Problem 1. (14 points)

Luke, Leia, Han and Chewie are stuck in the garbage compactor on the Death Star. When it begins to compact, the trash compactor is a cube. Find the rate at which the volume of the compactor is changing when the height of the compactor is 4, and the ceiling is moving towards them at rate \( \frac{2}{m/s} \), the width of the compactor is moving towards them at rate \( \frac{3}{m/s} \) and the length of room is moving in at rate \( \frac{1}{2} \frac{m}{s} \).

Solution. We have the following picture:

We know that \( H = L = W = 4 \) at this moment as the initial shape is a cube. We are also given that \( \frac{dH}{dt} = -2 \), \( \frac{dW}{dt} = -3 \), \( \frac{dL}{dt} = -\frac{1}{2} \). The volume of the compactor is given by \( V = LWH \). We want to find \( \frac{dV}{dt} \), so we use the method of related rates. Using the product rule we have:

\[
\frac{dV}{dt} = \frac{dL}{dt} WH + L \frac{dW}{dt} H + LW \frac{dH}{dt}
\]

\[
= \left( -\frac{1}{2} \right) (4)(4) + (4)(-3)(4) + (4)(4)(-2)
\]

\[
= -88
\]

Now, as our information had units, we need units for the answer to make sense. Thus, we conclude the volume of the compactor is changing at a rate of \(-88 \frac{m^3}{s}\).
Problem 2. (18 points)
Calculate the following limits. Carefully justify all steps in your calculations.

a) (6 points) \( \lim_{x \to \infty} \frac{\ln x + \tan^{-1} x}{x} \).

b) (6 points) \( \lim_{x \to 0^+} (\cos x)^{1/x} \).

c) (6 points) \( \lim_{x \to 0^+} (x)^{1/x} \).

Solution. a) We note that \( \ln x, x, \tan^{-1} x \) are all differentiable when \( x > 0 \), this limit also has indeterminate form \( \frac{\infty}{\infty} \), so we can apply L'Hôpital's Rule.

\[
\lim_{x \to \infty} \frac{\ln x + \tan^{-1} x}{x} = \lim_{x \to \infty} \frac{1}{x} + \frac{1}{1 + x^2} = 0
\]

b) This limit has indeterminate form \( 1^\infty \). If \( \lim_{x \to 0^+} (\cos x)^{1/x} = L \), we will find \( \ln(L) \) and exponentiate this. Thus we want to find

\[
\lim_{x \to 0^+} \frac{1}{x} \ln(\cos x) = \lim_{x \to 0^+} \frac{\ln(\cos x)}{x},
\]

Noting that \( \ln(\cos x), x \) are differentiable on an interval containing zero. (For instance \( \left( -\frac{\pi}{4}, \frac{\pi}{4} \right) \).) And this has indeterminate form \( \frac{0}{0} \), we can apply L'Hôpital's Rule.

\[
\lim_{x \to 0^+} \frac{\ln(\cos x)}{x} = \lim_{x \to 0^+} \frac{-\sin x}{\cos x} = \lim_{x \to 0^+} -\tan x = 0.
\]

Thus, as the exponential is continuous, we have

\[
\lim_{x \to 0^+} (\cos x)^{1/x} = e^{\lim_{x \to 0^+} \frac{\ln(\cos x)}{x}} = e^0 = 1.
\]

c) This has form \( ^0^\infty \), which is not indeterminate. In fact, this limit is zero.

\[
\lim_{x \to 0^+} (x)^{1/x} = 0.
\]
Problem 3. (14 points)
a) (6 points) State the Extreme Value Theorem.

b) (8 points) Find the absolute extrema of \( f(x) = e^{\sin x} \) on \( \left[ \frac{\pi}{4}, \frac{3\pi}{4} \right] \).

Solution. a) The Extreme Value Theorem states that a continuous function \( f \) defined on a closed, bounded interval \( [a, b] \) attains both an absolute maximum and an absolute minimum on that interval.

b) To find the absolute extrema we must find the critical numbers and compare the value of \( f \) at these points to \( f \) at the endpoints.

\[
f\left(\frac{\pi}{4}\right) = e^{\sqrt{2}/2} = f\left(\frac{3\pi}{4}\right)
\]

We need the derivative to find the critical numbers.

\[
f'(x) = e^{\sin x} \cos x
\]

This is a continuous function, so it always exists. Thus the only critical numbers will be when \( f'(x) = 0 \). As \( e^{\sin x} > 0 \) always, \( f'(x) = 0 \) when \( x = \frac{\pi}{2} \).

\[
f\left(\frac{\pi}{2}\right) = e^{1} = e
\]

Thus, \( f \) has an absolute maximum at \( x = \frac{\pi}{2} \) and absolute minimum at \( x = \frac{\pi}{4}, \frac{3\pi}{4} \).

\[\blacksquare\]

Problem 4. (12 points)
a) (6 points) State the Second Derivative Test with all hypotheses.

b) (6 points) Graphically explain why the Second Derivative Test makes sense. (Use a complete sentence as well.)

Solution. a) Suppose that \( f \) is continuous on the interval \((a, b)\) and \( f'(c) = 0 \) for some number \( c \) in \((a, b)\),

1. If \( f''(c) < 0 \), then \( f(c) \) is a local maximum.

2. If \( f''(c) > 0 \), then \( f(c) \) is a local minimum.

b) \( f'(c) = 0 \) means there is a horizontal tangent line. The value of the second derivative tells us the concavity of the function. If \( f''(c) < 0 \), the function is concave down and looks like the picture on the left, if \( f''(c) > 0 \), the function is concave up and looks like the picture on the right.
We can see that the picture on the left has a local maximum at $x = c$ and the picture on the right has a local minimum at $x = c$.

Problem 5. (12 points)
Given the following graphs of $f'(x)$ [left] and $f''(x)$ [right],

a) (4 points) Where is $f(x)$ increasing and decreasing?

b) (4 points) Where is $f(x)$ concave up and concave down?

c) (4 points) Classify all local extrema and inflection points. Be sure to justify your choices.

Solution. a) $f(x)$ is increasing when $f'(x) > 0$, which is on $(1, 5)$. $f(x)$ is decreasing when $f'(x) < 0$, which is on $(-\infty, 1) \cup (5, \infty)$.

b) $f(x)$ is concave up when $f''(x) > 0$, which is on $(-\infty, 3)$. $f(x)$ is concave down when $f''(x) < 0$, which is on $(3, \infty)$.

c) By the First Derivative Test, we have a local minimum at $x = 1$ (where the derivative goes from negative to positive) and a local maximum at $x = 5$ (where
the derivative goes from positive to negative.)

There is an inflection point at \((3, f(3))\) as this is where the second derivative changes signs.

**Problem 6. (12 points)**

To estimate the number \(\sqrt[3]{9}\),

\(\text{a) (6 points) State the function } f(x) \text{ and the point } x_0 \text{ you will use with Newton’s Method. Quickly explain your choices.} \)

\(\text{b) (6 points) Find } x_1 \text{ explicitly (i.e. give a simplified answer).} \)

**Solution.**

\(\text{a) Since Newton’s Method finds zeroes of functions and } \sqrt[3]{9} \text{ is a zero of } f(x) = x^3 - 9 \text{ and } f(2) = -1 \text{ is close to zero, we will choose } f(x) = x^3 - 9 \text{ and } x_0 = 2. \)

\(\text{b) We have } f'(x) = 3x^2, \text{ the formula for Newton’s Method gives us} \)

\[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \]

\[ = 2 - \frac{-1}{3(2)^2} \]

\[ = 2 + \frac{1}{12} = \frac{25}{12}. \]

**Problem 7. (14 points)**

Find the maximum area of a right triangle whose hypotenuse has length 5cm.

**Solution.** The situation can be described by the following picture.

As it is a right triangle, we have the relation \(x^2 + y^2 = 25\), and the area of the triangle is \(A = \frac{1}{2}xy\). We can assume that both \(x\) and \(y\) are positive as we want
to maximize the area, we have
\[ y = \sqrt{25 - x^2}, \]
\[ A(x) = \frac{1}{2} x \sqrt{25 - x^2}. \]

Further, we have that \(0 \leq x \leq 5\) so that our triangle has positive area. We first check the area at the endpoints,
\[ A(0) = 0 \]
\[ A(5) = 0 \]

We need to check the area at the critical numbers.
\[ A'(x) = \frac{1}{2} \sqrt{25 - x^2} + \left( \frac{1}{2} x \right) \left( \frac{1}{2} (25 - x^2)^{-1/2} (-2x) \right) \]
\[ = \left( \frac{1}{2} (25 - x^2)^{-1/2} \right) ((25 - x^2) - x^2) \]

This is undefined when \(x = 5\) and is zero when \(25 - 2x^2 = 0\), or \(x = \sqrt{\frac{25}{2}}\).
\[ A\left(\sqrt{\frac{25}{2}}\right) = \frac{1}{2} \sqrt{\frac{25}{2}} \sqrt{5 - \left(\sqrt{\frac{25}{2}}\right)^2} \]
\[ = \frac{25}{4}. \]

As this is positive, this is the maximum area of the triangle.

**Problem 8. (4 points)**

*Sketch a plausible graph given the following information.*

<table>
<thead>
<tr>
<th>(f''(x))</th>
<th>((-\infty, 4))</th>
<th>(4, 7)</th>
<th>(7, (\infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f'''(x))</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>