies \((k, d)\)-sparseness, which in turn implies the condition on \(\text{Mad}(G)\). Theorem 1.1 thus implies that \(\text{Arb}(G) \leq k + \frac{d}{k+d+1}\) suffices for \(G\) to be \((kF, D_d)\)-decomposable, but \(\text{Mad}(G) < 2k + \frac{2d}{k+d+1}\) might not. On the other hand, since \(k^2 = 1\) when \(k = 1\), the \((1, d)\)-sparseness condition is the same as the condition on \(\text{Mad}(G)\) solving the problem in [10], as mentioned earlier.
Theorem 1.1. If a graph $G$ is $(k, d)$-sparse, where $d > k$, then $G$ is $(kF, D_d)$-decomposable. If $d = k + 1$, then also $G$ is $(kF, F_d)$-decomposable.

Our proof of Theorem 1.1 in Section 2 is purely inductive, proving a technically stronger statement. An enhanced proof in Section 3 yields the NDT Conjecture for the case $d = k + 1$. In Sections 4–6, we prove the NDT Conjecture for $(k, d) = (1, d)$ with $d \leq 6$, in a form that requires only $(k, d)$-sparseness as long as small graphs violating Arb($G$) $\leq k + 1$ are forbidden. Meanwhile, the Strong NDT Conjecture asserts that Arb($G$) $\leq k + \frac{d}{k+d+1}$ guarantees a $(kF, F_d)$-decomposition where every component of the forest in $F_d$ has at most $d$ edges. We prove this for $(k, d) = (1, 2)$ in Section 7 (the result of [11] implies it for $(k, d) = (1, 1)$). These results use reducible configurations and discharging.

2 $(kF, D_d)$-decomposition

We prove Theorem 1.1 in a seemingly more general form, but we will see in Section 4 that it is equivalent to Theorem 1.1. Prior results in this area have been proved by the discharging method, which uses properties of a minimal counterexample $G$ to contradict the hypothesized sparseness. Replacing the constant bound $d$ on vertex degrees by an individual bound for each vertex permits a simple inductive proof without using discharging.

Definition 2.1. Fix positive integers $d$ and $k$. A capacity function on a graph $G$ is a function $f: V(G) \to \{0, \ldots, d\}$. Let $D_f$ be the family of graphs with vertex set $V(G)$ such that $d_D(v) \leq f(v)$ for each vertex $v$. For each vertex set $A$ in a graph $G$ with capacity function $f$, let

$$\beta_f(A) = (k + 1) \sum_{v \in A} (k + f(v)) - (k + d + 1) \|A\| - k^2.$$ 

A capacity function $f$ on $G$ is feasible if $\beta_f(A) \geq 0$ for all nonempty $A \subseteq V(G)$.

For $B \subseteq V(G)$, let $G_B$ denote the graph obtained by contracting $B$ into a new vertex $z$. The degree of $z$ in $G_B$ is the number of edges joining $B$ to $V(G) - B$ in $G$; edges of $G$ with both endpoints in $B$ disappear.

In this section, we use induction on the number of edges and vertices to show that a graph with a feasible capacity function $f$ is $(kF, D_f)$-decomposable. In Section 3 we modify the argument to obtain a stronger conclusion in a special case, proving the case $d = k + 1$ of the NDT Conjecture. We begin with several lemmas that will facilitate the induction proofs.

Lemma 2.2. If $f$ is a feasible capacity function on $G$, and $B$ is a proper subset of $V(G)$ such that $|B| \geq 2$ and $\beta_f(B) \leq k$, then $f'$ is a feasible capacity function on $G_B$, where $f'(z) = 0$ and $f'$ agrees with $f$ on $V(G) - B$. 

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Proof. For $A \subseteq V(G_B)$, we have $\beta_f(A) = \beta_f(A') \geq 0$ if $z \notin A$. When $z \in A$, we compute $\beta_f(A)$ by comparison with $\beta_f(A')$, where $A' = (A - \{z\}) \cup B$. Every edge in the induced subgraph $G[A']$ appears in exactly one of $G_B[A]$ and $G[B]$; hence the edges contribute the same to both sides of the equation below. Comparing the terms for constants and the terms for vertices (using $f'(z) = 0$) yields

$$\beta_f(A') = \beta_f(A) - (k+1)k + \beta_f(B) + k^2.$$ 

If $\beta_f(B) \leq k$, then $\beta_f(A) \geq \beta_f(A') \geq 0$. \hfill \Box

Our graphs have no loops, so when $B$ consists of a single vertex $v$ we have $\beta_f(B) = k + (k+1)f(v) \geq k$. If $\beta_f(B) = k$, then $f(v) = 0$ and $f' = f$. Thus the conclusion still holds, but we apply Lemma 2.2 only when $|B| \geq 2$.

**Lemma 2.3.** Let $f$ be a capacity function on a graph $G$ with vertex subset $B$. If $G[B]$ is $(k\mathcal{F}, \mathcal{D}_f)$-decomposable and $G_B$ is $(k\mathcal{F}, \mathcal{D}'_f)$-decomposable, where $f'$ is defined from $f$ as in Lemma 2.2, then $G$ is $(k\mathcal{F}, \mathcal{D}_f)$-decomposable.

Proof. Let $(F, D)$ be a $(k\mathcal{F}, \mathcal{D}_f)$-decomposition of $G[B]$, and let $(F', D')$ be a $(k\mathcal{F}, \mathcal{D}'_f)$-decomposition of $G_B$. Each edge of $G$ is in $G[B]$ or $G_B$, becoming incident to $z$ in $G_B$ if it joins $B$ to $V(G) - B$ in $G$. View $(F \cup F', D \cup D')$ as a decomposition of $G$ by pulling the edges incident to $z$ in $F'$ back to $G$.

The resulting decomposition is a $(k\mathcal{F}, \mathcal{D}_f)$-decomposition of $G$. Since $f'(z) = 0$, we have $d_{D'}(z) = 0$, and all edges joining $B$ to $V(G) - B$ lie in $F'$. Hence the restrictions from $f$ are satisfied by $D \cup D'$. For each forest $F_i$ among the $k$ forests in $F$, its union with the corresponding forest $F_i'$ in $F'$ is still a forest, since otherwise contracting the portion in $F_i$ of a resulting cycle would yield a cycle through $z$ in $F_i'$ when viewed as a forest in $G'$. \hfill \Box

**Theorem 2.4.** If $d > k$ and $G$ is a graph with a feasible capacity function $f$, then $G$ is $(k\mathcal{F}, \mathcal{D}_f)$-decomposable.

Proof. We use induction on the number of vertices plus the number of edges; the statement is trivial when there are at most $k$ edges. For the induction step, suppose that $G$ is larger.

We argue first that $\beta_f(B) \geq k + 1$ for every proper subset $B$ of $V(G)$ such that $|B| \geq 2$. If $\beta_f(B) \leq k$, then the capacity function $f'$ on $G_B$ that agrees with $f$ except for $f'(z) = 0$ is feasible, by Lemma 2.2. Since $G[B]$ is an induced subgraph of $G$, the restriction of $f$ to $B$ is feasible on $G[B]$. Since $G_B$ and $G[B]$ are smaller than $G$, the induction hypothesis yields that $G_B$ is $(k\mathcal{F}, \mathcal{D}_{f[B]})$-decomposable and $G_B$ is $(k\mathcal{F}, \mathcal{D}'_f)$-decomposable. By Lemma 2.3, $G$ is $(k\mathcal{F}, \mathcal{D}_f)$-decomposable.
Hence we may assume that $\beta_f(B) \geq k+1$ for all such $B$. Let $S = \{v \in V(G): f(v) > 0\}$; we will show that $S$ is independent. If $S$ contains adjacent vertices $u$ and $v$, then let $f'$ be the capacity function on $G - uv$ that agrees with $f$ except for $f'(u) = f(u) - 1$ and $f'(v) = f(v) - 1$. If $f'$ is feasible, then since $G - uv$ is smaller than $G$, it has a $(kF, D_f)$-decomposition, and we add $uv$ to the second subgraph to obtain a $(kF, D_f)$-decomposition of $G$.

To show that $f'$ is feasible, consider $A \subseteq V(G') = V(G)$. If $u, v \notin A$, then $\beta_f(A) = \beta_f(A)$. If $u, v \in A$, then the reduction in $f$ and loss of one edge yield $\beta_f(A) = \beta_f(A) - 2(k+1) + (k+d+1) \geq \beta_f(A)$, where the last inequality uses $d > k$. If exactly one of $\{u, v\}$ is in $A$, then $A$ is a proper subset of $V(G)$. If $|A| \geq 2$, then $\beta_f(A) = \beta_f(A) - (k+1) \geq 0$. If $|A| = 1$, then $\beta_f(A) \geq k$, since $G'$ has no loops.

Hence we may assume that $S$ is independent. In this case, we show that $G$ decomposes into $k$ forests, yielding a $(kF, D_f)$-decomposition of $G$ in which the last graph has no edges. If $\Upsilon(G) > k$, then $V(G)$ has a minimal subset $A$ such that $|A| \geq k(|A| - 1) + 1$ (note that $|A| \geq 2$). By this minimality, every vertex of $A$ has at least $k + 1$ neighbors in $A$. Let $A' = S \cap A$. Since $S$ is independent, $|A| \geq (k+1)|A'|$. Taking $k+1$ times the first lower bound on $|A|$ plus $d$ times the second yields

$$(k + 1 + d)|A| \geq (k + 1)k(|A| - 1) + (k + 1) + d(k + 1)|A'|.$$ 

Now we compute

$$\beta_f(A) = (k+1)k|A| + (k+1)\sum_{v \in A'} f(v) - (k+d+1)|A| - k^2$$

$$\leq (k+1)k|A| + (k+1)d|A'|(k+1)k(|A| - 1) - (k+1) - d(k+1)|A'| - k^2$$

$$= (k+1)k - (k+1) - k^2 = -1.$$ 

This contradicts the feasibility of $f$, and hence the desired decomposition of $G$ exists.

Sharpness of Theorem 2.4 is shown by the following example.

**Example 2.5.** We construct a bipartite graph $G$ with partite sets $X$ and $Y$ of sizes $s$ and $t$, respectively. Let $s = t(k+d) - k+1$, so $|V(G)| = t(k+d+1) - k+1$. With $X = \{x_0, \ldots, x_{s-1}\}$ and $Y = \{y_0, \ldots, y_{t-1}\}$, make $x_i$ adjacent to $y_i, \ldots, y_{i+k}$, where indices are taken modulo $t$. Every vertex in $X$ has degree $k+1$, so $|E(G)| = (k+1)(k+d)t - k^2 + 1$.

To see that $G$ has no $(kF, D_d)$-decomposition, observe that deleting at most $dt$ edges from $G$ leaves at least $k(k+d)t + kt - k^2 + 1$ edges. However, $k$ forests in $G$ cover at most $k[t(k+d+1) - k]$ edges.
On the other hand, the feasibility condition just barely fails. Let \( f(v) = d \) for all \( v \in V(G) \). If \( |A| = 1 \), then \( \beta_f(A) = kd + k + d \). Choose \( A \) to minimize \( \beta_f(A) \) among subsets of \( V(G) \) with size at least 2. If \( d_{G[A]}(v) \leq k \) for some \( x_i \in A \), then \( \beta_f(A - v) < \beta_f(A) \), so all neighbors of vertices in \( X \cap A \) are also in \( A \). With \( s' = |A \cap X| \) and \( t' = |A \cap Y| \), we have

\[
\beta_f(A) = (k + 1)(k + d)(s' + t') - (k + d + 1)(k + 1)s' - k^2 \\
= (k + 1)(k + d)t' - k^2 - s'(k + 1) = (k + 1)(k + d)t' - s' - k + 1 - 1.
\]

We conclude that \( \beta_f(A) > 0 \) if and only if \( s' \leq (k + d)t' - k \). When \( t' = t \), this yields \( \beta_f(A) \leq 0 \) if and only if \( A = V(G) \).

If \( t' < t \), then each vertex of \( Y - A \) forbids all its neighbors from \( A \). For fixed \( t' \), we maximize \( s' \) and minimize \( \beta_f(A) \) by letting \( Y \cap A = \{ y_0, \ldots, y_{t'-1} \} \). Writing \( i = qt + r \) with \( q \geq 0 \) and \( 0 \leq r < t \), this allows \( x_i \in A \) only when \( 0 \leq r \leq t' - k \). With \( s = t(k + d) - k + 1 \), we have \( s' \leq (k + d)(t' - k + 1) \leq (k + d)t' - k \). 

Although we need the generality of the capacity function to facilitate the inductive proof of Theorem 2.4, and the desired statement about \((kF, D_d)\)-decomposition is a special case, in fact the special case with \( f(v) = d \) for all \( v \) implies the general statement, making Theorem 1.1 and Theorem 2.4 equivalent.

Suppose that Theorem 1.1 holds and that \( f \) is feasible on \( G \), with \( f(v) \leq d \) for all \( v \). Form \( G' \) by giving \( d - f(v) \) new neighbors to each vertex \( v \), with each new neighbor adjacent to \( v \) via \( k + 1 \) edges. We say that a vertex of degree \( k + 1 \) having only one neighbor (via all \( k + 1 \) incident edges) is a ghost, and a neighbor of \( v \) that is a ghost is a ghost neighbor of \( v \).

Define \( f'(w) = d \) for all \( w \in V(G') \); we claim that \( f' \) is feasible. Consider \( A' \subseteq V(G') \). Increasing the capacity at a vertex increases the value of \( \beta \) on a set, as does adding a ghost to the set when its neighbor is not in the set. On the other hand, when \( v \in A' \cap V(G) \), adding a ghost neighbor of \( v \) to \( A' \) adds 1 to \( |A'| \) and \( k + 1 \) to \( \|A'\| \); this increases \( \beta \) by \((k + 1)(k + d) - (k + d + 1)(k + 1)\), which equals \(-(k + 1)\). Therefore, to prove feasibility of \( f' \) it suffices to consider \( A' \) containing all the ghost neighbors of vertices in \( A \), where \( A = A' \cap V(G) \). Since each \( v \in A \) gains \( d - f(v) \) ghost neighbors,

\[
\beta_{G'}(A') = \beta_G(A) + (k + 1) \sum_{v \in A} (d - f(v)) - \sum_{v \in A} (k + 1)(d - f(v)) = \beta_G(A) \geq 0.
\]

Since we have assumed Theorem 1.1, \( G' \) now has a \((kF, D_d)\)-decomposition. Deleting the added ghost vertices yields a \((kF, D_f)\)-decomposition of \( G \).

In essence, we have shown that ghosts have the same effect as reduced capacity on the existence of decompositions. In Section 4, ghosts will provide an important tool for obtaining properties of minimal counterexamples.
3 The NDT Conjecture for $d = k + 1$

For the special case $d = k + 1$, we can strengthen the approach of Section 2 to require that the leftover graph also be a forest. Let $\mathcal{F}_f = \mathcal{F} \cap \mathcal{D}_f$. When seeking a decomposition in which the degree-bounded graph is also a forest, feasibility of $f$ is not sufficient.

In particular, if $G$ consists of two vertices and an edge of multiplicity $k + 2$, and $f(u) = f(v) = d$, then $\beta(A) \geq 0$ for all $A$, but $G$ does not decompose into $k + 1$ forests. We will need another auxiliary function that excludes such examples, and to facilitate the inductive proof we will place another technical requirement on the desired $(k\mathcal{F}, \mathcal{F}_f)$-decomposition.

**Definition 3.1.** Given a capacity function $f$ on $V(G)$ using capacities at most $d$, let $S = \{v \in V(G) : f(v) = d\}$. Let $\alpha_f(A) = k |A| - k - |A| + |A \cap S|$. Say that $f$ is strongly feasible when both $\beta_f(A)$ and $\alpha_f(A)$ are nonnegative for all nonempty $A \subseteq V(G)$, and $\alpha_f(A) \geq 1$ whenever $A \subseteq S$. Let $\hat{f}(A) = \min\{f(x) : x \in A\}$. A graph is strongly $(k\mathcal{F}, \mathcal{F}_f)$-decomposable if it has a $(k\mathcal{F}, \mathcal{F}_f)$-decomposition in which each component of the degree-bound ed forest has at most one vertex $v$ with $f(v) < d$.

With these definitions, we can state the main result of this section.

**Theorem 3.2.** If $d = k + 1$ and $f$ is a strongly feasible capacity function on a graph $G$, then $G$ has a strong $(k\mathcal{F}, \mathcal{F}_f)$-decomposition.

Before beginning the proof, we explain the necessity of the condition on $\alpha$ and the motivation for limiting each component of the last forest $D$ to at most one vertex outside $S$. Nonnegativity of $\alpha(A)$ states that $A$ has at most $|A \cap S|$ edges plus the number that $k$ forests can absorb. Each vertex of $A$ in $S$ permits one more edge in $D$, by allowing an edge joining two components. If $A \subseteq S$, then we reach the allowable spanning tree in $G[A]$ before the last vertex, so the the requirement must increase to $\alpha(A) \geq 1$ when $A \subseteq S$.

The “strong” decomposition condition facilitates restoring an edge $uv$ to a decomposition $(F, D)$ of $G - uv$. In Theorem 1.1, we deleted $uv$, reduced the capacities of $u$ and $v$, and added $uv$ to $D$ in the decomposition given by the induction hypothesis. Reduced capacity controls the vertex degrees, but it does not prevent cycles in $D$ when we replace $uv$.

The strong decomposition condition controls the cycles. We delete $uv$ only when one endpoint has capacity $d$. In $G - uv$, both endpoints have capacity less than $d$ and will be the only such vertex in their components in $D$; in particular, they will be in different components. We can thus add $uv$ to $D$; since one endpoint returns to capacity $d$, the strong condition continues to hold. This inductive approach will allow us to assume that no edge joins a vertex with capacity $d$ to a vertex with positive capacity. For such graphs, the hypotheses will yield a decomposition into $k$ forests, as in the final step of Theorem 1.1.
Since we consider now the case $d = k + 1$, we simplify the formula for $\beta_f$ to

$$
\beta_f(A) = (k + 1)[k|A| + f(A) - 2\|A\|] - k^2,
$$

where $f(A) = \sum_{v \in A} f(v)$. Recall that $\hat{f}(A) = \min\{f(x) : x \in A\}$. We prove a useful bound on $\beta_f$ in terms of $\alpha_f$.

**Lemma 3.3.** For a capacity function $f$ on a graph $G$ and $A \subseteq V(G)$,

$$
\beta_f(A) \leq (k + 1)[2\alpha_f(A) + \hat{f}(A) - |A \cap S|] + k.
$$

In particular, if $\alpha_f(A) \leq 0$ and $\beta_f(A) \geq 0$ with $A \nsubseteq S$, then $f(x) \geq |A \cap S|$ for all $x \in A$.

**Proof.** With $d = k + 1$, summing capacities over $x \in A$ yields $f(A) \leq (k + 1)|A \cap S| + \hat{f}(A) + k(|A - S| - 1)$ (the inequality is strict when $A \subseteq S$). Substituting this bound and the formula $|A| = k|A| - k - \alpha_f(A) + |A \cap S|$ into the formula for $\beta_f(A)$ (and simplifying) completes the proof. \hfill \Box

We need an analogue of Lemma 2.2, with $G_B$ as defined there.

**Lemma 3.4.** If $f$ is a strongly feasible capacity function on $G$, and $B$ is a proper subset of $V(G)$ with $|B| \geq 2$ such that $\alpha_f(B) = 0$, then $f'$ is strongly feasible on $G_B$, where $f'(z) = \hat{f}(B) - |B \cap S|$ and $f'$ agrees with $f$ on $V(G) - B$.

**Proof.** As observed in Lemma 3.3, $\hat{f}(B) \geq |B \cap S|$. Hence $f'(z) \geq 0$, and $f'$ is a capacity function. Also, $\hat{f}(B) = k + 1$ only if $B \subseteq S$, so $f'(z) \leq k$.

For $A \subseteq V(G_B)$, we have $\beta_{f'}(A) = \beta_f(A) \geq 0$ and $\alpha_{f'}(A) = \alpha_f(A) \geq 0$ if $z \notin A$. When $z \in A$, we compute the new values for $A$ by comparison with the old values for $A'$, where $A' = (A - \{z\}) \cup B$. As in Lemma 2.2, $|A'| = |A| - 1 + |B|$ and $\|A'\| = \|A\| + \|B\|$, where $\|A\|$ counts edges in $G_B$. Thus

$$
\alpha_f(A') = \alpha_{f'}(A) + \alpha_f(B);
\beta_f(A') = \beta_{f'}(A) - (k + 1)[k + f'(z)] + \beta_f(B) + k^2.
$$

Since $\alpha_f(B) = 0$, we obtain $\alpha_{f'}(A) \geq \alpha_f(A') \geq 0$, which suffices since $f'(z) < d$. By Lemma 3.3, $\alpha_f(B) = 0$ implies $\beta_f(B) \leq k + (k + 1)[\hat{f}(B) - |B \cap S|]$. Now the value of $f'(z)$ yields $\beta_{f'}(A) \geq \beta_f(A') \geq 0$. \hfill \Box

**Lemma 3.5.** Let $f$ be a capacity function on a graph $G$ with vertex subset $B$. If $G[B]$ is strongly $(kF, F_{[B]})$-decomposable and $G_B$ is strongly $(kF, F_{f'})$-decomposable, where $f'$ is defined from $f$ as in Lemma 3.4, then $G$ is strongly $(kF, F_f)$-decomposable.
Proof. Let \((F, D)\) be a strong \((k\mathcal{F}, \mathcal{F}_f)\)-decomposition of \(G[B]\), and let \((F', D')\) be a strong \((k\mathcal{F}, \mathcal{F}_f)\)-decomposition of \(G_B\). Each edge of \(G\) is in \(G[B]\) or \(G_B\), becoming incident to \(z\) in \(G_B\) if it joins \(B\) to \(V(G) - B\) in \(G\). Viewing \(F'\) and \(D'\) as subgraphs of \(G\), we show that \((F \cup F', D \cup D')\) is a strong \((k\mathcal{F}, \mathcal{F}_f)\)-decomposition of \(G\).

As in Lemma 2.3, the union of any forest \(F_i\) among the \(k\) forests in \(F\) with the corresponding forest \(F_i'\) in \(F'\) is still a forest, since otherwise contracting the portion in \(F_i\) of a resulting cycle would yield a cycle through \(z\) in \(F_i'\) when viewed as a forest in \(G'\). This argument applies also to \(D \cup D'\).

If \(f'(z) = d\), then \(\hat{f}(B) = d\) and \(B \subseteq S\), which yields \(f'(z) < d\). Since \((F', D')\) has the strong property, \(f'(z) < d\) implies that all other vertices in the component of \(D'\) containing \(z\) have capacity \(d\). Therefore, each component of \(D \cup D'\) in \(G\) has at most one vertex with capacity less than \(d\).

Since each component of \(D\) (within \(G[B]\)) has at most one vertex with capacity less than \(d\), in \(D\) each vertex \(v\) of \(B\) has at most \(|B \cap S|\) neighbors. By the definition of \(f'(z)\), it gains at most \(\hat{f}(B) - |B \cap S|\) neighbors in \(D'\); together \(v\) has at most \(f(v)\) neighbors in \(D \cup D'\). □

Proof of Theorem 3.2: If \(d = k + 1\) and \(f\) is a strongly feasible capacity function on a graph \(G\), then \(G\) has a strong \((k\mathcal{F}, \mathcal{F}_f)\)-decomposition.

Proof. We use induction on the number of vertices plus the number of edges; the statement is trivial when there are at most \(k\) edges. For the induction step, suppose that \(G\) is larger.

Recall that \(S = \{v \in V(G) : f(v) = d\}\). Let \(R = \{v \in V(G) : f(v) = 0\}\), and let \(T = V(G) - S - R\). We prove the structural claim that if \(G\) has no strong \((k\mathcal{F}, \mathcal{F}_f)\)-decomposition, then \(S\) is independent and no edge joins \(S\) and \(T\).

Suppose that \(G\) has an edge \(uv\) such that \(u \in S\) and \(v \in S \cup T\). We choose such an edge with \(v \in T\) if one exists; otherwise, \(v \in S\). Let \(G' = G - uv\), and let \(f'\) be the capacity function on \(G'\) that agrees with \(f\) except for \(f'(u) = f(u) - 1\) and \(f'(v) = f(v) - 1\). Note that \(u \notin \{x : f'(x) = d\}\). If \(f'\) is strongly feasible, then since \(G - uv\) is smaller than \(G\), it has a strong \((k\mathcal{F}, \mathcal{F}_f)\)-decomposition \((F, D)\). Since \(f'(u) < d\) and \(f(u) = d\), adding the edge \(uv\) to \(D\) yields a strong \((k\mathcal{F}, \mathcal{F}_f)\)-decomposition of \(G\).

To prove the structural claim, it thus suffices to show that \(f'\) is strongly feasible. We consider \(\alpha_{f'}(A)\) and \(\beta_{f'}(A)\). If \(|A| = 1\), then \(\alpha_{f'}(A) = |A \cap S|\), which suffices. Also, \(\beta_{f'}(A) = k + (k + 1)\hat{f}(A) \geq k > 0\).

Next consider \(A = V(G)\). With \(u, v \in A\), we compute \(\beta_{f'}(A) = \beta_{f}(A) \geq 0\). Also, \(\alpha_{f'}(A) < \alpha_{f}(A)\) requires \(u, v \in S\). Not all vertices satisfy \(f'(x) = d\), since \(f'(u) < d\). Therefore, having \(\alpha_{f}(A) \geq 1\) and \(\alpha_{f'}(A) \geq 0\) suffices, so we may assume that \(\alpha_{f}(A) = 0\).

Since \(A = V(G)\), we have \(u, v \in S\), and now the choice of \(uv\) in defining \(f'\) implies that no edges join \(S\) and \(T\). Since \(\alpha_{f}(A) = 0\) implies \(A \not\subseteq S\), we have \(R \cup T \neq \emptyset\). If \(R \neq \emptyset\), then
\[ \hat{f}(A) = 0; \] since \(|A \cap S| \geq 2\), Lemma 3.3 yields \( \beta_f(A) \leq (k + 1)[-2] + k < 0\), contradicting feasibility. Hence \( R = \emptyset \). Since no edges join \( S \) and \( T \), now \( G \) is disconnected, and we can combine strong decompositions of the components obtained from the induction hypothesis.

Finally, suppose \( 2 \leq |A| < |V(G)| \). If \( \alpha_f(A) = 0 \), then the capacity function \( f_A \) on \( G_A \) that agrees with \( f \) except for \( f_A(z) = \hat{f}(A) - |A \cap S| \) is strongly feasible, by Lemma 3.4. Also, the restriction of \( f \) to \( A \) is strongly feasible on \( G[A] \). Since \( G_A \) and \( G[A] \) are smaller than \( G \), by the induction hypothesis \( G[A] \) is strongly \((kF, F_{|A})\)-decomposable and \( G_A \) is strongly \((kF, F_{|A})\)-decomposable. By Lemma 3.5, \( G \) is strongly \((kF, F_f)\)-decomposable.

Hence we may assume that \( \alpha_f(A) > 0 \), and hence \( \alpha_f(A) \geq 0 \). If \( \beta_f(A) \geq 0 \), then \( A \) causes no problem. Otherwise, \( \beta_f(A) \leq k \), since reduction of \( \beta \) requires \( A \) to contain exactly one of \( \{u, v\} \), and the reduction is then by \( k + 1 \). Now let \( f^* \) be the capacity function on \( G_A \) that agrees with \( f \) except for \( f^*(z) = 0 \); note that \( \{x \in V(G_A) : f^*(x) = d\} = S - A \). By Lemma 2.2, \( f^* \) is feasible. For \( C \subseteq V(G_A) - \{z\} \), we have \( \alpha_f(C) = \alpha(C) \geq 0 \). For \( z \in C \subseteq V(G_A) \), we have \( f^*(C) = 0 \) and \( C \not\subseteq S - A \). If \( \alpha_f(C) < 0 \), then Lemma 3.3 yields \( \beta_f(C) \leq (k + 1)[-2 - |C \cap (S - A)|] + k < 0 \); contradicting that \( f^* \) is feasible. Thus \( f^* \) is strongly feasible. By the induction hypothesis, \( G_A \) has a strong \((kF, F_{f^*})\)-decomposition \((F, D)\) such that \( D \) has no edges incident to \( z \). Combining \((F, D)\) with a strong \((kF, F_{f|A})\)-decomposition of \( G[A] \) provided by the induction hypothesis yields a strong \((kF, F_f)\)-decomposition of \( G \) as in Lemma 3.5.

Hence we may assume that \( S \) is independent and that no edge joins \( S \) and \( T \). As in Theorem 2.4, we claim that \( G \) decomposes into \( k \) forests, completing the desired decomposition. If \( \Upsilon(G) > k \), then \( V(G) \) has a minimal subset \( A \) such that \( |A| \geq k(|A| - 1) + 1 \) (note that \( |A| \geq 2 \)). The minimality of \( A \) implies that every vertex of \( A \) has at least \( k + 1 \) neighbors in \( A \). Also, for any nonempty proper subset \( B \) of \( A \), it yields \( |A| > k(|A| - 1) \) and \( |A - B| \leq k(|A - B| - 1) \), and hence more than \( k|B| \) edges of \( A \) are incident to \( B \).

From the independence of \( S \), the absence of edges joining \( S \) and \( T \), and the fact that at least \( k |A \cap T| \) edges of \( A \) are incident to \( T \), we obtain \( |A| > (k + 1)|A \cap S| + k|A \cap T| \). Summing the two lower bounds on \( |A| \) yields

\[
2|A| > (k + 1)|A| - 1 + (k + 1)|A \cap S| + k|A \cap T|.
\]

Using the bounds on \( f(v) \) for \( v \) in each of \( R, S, T \), we compute

\[
\beta_f(A) = (k + 1)[k|A| + f(A) - 2|A|] - k^2
\]
\[
< (k + 1)[k - 1 + f(A) - (k + 1)|A \cap S| - k|A \cap T|] - k^2 \leq -1,
\]

which contradicts the feasibility of \( f \). \(\square\)
Corollary 3.6. The NDT Conjecture holds when \( d = k + 1 \).

Proof. We apply Theorem 3 with \( f(v) = d = k + 1 \) for all \( v \), so \( |A \cap S| = |A| \) for all \( A \subseteq V(G) \). As observed earlier, the condition \( \text{Arb}(G) \leq k + \frac{d}{k+d+1} = k + 1/2 \) implies that \( \beta_f(A) \geq 0 \) for all \( A \). In addition, with \( |A \cap S| = |A| \), the condition \( \alpha_f(A) \geq 1 \) becomes \( \|A\| \leq (k+1)(|A| - 1) \), which holds for all \( A \) when \( \text{Arb}(G) < k + 1 \).

\[ \square \]

4 Approach to \((k,F,F_d)\)-decomposition

For our remaining stronger conclusions in which the “leftover” subgraph \( D \) must also be a forest, the highly local approach of Section 2 that reserves one edge for \( D \) by reducing the degree capacity of its endpoints is not adequate. When \( d > k + 1 \), it becomes harder to avoid creating a cycle when replacing a reserved edge.

We use the inductive approach of obtaining reducible configurations (structures that are forbidden from minimal counterexamples) and then the discharging method, showing that the average degree in any graph avoiding the reducible configurations is too high. This method can also be used to prove Theorem 1.1, but such a proof would be lengthier than that in the previous section. On the other hand, it may settle the case \( k = d \) for \((k,F,F_d)\)-decomposition.

For this discussion, we modify \( \beta \) by removing the term independent of \( A \), and we drop the notation for the capacity function because each vertex will have capacity \( d \).

Definition 4.1. Let \( m_{k,d} = 2k + \frac{2d}{k+d+1} \). For a set \( A \) of vertices in a graph \( G \), the sparseness \( \beta_G(A) \) is defined by \( \beta_G(A) = (k+1)(k+d) |A| - (k+d+1) \|A\| \).

The term “sparseness” here is natural, because if \( \beta_G(A) \) is sufficiently large for all \( A \), then \( G \) is sufficiently sparse to satisfy the relevant bound on \( \text{Mad}(G) \) or \( \text{Arb}(G) \). Sparseness also distinguishes between the conditions on \( \text{Mad}(G) \) and \( \text{Arb}(G) \). As mentioned previously, \( \text{Arb}(G) \leq m_{k,d}/2 \) may fail when \( \text{Mad}(G) < m_{k,d} \) holds. The former requires a set \( A \) such that \( \beta_G(A) < (k+1)(k+d) \), while the latter requires only that \( \beta_G(A) \geq 1 \) for all \( A \).

Example 4.2. Let \( H \) be the (multi)graph consisting of \( q + 1 \) vertices in which one vertex has degree \((k+1)q\) and the others have degree \( k + 1 \) and form an independent set. We have \( \text{Arb}(H) = k + 1 \), but \( \text{Mad}(H) = 2(q(k+1))/(q + 1) \). If \( d < q < k + d \), then \( \text{Mad}(H) < m_{k,d} \), but \( H \) has no \((k,F,F_d)\)-decomposition.

This graph \( H \) can be excluded by requiring \((k,d)\)-sparseness (note that \( d < q < k + d \) requires \( k \geq 2 \), which is where \((k,d)\)-sparse and \( \text{Mad}(G) < m_{k,d} \) differ). For \( H \), we have \((k+1)(k+d) ||V(H)|| - (k+d+1) \|V(H)\| = (k+1)(k+d-q) \), which violates \((k,d)\)-sparseness if and only if \( q > d \). Furthermore, \( q > d \) if and only if \( H \) has no \((k,F,F_d)\)-decomposition.
Even $\beta_G(A) \geq k^2 ((k,d)\text{-sparseness})$ allows $\Upsilon(G) \leq k + 1$ to fail, but only on a small subgraph. Violating $\Upsilon(G) \leq k + 1$ requires an $r$-vertex subgraph with at least $(k+1)(r-1)+1$ edges. If such a graph is also $(k,d)$-sparse, then

$$(k+1)(k+d)r - (k+d+1)((k+1)(r-1)+1) \geq k^2,$$

which simplifies to $r \leq \frac{k}{k+1}(d+1)$.

In the cases where we can guarantee a $(kF,F_d)$-decomposition, we obtain a stronger statement than the case $(k,d)$ of the NDT Conjecture by weakening the hypothesis to require only $(k,d)$-sparseness, while excluding multigraphs with at most $(d+1)k/(k+1)$ vertices that satisfy this bound but fail to decompose into $k+1$ forests.

**Definition 4.3.** Fix $k,d \in \mathbb{N}$. A graph $G$ is feasible if $\beta_G(A) \geq k^2$ for all nonempty $A \subseteq V(G)$. A set $A \subseteq V(G)$ is overfull if $\|A\| > (k+1)(|A|-1)$.

Now that we are fixing $(k,d)$, “feasible” is a convenient abbreviation for “$(k,d)$-sparse”. Theorem 1.1 showed that feasible graphs are $(kF,D_d)$-decomposable (when $d > k$), and by Example 2.5 this condition on $\beta_G$ is sharp. Graphs with overfull sets are not $(kF,F_d)$-decomposable. We have noted that the bound $\text{Arb}(G) \leq m_{k,d}/2$ both implies feasibility and prohibits overfull sets. Furthermore, feasibility prohibits overfull sets with more than $(d+1)k/(k+1)$ vertices. Hence the conjecture below is equivalent to the NDT Conjecture.

**Conjecture 4.4.** Fix $k,d \in \mathbb{N}$. If $G$ is feasible and has no overfull set with at most $(d+1)k/(k+1)$ vertices, then $G$ is $(kF,F_d)$-decomposable.

We will prove Conjecture 4.4 when $k = 1$ and $d \leq 6$. The advantage we gain when $k = 1$ is that $k^2 = 1$, so the feasibility condition reduces to $\beta_G(A) > 0$ for all $A$. We can then bring a variety of techniques to bear, including properties of submodular functions.

The basic framework of the proof holds for general $k$, so we maintain the general language throughout this section before specializing to $k = 1$. We do this to suggest the generalization to larger $k$ and because the proofs of these lemmas are as short for general $k$ as for $k = 1$.

We typically use $(F,D)$ to denote a $(kF,F_d)$-decomposition of $G$, where $F$ is a disjoint union of $k$ forests and $D$ is a forest with maximum degree at most $d$. Note that the hypotheses of Conjecture 4.4 remain satisfied under discarding edges or vertices.

**Definition 4.5.** A $j$-vertex is a vertex of degree $j$. Among the non-($F,F_d$)-decomposable graphs satisfying the hypotheses of Conjecture 4.4, a minimal counterexample is one that has the fewest ghosts among those with the fewest non-ghosts.
Ghosts help control \((k\mathcal{F}, \mathcal{F}_d)\)-decompositions, because such a decomposition must put one edge at a ghost into \(D\). Without loss of generality, the other \(k\) edges at the ghost may be placed arbitrarily into the forests in \(F\).

**Lemma 4.6.** A minimal counterexample \(G\) is \((k + 1)\)-edge-connected (and hence has minimum degree at least \(k + 1\)).

**Proof.** If \(G\) has an edge cut \(Q\) with size at most \(k\), then \((k\mathcal{F}, \mathcal{F}_d)\)-decompositions of the components of \(G - Q\) combine to form an \((k\mathcal{F}, \mathcal{F}_d)\)-decomposition of \(G\) by allowing each forest to acquire at most edge of the cut.

**Corollary 4.7.** In a minimal counterexample \(G\), a vertex with degree at most \(2k + 1\) cannot be a neighbor of a ghost.

**Proof.** If such a vertex \(v\) is also a ghost, then \(G\) has two vertices and is \((k\mathcal{F}, \mathcal{F}_d)\)-decomposable. Otherwise, the edges incident to \(v\) and not incident to the neighboring ghost form an edge cut of size at most \(k\), contradicting Lemma 4.6.

**Definition 4.8.** A \(j\)-neighbor of a vertex is a neighbor that is a \(j\)-vertex. A ghost neighbor of a vertex is a neighbor that is a ghost. Adding a ghost neighbor at a vertex \(v\) means adding to the graph a vertex of degree \(k + 1\) whose only neighbor is \(v\). For a vertex set \(A\) in a graph \(G\), contracting \(A\) to a vertex \(v^*\) means deleting all edges within \(A\) and replacing \(A\) with a single vertex \(v^*\) incident to all edges that joined \(A\) to \(V(G) - A\). Let \(G_A\) denote the graph obtained from \(G\) by contracting \(A\) to \(v^*\) and adding \(d\) ghost neighbors at \(v^*\).

**Lemma 4.9.** If \(G\) is feasible and \(\beta_G(A) \leq k(k + 1)\), then \(G_A\) is feasible.

**Proof.** For \(X \subseteq V(G_A)\), we show that \(\beta_{G_A}(X) \geq k^2\). Let \(S\) be the set of \(d\) ghost neighbors added at \(v^*\). If \(v^* \notin X\), then the inequality is hardest when \(S \cap X = \emptyset\), since each vertex of \(S\) adds \((k + 1)(k + d)\) to the sparseness of \(X - S\). With \(S \cap X = \emptyset\), we have \(\beta_{G'}(X) = \beta_G(X) \geq k^2\).

If \(v^* \in X\), then the inequality is hardest when \(S \subseteq X\), since each addition of a ghost to a set containing its neighbor reduces the sparseness by \(k + 1\). Before adding \(S\), contracting \(A\) to \(v^*\) loses \(|A| - 1\) vertices and \(|A|\) edges. Let \(X' = A \cup (X - S - v^*)\); note that \(X' \subseteq V(G)\). We compute

\[
\beta_{G_A}(X) = \beta_G(X') - (k + 1)(k + d)(|A| - 1) + (k + d + 1)|A| - d(k + 1)
\]

\[
= \beta_G(X') + k(k + 1) - \beta_G(A) \geq \beta_G(X') \geq k^2.
\]

**Lemma 4.10.** If \(G\) is a minimal counterexample, \(A \subseteq V(G)\), and \(G_A\) is \((k\mathcal{F}, \mathcal{F}_d)\)-decomposable, then \(G\) is \((k\mathcal{F}, \mathcal{F}_d)\)-decomposable.
Proof. Let \((F, D)\) be a \((k\mathcal{F}, \mathcal{F}_d)\)-decomposition of \(G_A\). Since \(v^*\) has \(d\) ghost neighbors in \(G_A\), its neighbors in \(D\) are only those ghosts; no edges of \(D\) join \(v^*\) to vertices of \(G\).

An induced subgraph of \(G\) cannot have more non-ghosts than \(G\), so \(G[A]\) has a \((k\mathcal{F}, \mathcal{F}_d)\)-decomposition \((F', D')\). Combining \((F', D')\) and \((F, D)\) (after deleting the ghost neighbors of \(v^*\)) forms a \((k\mathcal{F}, \mathcal{F}_d)\)-decomposition of \(G\). All edges joining \(v^*\) to \(V(G) - A\) lie in \(F\) and are incident to various vertices of \(A\). Since \(v^*\) lies on no cycle in \(F\), adding the edges of \(F'\) does not complete a cycle. That is, each forest in a \(k\mathcal{F}\)-decomposition of \(F\) can be combined with any one of the forests in a \(k\mathcal{F}\)-decomposition of \(F'\).

\(\square\)

**Definition 4.11.** Let \(d_G(v)\) denote the degree of a vertex \(v\) in a graph \(G\). A set \(A \subseteq G\) is *nontrivial* if \(A\) contains at least two non-ghosts but not all non-ghosts in \(G\).

We avoid confusion between the overall parameter \(d\) and the degree function by always using the relevant graph as a subscript when discussing individual vertex degrees.

**Lemma 4.12.** Let \(A\) be a vertex set in a minimal counterexample \(G\). If \(A\) is nontrivial, then \(\beta_G(A) > k(k+1)\). If \(A\) is trivial with exactly one non-ghost vertex \(v\), and \(\beta_G(A) \leq k(k+1)\), then \(d_G(v) \geq (k+1)(d+1)\).

**Proof.** Suppose that \(\beta_G(A) \leq k(k+1)\). By Lemma 4.9, \(G_A\) is feasible. If \(A\) is nontrivial, then \(G_A\) has fewer non-ghosts than \(G\). Since \(G\) is a minimal counterexample, \(G_A\) is \((k\mathcal{F}, \mathcal{F}_d)\)-decomposable. By Lemma 4.10, \(G\) is \((k\mathcal{F}, \mathcal{F}_d)\)-decomposable and hence is not a counterexample.

Now suppose that \(A\) is trivial with non-ghost vertex \(v\), so \(A\) consists of \(v\) and some number \(h\) of ghost neighbors of \(v\). Now \(\beta_G(A) = (k+1)(k+d-h)\), so \(\beta_G(A) \leq k(k+1)\) requires \(h \geq d\). If \(h > d\), then already \(d_G(v) \geq (k+1)(d+1)\). If \(h = d\) and \(A = V(G)\), then \(G\) is explicitly \((k\mathcal{F}, \mathcal{F}_d)\)-decomposable. In the remaining case, \(G\) has vertices outside \(A\), and the only vertex of \(A\) with outside neighbors is \(v\). Since \(G\) is \((k+1)\)-edge-connected (by Lemma 4.6), we again have \(d_G(v) \geq (k+1)(d+1)\). \(\square\)

**Lemma 4.13.** If \(v\) is a vertex in a minimal counterexample \(G\), and \(d_G(v) < (k+1)(k+d)\), then \(v\) has no non-ghost \((k+1)\)-neighbor.

**Proof.** Let \(u\) be a non-ghost \((k+1)\)-neighbor of \(v\), and let \(W\) be the set of other neighbors of \(u\). Since \(d_G(u) = k+1\), no vertex in \(W \cup \{v\}\) is a ghost. Form \(G'\) from \(G\) by deleting the edges incident to \(u\) and then adding \(k+1\) edges joining \(u\) to \(v\); this makes \(u\) a ghost neighbor of \(v\) in \(G'\). Note that \(G'\) and \(G\) have the same numbers of edges and vertices, but \(G'\) has fewer non-ghost vertices than \(G\), since \(u\) and its neighbors are non-ghosts in \(G\) and at least \(u\) becomes a ghost in \(G'\).
If $G'$ is feasible, then the choice of $G$ implies that $G'$ has an $(k\mathcal{F},\mathcal{F}_d)$-decomposition $(F,D)$. Now modify $(F,D)$: delete the copies of $uv$ in $F$ (keeping the copy in $D$), and add the $k$ other edges at $u$ in $G$ to the $k$ forests in $F$. This yields a $(k\mathcal{F},\mathcal{F}_d)$-decomposition of $G$.

It thus suffices to show that $G'$ is feasible. We need only consider $A$ such that $u,v \in A$ and $W \not\subseteq A$; otherwise, $\beta_{G'}(A) \geq \beta_G(A) \geq k^2$, since $G$ is feasible. With $u \in A$, we have $\beta_{G'}(A) = \beta_G(A-u) - (k+1)$, since adding a ghost neighbor costs $k+1$. We worry only if $\beta_G(A-u) \leq k(k+1)$. Since $W \not\subseteq A$, the set $A$ does not contain all non-ghosts in $G$. If $v$ is the only non-ghost in $A-u$, then $d_G(v) \geq (k+1)(k+d)$, by Lemma 4.12. Since our hypothesis is $d_G(v) < (k+1)(k+d)$, we conclude that $A-u$ is nontrivial, and now Lemma 4.12 yields $\beta_G(A-u) > k(k+1)$.

**Lemma 4.14.** If a minimal counterexample $G$ has a vertex $v$ with $q$ ghost neighbors, where $q \geq 1$, then $d_G(v) > kq + k + d$.

*Proof.* Form $G'$ from $G$ by deleting the ghost neighbors of $v$. Since $G'$ is an induced subgraph of $G$, it is feasible. Forming $G'$ does not increase the number of non-ghost vertices, but it decreases the numbers of vertices and edges, so $G'$ has an $(k\mathcal{F},\mathcal{F}_d)$-decomposition $(F',D')$.

By Lemma 4.6, $d_{G'}(v) \geq k+1$. We may assume that $d_{D'}(v) \leq d_{G'}(v) - k$, since edges of $D'$ at $v$ can be moved arbitrarily to $F'$ until $F'$ has at least $k$ edges at $v$. Now restore each ghost vertex by adding one incident edge to each forest in $F'$ and the remaining incident edge to $D'$, yielding $(F,D)$.

Since $F \in k\mathcal{F}$ and $D \in \mathcal{F}$, it suffices to check $d_D(v)$. We have $d_D(v) = d_{D'}(v) + q \leq d_{G'}(v) - k + q = d_G(v) - kq - k$. Thus $d_D(v) \leq d$ unless $d_G(v) > kq + k + d$. \hfill \Box

If $v$ has $q$ ghost neighbors, then $d_G(v) \geq (k+1)q$. Hence the lower bound in Lemma 4.14 strengthens the trivial lower bound when $q \leq k + d$.

**Lemma 4.15.** If $G$ is a minimal counterexample, then two vertices in $G$ are joined by $k+1$ edges only when one of them is a ghost.

*Proof.* Since $G$ has no overfull set, edge-multiplicity is at most $k+1$. If two ghosts are adjacent, then $G$ has two vertices and is $(k\mathcal{F},\mathcal{F}_d)$-decomposable.

Suppose that non-ghosts $u$ and $v$ are joined by $k+1$ edges. Obtain $G'$ from $G$ by contracting these edges into a single vertex $v^*$ and adding a ghost neighbor $w$ to $v^*$.

We claim that $G'$ is feasible and has no overfull set. If $A \subseteq V(G') - \{v^*\}$, then $\beta_{G'}(A) \geq \beta_G(A - \{w\}) \geq k^2$. If $v^* \in A \subseteq V(G')$, then $\beta_{G'}(A) \geq \beta_{G'}(A \cup \{w\}) = \beta_G(A') \geq k^2$, where $A' = (A - \{v^*,w\}) \cup \{u,v\}$. Hence $G'$ is feasible.

Since $G$ has no overfull set, an overfull set in $G'$ must contain $v^*$, and a smallest such set $A$ does not contain $w$. Let $A' = (A - \{v^*\}) \cup \{u,v\}$. Now $A'$ has one more vertex than $A$
and induces $k + 1$ more edges in $G$ than $A$ induces in $G'$. Hence $A'$ is overfull if and only if $A$ is overfull. We conclude that $G'$ has no overfull set.

Since $G'$ has the same numbers of vertices and edges as $G$, but $G'$ has fewer non-ghosts than $G$, minimality of $G$ now implies that $G'$ has a $(k\mathcal{F}, \mathcal{F}_d)$-decomposition $(F, D)$. At $w$ there is one edge in each forest in $F$ and one edge in $D$. Replacing these with the edges joining $u$ and $v$ (one in each forest) yields a $(k\mathcal{F}, \mathcal{F}_d)$-decomposition of $G$, since the new degree of $u$ or $v$ in $D$ is at most $d_D(v^*)$, and an edge joining $u$ and $v$ completes a cycle in its forest only if contracting that edge yields a cycle in the corresponding forest in $(F, D)$. □

5 Discharging Argument and Submodularity

The lemmas of Section 4 provide a framework for a discharging argument. We would like to show that if $G$ has the structural properties of a minimal counterexample, then $\text{Mad}(G) \geq m_{k,d}$; this would prove the conjecture. We have not yet proved sufficient structural properties to complete the argument. By outlining a discharging argument, we will suggest what else is needed. Section 6 will complete the proof for $k = 1$ and $d \leq 6$.

Let $G$ be a minimal counterexample. Since $G$ is feasible, $\text{Mad}(G) < m_{k,d} = 2k + \frac{2d}{k+d+1}$. Give each vertex an initial charge equal to its degree in $G$ (by Lemma 4.6, each vertex has degree at least $k + 1$). We aim to redistribute charge to obtain a final charge $\mu(v)$ for each vertex $v$ such that $\mu(v) \geq m_{k,d}$. This motivates our first discharging rule.

**Rule 1:** A vertex of degree $k + 1$ takes charge $m_{k,d}/(k + 1) - 1$ along each incident edge from the other endpoint of that edge. This amount equals $\frac{k+d-1}{k+d+1}$.

In particular, a ghost takes total charge $m_{k,d} - (k + 1)$ from its neighbor. By force, Rule 1 increases the charge of each $(k + 1)$-vertex to $m_{k,d}$, since Lemma 4.13 implies that $(k + 1)$-vertices are not adjacent unless $G$ has just two vertices.

If all neighbors of $v$ have degree $k + 1$, then $\mu(v) = d_G(v)\frac{2}{k+d+1}$, since each edge takes $\frac{k+d-1}{k+d+1}$. In this case, $\mu(v) \geq m_{k,d}$ if and only if $d_G(v) \geq (k + 1)(k + d)$.

The problem is how to handle vertices with degree between $k + 1$ and $(k + 1)(k + d)$. Vertices with degree at most $2k$ need additional charge (as do vertices with degree $2k + 1$ when $d > k + 1$), though they do not need as much as $(k + 1)$-vertices need. Vertices with degree less than $(k + 1)(k + d)$ cannot afford to give away too much. The principle we need to quantify is that lower-degree vertices must have higher-degree neighbors.

A vertex $v$ with degree less than $(k + 1)(k + d)$ cannot be adjacent only to $(k + 1)$-vertices. By Lemma 4.13, $v$ has no non-ghost $(k + 1)$-neighbor. If $v$ has only ghost neighbors, then $G$ consists of one vertex plus ghost neighbors, but such a graph has the desired decomposition or is infeasible (see Example 4.2). Hence $v$ has some neighbors with higher degrees and will
not need to give away as much. More information is needed about the degrees of neighboring vertices to complete a proof.

When \((k, d) = (1, 1)\), only 2-vertices need charge. By Lemma 4.13, their neighbors have high enough degree that Rule 1 suffices to complete the discharging argument. Since a forest with maximum degree 1 is a matching, this proves the result of [11] that the Strong NDT Conjecture holds when \((k, d) = (1, 1)\).

When \(k = 1\) and \(d > 1\), only 2-vertices and 3-vertices need charge. This leads to a sufficient condition for completing the discharging argument.

**Theorem 5.1.** For \(d > k = 1\), let \(G\) be a minimal counterexample in the sense of Section 4. If each 3-vertex in \(G\) has a neighbor with degree at least \(d + 2\), then \(\text{Mad}(G) \geq m_{1,d} = 2 + \frac{d}{d+2}\).

**Proof.** In addition to the special case for \(k = 1\) of Rule 1 stated above, in which each 2-vertex receives \(\frac{d}{d+2}\) along each edge, we add a rule to satisfy 3-vertices.

**Rule 2:** If \(d_G(v) = 3\), and \(v\) has neighbor \(u\) with \(d_G(u) \geq d + 2\), then \(v\) receives \(\frac{d-2}{d+2}\) from \(u\).

We show that the final charge of each vertex is at least \(m_{1,d}\). Rules 1 and 2 ensure that \(\mu(v) \geq m_{1,d}\) when \(d_G(v) \in \{2, 3\}\) (since \(3 + \frac{d-2}{d+2} = 2 + \frac{2d}{d+2}\)). Since \(\frac{d-2}{d+2} < \frac{d}{d+2}\), the general argument for vertices with degree at least \(2d + 2\) also remains valid.

If \(4 \leq d_G(v) \leq 2d+1\), then \(v\) has no non-ghost 2-neighbor, by Lemma 4.13. If \(v\) has \(q\) ghost 2-neighbors with \(q \geq 1\), then \(d_G(v) \geq q + d + 2\), by Lemma 4.14. Hence \(\mu(v) = d_G(v) > m_{1,d}\) if \(4 \leq d_G(v) \leq d + 1\), since Rule 2 takes no charge from \(v\).

If \(d + 2 \leq d_G(v) \leq 2d + 1\), then \(v\) may give charge to \(q\) ghost neighbors (to each along two edges) and to \(d_G(v) - 2q\) neighbors of degree 3. Using Lemma 4.14,

\[
\mu(v) \geq d_G(v) - \frac{d}{d+2}2q - [d_G(v) - 2q]\frac{d-2}{d+2} = \frac{4(d_G(v) - q)}{d + 2} \geq \frac{4(d+2)}{d + 2} = 4 > m_{1,d}.
\]

The final charge at each vertex is at least \(m_{1,d}\), so no minimal counterexample is feasible. \[\Box\]

This reduces Conjecture 4.4 for the case \(k = 1\) to proving that in a minimal counterexample \(G\), each 3-vertex has a neighbor with degree at least \(d + 2\). Our proofs of this fact depend on \(d\). In each case, we will use submodularity properties of the function \(\beta_G\).

**Definition 5.2.** A function \(\beta\) on the subsets of a set is submodular if \(\beta(X \cap Y) + \beta(X \cup Y) \leq \beta(X) + \beta(Y)\) for all subsets \(X\) and \(Y\). When \(G'\) is an induced subgraph of \(G\), define the potential function \(\rho_{G'}\) by \(\rho_{G'}(X) = \min\{\beta_G(W) : X \subseteq W \subseteq V(G')\}\).

**Lemma 5.3.** For any graph \(G\) and any induced subgraph \(G'\) of \(G\), the sparseness function \(\beta_G\) on the subsets of \(V(G)\) is submodular.

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Proof. To compare \( \beta_G(X \cap Y) + \beta_G(X \cup Y) \) with \( \beta_G(X) + \beta_G(Y) \), note first that \( |X \cup Y| + |X \cap Y| = |X| + |Y| \). Hence it suffices to show that \( \|X \cup Y\| + \|X \cap Y\| \geq \|X\| + \|Y\| \). All edges contribute equally to both sides except edges joining \( X - Y \) and \( Y - X \), which contribute 1 to the left side but 0 to the right.

\[ \square \]

6 Neighbors of 3-vertices when \( k = 1 \)

Throughout this section, \( k = 1 \). For \( k = 1 \), feasibility reduces to the statement that \( \beta_G(A) = (2d + 2)|A| - (d + 2)\|A\| \geq 1 \) for \( A \subseteq V(G) \). When \( G \) is a minimal counterexample, Lemma 4.12 implies that \( \beta_G(A) \geq 3 \) when \( A \) is nontrivial (contains at least two non-ghosts but not all non-ghosts). Furthermore, if \( d \) is even, then always \( \beta_G(A) \) is even, so in that case we may assume \( \beta_G(A) \geq 4 \) when \( A \) is nontrivial. By Theorem 5.1, to prove the NDT Conjecture when \( k = 1 \) it suffices to prove that every 3-vertex in a minimal counterexample has a neighbor with degree at least \( d + 2 \).

Lemma 6.1. Fix \( d \) with \( 2 \leq d \leq 6 \), and let \( G \) be a minimal counterexample. If \( v \) is a 3-vertex in \( G \) and has no neighbor with degree at least \( d + 2 \), then \( v \) has two neighbors \( u \) and \( u' \) such that \( \rho_{G'}(\{u, u'\}) \geq d + 3 \), where \( G' = G - v \).

Proof. Together, Corollary 4.7 and Lemma 4.15 imply that every 3-vertex has three distinct neighbors. Let \( U \) be the neighborhood of \( v \), with \( U = \{u_1, u_2, u_3\} \). Let \( Z_i = U - \{u_i\} \). Suppose that \( \rho_{G'}(U_i) \leq d + 2 \) for all \( i \).

For each \( i \), let \( X_i \) be a subset of \( V(G') \) such that \( \rho_{G'}(Z_i) = \beta_G(X_i) \). For any permutation \( i, j, k \) of \( \{1, 2, 3\} \),

\[
2d + 4 \geq \beta_G(X_i) + \beta_G(X_j) \geq \beta_G(X_i \cup X_j) + \beta_G(X_i \cap X_j).
\]

For \( X' \subseteq V(G') \), let \( X = X' \cup \{v\} \). If \( U \subseteq X' \subseteq V(G') \), then \( \beta_G(X') = \beta_G(X) + d + 4 \). If \( X' \neq V(G') \), then \( X \neq V(G) \), and \( X \) is nontrivial if it has at least two non-ghosts, which by Lemma 4.12 would yield \( \beta_G(X') \geq d + 7 + \epsilon \), where \( \epsilon = 1 \) if \( d \) is even and \( \epsilon = 0 \) if \( d \) is odd. However, if \( X' = V(G') \), then we only have \( \beta_G(X') \geq d + 5 + \epsilon \).

Since each edge \( vu_i \) has multiplicity 1, no vertex in \( U \) is a ghost, and neither is \( v \). Since \( u_k \in X_i \cap X_j \) and \( d_G(u_k) < d + 2 \), Lemma 4.12 implies \( \beta_G(X_i \cap X_j) \geq 3 + \epsilon \). Since \( U \subseteq X_i \cup X_j \), we also conclude \( \beta_G(X_j) + \beta_G(X_j) \geq d + 8 + 2\epsilon \) for all \( d \), and the lower bound increases by 2 if \( X_i \cup X_j \neq V(G') \).

Thus \( \rho_{G'}(X_i) + \rho_{G'}(X_j) \geq d + 8 + 2\epsilon \). If \( d \leq 4 \), then \( d + 8 + 2\epsilon > 2d + 4 \), and the desired conclusion follows. Hence we may assume \( d \in \{5, 6\} \); furthermore, \( X_i \cup X_j = V(G') \) for all \( i, j \), since otherwise the lower bound on \( \beta_G(X_i) + \beta_G(X_j) \) again exceeds \( 2d + 4 \).
In more detail, the computation of Lemma 5.3 is

\[ \beta_G(X_i) + \beta_G(X_j) = \beta_G(X_i \cup X_j) + \beta_G(X_i \cap X_j) + (k + d + 1)m, \]

where \( m \) is the number of edges joining \( X_i - X_j \) and \( X_j - X_i \). If \( m \geq 1 \), then we obtain \( \beta_G(X_i) + \beta_G(X_j) \geq 2d + 10 > 2d + 4 \), which yields the desired conclusion. Hence \( m = 0 \) in each case. That is, each \( X_i \cap X_j \) is a separating set in \( G' \). (If \( G' \) is disconnected, then some edge incident to \( v \) is a cut-edge, which contradicts Lemma 4.6.) Furthermore,

\[ \beta_G(X_i \cap X_j) = \beta_G(X_i) + \beta_G(X_j) - \beta_G(X_i \cup X_j) \leq 2d + 4 - (d + 5 + \epsilon) = d - 1 - \epsilon. \]

Now let \( Z = X_1 \cap X_2 \cap X_3 \). Since \( X_i \cup X_j = V(G') \), any vertex of \( V(G') - Z \) misses exactly one of the three sets, so \( \{Z, \overline{X}_1, \overline{X}_2, \overline{X}_3\} \) is a partition of \( V(G') \). Since \( \beta_G(X_i) \leq d + 2 \) and \( \beta_G(V(G')) \geq d + 5 \), each \( \overline{X}_i \) is nonempty. So \( Z \neq V(G') \). If \( Z \) contains only one non-ghost, then feasibility requires it to have at most \( d \) ghost neighbors, and \( \beta_G(Z) \geq 2 \). Otherwise, since \( v \notin Z \), we conclude that \( Z \) is nontrivial, and hence \( \beta_G(Z) \geq 3 \).

Now, since \( \overline{X}_i \subseteq X_j \cap X_k \), submodularity yields

\[ 2d + 1 - \epsilon \geq \beta_G(X_i) + \beta_G(X_j \cap X_k) \geq \beta_G(V(G')) + \beta_G(Z) \geq d + 7. \]

We conclude that \( d \geq 6 + \epsilon \), which completes the proof for \( d \leq 6 \). \( \square \)

**Lemma 6.2.** If \( 3 \leq d \leq 6 \) and \( G \) is a minimal counterexample, then every 3-vertex has a neighbor with degree at least \( d + 2 \).

**Proof.** Let \( u_1, u_2, u_3 \) be the neighbors of a 3-vertex \( v \), and let \( U = \{u_1, u_2, u_3\} \). Suppose that \( d_G(u) \leq d + 1 \) for \( u \in U \). Since each edge \( vu_i \) has multiplicity 1, no vertex in \( U \) is a ghost vertex, and any edge induced by \( U \) has multiplicity 1 (Lemma 4.15).

Let \( G' = G - v \). By Lemma 6.1, we may assume by symmetry that \( \rho_{G'}(\{u_1, u_2\}) \geq d + 3 \). Form \( H \) from \( G' \) by adding an extra edge joining \( u_1 \) and \( u_2 \). For \( A \subseteq V(H) = V(G') \), we have \( \beta_H(A) = \beta_G(A) \) unless \( u_1, u_2 \in A \), but in the remaining case \( \rho_{G'}(\{u_1, u_2\}) \geq d + 3 \) yields \( \beta_H(A) \geq 1 \).

Hence \( H \) is feasible, and it has fewer non-ghosts than \( G \). To have an \((\mathcal{F}, \mathcal{F}_d)\)-decomposition of \( H \), we need only exclude overfull sets of size at most \( (d + 1)/2 \), which is at most 3. There are no triple-edges in \( H \), since \( G \) has no double-edges within \( U \). An overfull triple must include \( u_1 \) and \( u_2 \), since \( G \) has no overfull triple. The third vertex \( w \) must be adjacent to \( u_1 \) or \( u_2 \) by two edges in \( G \). Since those vertices are also adjacent to \( v \), we have contradicted \( d_G(u_1) = d_G(u_2) = 3 \).

Let \((F, D)\) be an \((\mathcal{F}, \mathcal{F}_d)\)-decomposition of \( H \). Obtain a decomposition of \( G \) by (1) replacing the added edge \( u_1u_2 \) with \( vu_1 \) and \( vu_2 \) in whichever of \( F \) and \( D \) contains it, and
(2) placing $v u_3$ in the other subgraph. The degree in $D$ of $u_1$ and $u_2$ is the same as a subgraph of $H$ or $G$, and cycles through $v$ would correspond to cycles in the decomposition of $H$. The only worry is $d_D(u_3)$, since we have increased this by 1 if the added edge in $H$ belonged to $F$. If $d_D(u_3)$ has increased to $d + 1$, then we have the desired conclusion unless $d_G(u_3) = d + 1$, but now we can move any one edge incident to $u_3$ from $D$ to $F$ to complete a $(F, F_d)$-decomposition of $G$. \[\square\]

7 The Strong NDT Conjecture for $(k, d) = (1, 2)$

In this section we prove our strongest conclusion for our most restrictive hypothesis. Many of the steps are quite similar to our previous arguments, so we put them all together in a single proof.

**Theorem 7.1.** The Strong NDT Conjecture holds when $(k, d) = (1, 2)$. That is, if $G$ is feasible, then $G$ has an $(F, F_d)$-decomposition $(F, D)$ in which every component of $D$ has at most two edges (a strong decomposition).

**Proof.** Since $m_{1,2} = 3$, feasibility is equivalent to $\text{Mad}(G) < 3$. Let $G$ be a counterexample with the fewest non-ghosts. By the argument of Lemma 4.6, $G$ is 2-edge-connected.

If $G$ has adjacent 2-vertices $u$ and $v$, then at least one is not a ghost. Letting $G' = G - \{u, v\}$, the minimality of $G$ yields a strong decomposition $(F, D)$ of $G'$. Adding the edge $uv$ to $D$ and the other edges incident to $u$ and $v$ to $F$ yields a strong decomposition of $G$.

If $G$ has a vertex with three ghost neighbors, then $G$ is infeasible, so every vertex has at most two ghost neighbors. If $G$ has only one non-ghost, then $G$ explicitly has a strong decomposition. Hence we may assume that $G$ has at least two non-ghosts.

Since $d$ is even, always $\beta_G$ is even, so feasibility can be stated as $\beta_G(A) \geq 2$ for $A \subseteq V(G)$ (here $\beta_G(A) = 6|A| - 4\|A\|$). A set $A$ is tight if $\beta_G(A) = 2$. A set consisting of a vertex with two ghost neighbors is a trivial tight set.

By Lemma 4.9, if $A$ is a tight set, then $G_A$ is feasible. The same argument as in Lemma 4.10 shows that if $G$ is a minimal counterexample, $A \subseteq V(G)$, and $G_A$ has a strong decomposition, then $G$ has a strong decomposition. Hence we may assume, as in the earlier proofs, that $\beta_G(A) \geq 4$ for every nontrivial set $A$.

Suppose that $G$ has a non-ghost 2-vertex $v$. Each neighbor of $v$ has degree at least 3. If a neighbor $u$ of $v$ has at most one ghost neighbor, then form $G'$ from $G - v$ by giving $u$ one additional ghost neighbor $w$. Now $G$ and $G'$ have the same numbers of vertices and edges, but $G'$ has fewer non-ghost vertices.

We claim also that $G'$ is feasible. If $u \notin A \subseteq V(G')$, then $\beta_{G'}(A)$ is minimized when $w \notin A$, and then $\beta_{G'}(A) = \beta_G(A) \geq 2$. If $u \in A \subseteq V(G')$, then $\beta_{G'}(A)$ is minimized when
$w \in A$, and then $\beta_{G'}(A) \geq \beta_{G}(A - \{w\} \cup \{v\}) - 2 \geq 2$, since $A - \{w\} \cup \{v\}$ is nontrivial.

We conclude that $G'$ has a strong decomposition $(F, D)$, by the minimality of $G$. Each of $F$ and $D$ must have one edge incident to $w$. We obtain a strong decomposition of $G$ by deleting $w$, adding $vu$ to $D$, and adding the other edge at $v$ to $F$.

We may therefore assume that every neighbor of a non-ghost 2-vertex has at least two ghost neighbors. Since $G$ is 2-edge-connected, a $q$-vertex cannot have $(q - 1)/2$ ghost neighbors. In particular, a vertex with at least two ghost neighbors must have degree at least 6, so every neighbor of a non-ghost 2-vertex has degree at least 6.

Once again we have derived many properties of a minimal counterexample. We complete the proof by using discharging to show that if $G$ has these properties, then $\text{Mad}(G) \geq 3$. This contradicts feasibility, which is equivalent to $\text{Mad}(G) < 3$; hence there is no minimal counterexample.

The initial charge of each vertex is its degree; we manipulate charge so that the final charge $\mu(v)$ of each vertex $v$ is at least 3. The only discharging rule is that a 2-vertex takes charge $1/2$ along each incident edge from the other endpoint of that edge. Hence the final charge of a 2-vertex is 3.

Since each neighbor of a non-ghost 2-vertex has degree at least 6, vertices of degree 3, 4, or 5 give charge only to ghosts. If $d_G(v) = 3$, then $v$ has no ghost neighbors, and $\mu(v) = 3$. If $d_G(v) \in \{4, 5\}$, then $v$ has at most one ghost neighbor, and $\mu(v) \geq d_G(v) - 1 \geq 3$. If $d_G(v) \geq 6$, then $v$ gives at most $1/2$ along each edge, so $\mu(v) \geq d_G(v) - d_G(v)/2 \geq 3$. \qed

References


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