Mapping Schemes Realizable by Obstructed Topological Polynomials

Gregory A. Kelsey

UIUC

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A **branched covering map** $f$ is a covering map at all but a finite number of points in the range (called **critical values**). At the preimages of the critical values, $f$ locally acts like $z \mapsto z^d$ at the origin, for some $d \geq 1$. If $d \geq 2$, then we call that preimage a **critical point**.
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A branched covering map $f$ is post-critically finite if for $C_f$ the set of critical points of $f$, then the post-critical set $P_f := \bigcup_{\omega \in C_f} \bigcup_{n \geq 1} f^n(\omega)$ is finite ($f^n$ denotes the composition of $f$ with itself $n$ times).
Theorem (Thurston’s Theorem ... for our purposes)

A topological polynomial $f$ is Thurston equivalent to a complex polynomial if and only if $f$ does not admit a Levy cycle.
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A Levy cycle for a topological polynomial $f$ is a multicurve $\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{l-1}\}$ on $S^2 \setminus P_f$ such that:

- each $f^{-1}(\gamma_i)$ has exactly one non-peripheral component $\tilde{\gamma}_{i-1}$ (subtracting mod $l$) in $\Gamma$,
- $\tilde{\gamma}_{i-1}$ is homotopic to $\gamma_{i-1}$,
- the map $f : \tilde{\gamma}_{i-1} \to \gamma_i$ has degree 1.
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So topological polynomials are either complex polynomials or obstructed polynomials.
Obstructed polynomials have remained relatively unstudied for the past 20-30 years. Recent work of Bonk and Meyer and also Haissinsky and Pilgrim have used obstructed polynomials to construct interesting metrics on the sphere.

So we would like to better understand obstructed topological polynomials.
The **mapping scheme** of a topological polynomial $f$ is a finite directed graph with vertex set equal to $C_f \cup P_f$. For each vertex $\omega$, the number of directed edges from $\omega$ to $f(\omega)$ equals the local degree of $f$ at $\omega$.

**Examples**

- $z^2$
  - $\mathbb{C}_\infty \rightarrow \mathbb{C}_0$

- $z^2 - 1$
  - $\mathbb{C}_\infty \rightarrow 0 \rightarrow -1$

- $z^2 + i$
  - $\mathbb{C}_\infty \rightarrow i \rightarrow i - 1 \rightarrow -i$
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A directed cycle in a mapping scheme is an **attractor** if it contains a critical point.
Question

Which mapping schemes are realizable by obstructed polynomials, and which are only realizable by complex polynomials?
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Theorem (Berstein-Levy, 1985)

A topological polynomial with all attractor cycles in its mapping scheme does not admit a Levy cycle.

Some similar (but much easier) topological arguments exist in very specific cases.
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Theorem (Koch private communication)

Given a preperiodic mapping scheme with a single finite critical point and the period having length \( n \geq 2 \), there exists a topological polynomial \( f \) with this mapping scheme that admits a Levy cycle.
Main Result

Theorem (K)

Suppose that a polynomial mapping scheme satisfies one of the following conditions:

1. at least one (non-attractor) period of length at least two and not containing critical values,
2. at least two (non-attractor) periods not containing critical values,
3. at least two non-attractor periods both of length at least two, or
4. at least four non-attractor periods.

Then this scheme is realized by an obstructed topological polynomial.
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Perhaps more importantly, the proof of this theorem allows us to construct a wealth of examples of obstructed polynomials.
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- Determine the equivalence classes of the constructed polynomials.
- Extend this method to maps that topologically resemble complex rational maps.