Math 220 A1 Exam III
Full NAME: (Please Print)

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**DIRECTIONS:**

- **DO NOT OPEN YOUR EXAM UNTIL YOU ARE TOLD TO DO SO!**

- Each exam is worth 100 points and has a total of 11 pages including cover.

- No work means no points.

- This is a closed book, closed notes exam. No calculators allowed.

- No hats or dark sunglasses.

- No cell phones. Turn them OFF now. If you are seen with a cell phone in hand during the exam or if your cell phone is heard, it will be considered cheating and you will be asked to leave.

- No other electronic devices – MP3 players, PDAs, etc. Same rules as phones.

- If you have a question, raise your hand and a proctor will come to you. Once you stand up, you are done with the exam. You will not be permitted to leave the room and return during the exam.

- Be sure to write your proper name CLEARLY and circle the discussion section for which you are registered.

- If you finish early, quietly and respectfully get up and hand in your exam. You need to show a picture ID with a clear picture when you turn in your exam.

- When time is up, you will be instructed to put down your writing utensil and close your exam. Anyone seen continuing to write after this announcement will have their exam marked and will lose all points for the page on which they are writing.
1. Given the following graphs of $f'(x)$ [left] and $f''(x)$ [right],

(a) (3 points) Where is $f(x)$ increasing and decreasing?

\[ (1, 5) \text{ increasing} \]
\[ (\infty, 1) \cup (5, \infty) \text{ decreasing} \]

(b) (2 points) Where is $f(x)$ concave up and concave down?

\[ (-\infty, 3) \text{ conc up} \]
\[ (3, \infty) \text{ conc down} \]

(c) (5 points) Classify all local extrema and inflection points. Be sure to justify your choices.

- **local min** $\leq x = 1$ by first der. test
- **local max** $\leq x = 5$ by first der. test
- **inflection points** $x = 3$ since $f'' > 0$ for $x < 3$ and $f'' < 0$ for $x > 3$
(d) (5 points) Sketch a possible graph of $f$. Make sure to identify the $x$ coordinate of all local extrema and inflection points.
2. (a) (10 points) Suppose that \( f(x) = \sin x \). Find the linear approximation of \( f(x) \) at \( x = 0 \).

\[
\begin{align*}
\frac{f'(x)}{x=0} &= \cos x = 1 \\
L(x) &= f(0) + f'(0)(x-0) \\
&= 0 + 1(x-0) = x \\
L(x) &= x.
\end{align*}
\]

(b) (10 points) Use Newton's Method to find the first approximation to the root of the equation \( e^x = -x \), starting with \( x = 0 \). Fully simplify your answer.

\[
\begin{align*}
f(x) &= e^x + x \\
f'(x) &= e^x + 1 \\
f'(0) &= e^0 + 1 = 2 \\
f(0) &= e^0 + 0 = 1
\end{align*}
\]

\[
\begin{align*}
x_1 &= 0 - \frac{f(0)}{f'(0)} = 0 - \frac{1}{2} = \frac{1}{2} \\
x_1 &= \frac{1}{2}.
\end{align*}
\]
3. Evaluate the following limits.

(a) (5 points) \( \lim_{x \to 0^+} \left( \frac{1}{x} \right)^{\frac{1}{x}} \Rightarrow \infty \Rightarrow \infty \).

(b) (5 points) \( \lim_{t \to \infty} (te^t - t) \Rightarrow \infty - \infty \).

\[
\begin{align*}
= \lim_{t \to \infty} t (e^t - 1) & \Rightarrow \infty \cdot 0 \\
= \lim_{t \to \infty} \frac{e^t - 1}{1/t} & \Rightarrow \frac{0}{0} \\
\xrightarrow{\text{Hosp}} \lim_{t \to \infty} \frac{e^t \cdot (\frac{1}{t^2})}{-\frac{1}{t^2}} &= e^0 = 1
\end{align*}
\]
4. (10 points) **Glass is a good insulator**  
Insulator is a material that resists the flow of electric current. Materials such as glass, paper or teflon are very good electrical insulators. These materials are used to support or separate electrical conductors without passing current through themselves.

B & W Microchips builds microchip boards. They want to use glass to insulate a rectangular area of \( \frac{3}{2} \) square inches with a glass border, and then divide it in half with a glass segment, as to have two insulated rectangles next to each other (the glass segments are only on the border and the dividing line). Find the dimension that will minimized the amount of glass used.

\[
\begin{align*}
\text{P} &= 3x + 2y \\
A &= xy = \frac{3}{2} \\
\frac{d}{d x} y &= \frac{3}{2} x \\
\text{P} &= 3x + 2\left(\frac{3}{2} x\right) = 3x + \frac{16}{x} \\
\frac{dP}{dx} &= 3 - \frac{6}{2x^2} = 0 \\
x^2 &= \frac{1}{2} \\
x &= \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \\
y &= \frac{3}{2} \\
\text{length} \to x &= \frac{3}{2} \text{ in} \\
\text{width} \to y &= \frac{1}{\sqrt{2}} \text{ in}
\end{align*}
\]

\[
p''(x) = \frac{6}{x^2} > 0
\]

so \( P(1) > 0 \) \( \Rightarrow \) \( x = 1 \) is a min.
5. (a) (4 points) State the two versions of the Fundamental Theorem of Calculus with all hypotheses.

\[ V1. \quad \text{If } f(x) \text{ is continuous on } [a, b], \text{ then } g(x) = \int_a^b f(t) \, dt \text{ is continuous on } [a, b], \text{ and for } x \in (a, b) \text{ differentiate } \frac{d}{dx} g(x) = f(x). \]

\[ V2. \quad \text{If } f \text{ is continuous on } [a, b], \text{ then } \int_a^b f(x) \, dx = F(b) - F(a), \text{ where } F \text{ is any antiderivative of } f. \]

(b) (6 points) At what point \( x \) does the function \( g(x) = \int_0^x \cos \frac{\pi}{2} \, dt \) have a horizontal tangent line? (Hint: Fundamental Theorem of Calculus)

\[ \frac{d}{dx} g(x) = \left( \frac{d}{du} \int_0^u \cos \frac{\pi}{2} \, dt \right) \cdot \frac{du}{dx} = \frac{\pi}{2} \cos \left( \frac{\pi x}{2} \right) = 0 \]

\[ \cos \left( \frac{\pi x}{2} \right) = 0 \Rightarrow \frac{\pi x}{2} = \frac{\pi}{2} \Rightarrow x = 1 \pm 2k \text{ for } k \text{ integer } \]

\( x = 1 \) Acceptable Answer.
6. Let \( v(t) = \cos t \) be the velocity function of an object that moves vertically along a straight line.

(a) (2 points) Sketch a graph of the velocity function over the time interval \([0, 3\pi/2]\).

\[
\begin{align*}
\text{Graph of } v(t) = \cos t
\end{align*}
\]

(b) (4 points) Find the total displacement of the object over the time interval

\[
\int_{0}^{3\pi/2} \cos t \, dt = \sin t \bigg|_{0}^{3\pi/2} = \sin \frac{3\pi}{2} - \sin 0 = -1
\]

(c) (4 points) Compute the total distance traveled by the object over the time interval.

\[
\left| \int_{0}^{\pi/2} \cos t \, dt \right| + \left| \int_{\pi/2}^{3\pi/2} \cos t \, dt \right|
\]

\[
\begin{align*}
&= \Big| \sin t \bigg|_{0}^{\pi/2} + \sin t \bigg|_{\pi/2}^{3\pi/2} \\
&= \sin \frac{\pi}{2} - \sin 0 + \sin \frac{3\pi}{2} - \sin \frac{\pi}{2} \\
&= 1 - 1 - 1 - 1 = -2
\end{align*}
\]

\[\omega = 1 + 2 = 3.\]
7. Evaluate the following integrals

(a) (5 points) \( \int (e^x + \frac{1}{2} - \sec^2 x) \, dx \)

\[ e^x + \ln |x| - \tan x + C. \]

(b) (5 points) \( \int t(t - 1)^{2010} \, dt \)

\[ u = t - 1 \]
\[ du = dt \]

\[ \int (u + 1) u^{2010} \, du = \int (u + u^{2011}) \, du \]

\[ = \frac{u^{2012}}{2012} + \frac{u^{2011}}{2011} + C \]

\[ = \frac{(t - 1)^{2012}}{2012} + \frac{(t - 1)^{2011}}{2011} + C. \]

(c) (5 points) A bird population starts with 100 birds and increases at a rate of \( n'(t) \) birds per semester. What does the quantity below represent? Justify your answer with a convincing argument based on the concepts learned in class.

**By the net change theorem** \( 100 + \int_0^{20} n'(t) \, dt \)

It represents the total population of birds 10 years from now. \( \frac{5}{2} \)
8. In this problem, hypothesis are given for both the Mean Value Theorem (MVT) and Rolle's Theorem (RT).

(a) I identify each theorem by writing MVT or RT next to each of them, and complete each theorem with the appropriate conclusion.

(a) (3 points) **RT** Let $f$ be a function that satisfies the following hypothesis:
   a) $f$ is continuous on the closed interval $[a, b]$
   b) $f$ is differentiable on the open interval $(a, b)$
   c) $f(a) = f(b)$, then...

   There is a $c \in (a, b)$ where $f'(c) = 0$.

(b) (3 points) **MVT** Let $f$ be a function that satisfies the following hypothesis:
   a) $f$ is continuous on the closed interval $[a, b]$
   b) $f$ is differentiable on the open interval $(a, b)$ then...

   There is a $c \in (a, b)$ where $f'(c) = \frac{f(b) - f(a)}{b - a}$

(c) (4 points) Choose one of these theorem and provide a graph and a short written argument to explain what's the meaning of the theorem you chose.

*Rolls:*

Since $f(a) = f(b)$ and $f$ is continuous and differentiable on $[a, b]$ there's no way to avoid having at least one horizontal tangent line.

*Mean Value Theorem:*

Since $f$ is continuous and differentiable on $(a, b)$ there has to be at least one $c$ where the tangent line has a slope equal to the slope of the line through $A$ and $B$.  

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Please go on to the next page...
9. This is an extra credit question for a maximum of 5 points total. Don’t do it unless you are sure you did everything possible from the other problems.

(a) Show that for all $0 \leq x \leq \frac{\pi}{2}$, the following inequality holds:

$$\tan x > x$$

(Hint: Prove two things: one of them is that the function $f(x) = \tan x - x$ is increasing on that interval.)

1. $f(0) = \tan(0) - 0 = 0$

2. $f'(x) = \sec^2 x - 1 > 0$ for all $x > 0$.

Since $f$ starts at $0$ and increases from there as $x$ gets bigger, it means $f$ is positive for $x$ on the given interval.

$\Rightarrow f(x) > 0 \Rightarrow \tan x > x$. 