1. Parabolas

A **parabola** is the graph of all points that are the same distance from a fixed point (the **focus**) as they are from a fixed line (the **directrix**). See the picture below:

The **vertex** of a parabola is the midpoint of the line segment through the focus and perpendicular to the directrix.

Suppose that a parabola has vertex \((b, c)\), focus \((b, c + \frac{1}{4a})\), and directrix \(y = c - \frac{1}{4a}\).

a) For \((x, y)\) a point on the parabola, what is its distance to the focus?

b) For \((x, y)\) a point on the parabola, what is its distance to the directrix?
c) By the definition of a parabola, your answers to a) and b) must be equal. Thus, their squares must be equal to each other, as well. Use this fact to obtain a general form for such a parabola.

d) Use the general form you found in part c) to find an equation for the parabola with focus (3, 5) and directrix $y = 3$.

The general form you found in part c) is for a parabola with directrix parallel to the $x$-axis. For a parabola parallel to the $y$-axis with vertex $(c, b)$, focus $c + \frac{1}{4a}, b)$, and directrix $x = c - \frac{1}{4a}$, the general equation is $x = a(y - b)^2 + c$

e) In which direction does the parabola described by $12x + 6y + 24 = 0$ open? What is its focus? Directrix?
2. Ellipses

An ellipse is the graph of all points whose distance from two fixed points (called the foci) sum to the same constant $K$. The center of an ellipse is the midpoint of the line segment connecting the two foci. Extending this segment to where it meets the ellipse gives us the major axis; the minor axis is the segment perpendicular to the major axis through the center. The vertices are the points where the ellipse meets the major axis. See the picture below:

You’ve already seen that the equation for an ellipse centered at $(h, k)$ and major axis parallel to the $x$-axis is: 
\[
\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1, \quad \text{where } a > b > 0.
\]

a) What are the vertices of this ellipse?

b) At what points does the minor axis meet the ellipse?
c) Suppose the foci for such an ellipse are at \((h - c, k)\) and \((h + c, k)\). Using your answers from parts a) and b) and the definition of an ellipse, find the value of \(c\).

For an ellipse with major axis parallel to the \(y\)-axis, we use the general form \(\frac{(y - k)^2}{a^2} + \frac{(x - h)^2}{b^2} = 1\) in order to preserve the relation \(a > b > 0\)

d) Where are the vertices for this ellipse?

e) If this ellipse has foci at \((h, k \pm c)\), what is the equation for \(c\)?

f) Find the equation for the ellipse with foci \((2, 3)\) and \((6, 3)\) and vertices \((0, 3)\) and \((8, 3)\).
3. Hyperbolas

A hyperbola is the graph of all points whose distance from two fixed points (called the foci) have difference equal to the same constant $K$. So, this is the same as the ellipse from Problem 2, except that we are taking the difference of the distances, rather than their sum (it may be helpful to think of a hyperbola as an “inverted ellipse”).

For a hyperbola whose foci lie on a line parallel to the $x$-axis, the general form for the equation in rectangular coordinates is: \[
\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1, \]
with vertices at $(h \pm a, k)$.

a) Why is this form unsurprising? What do the vertices represent?

b) If the hyperbola has foci at $(h - c, k)$ and $(h + c, k)$, find the value of $c$.

Hyperbolas have an interesting feature that ellipses lack: asymptotes.

c) Write the equation of the hyperbola into a form giving $(y - k)^2$ as a function of $(x - h)^2$ (if you like, you may assume $h = k = 0$ for this section and then translate the hyperbola back to the correct center when you finish).
d) Using your answer to part c), find \( \lim_{x \to \pm \infty} \frac{(y - k)^2}{(x - h)^2} \).

e) How does your answer to part d) show that the **asymptotes** of the hyperbola are \( y = \pm \frac{b}{a}(x - h) + k \)?

f) What would be the general form and formula for foci and asymptotes of a hyperbola whose foci are on a line parallel to the \( y \)-axis?

g) Find the equation for the hyperbola with foci \((0, 1), (4, 1)\) and vertices \((1, 1), (3, 1)\).
4. For each equation, identify which type of conic section it describes and find all relevant data (foci, center, directrix, etc.)

a) \( y = 3(x - 4)^2 - 1 \)

b) \( \frac{(x + 2)^2}{9} + \frac{(y - 2)^2}{25} = 1 \)

c) \( \frac{x^2}{16} - \frac{(y + 1)^2}{9} = 1 \)

d) \( (x + 1)^2 + 4y^2 = 4 \)
5. Consider the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

a) Suppose $A > 0$ and $C > 0$. What types of conic sections could this equation describe?

b) Suppose $A > 0$ but $C < 0$. What types of conic sections could this equation describe?

c) What if $A < 0$, $C > 0$?

d) $A < 0$, $C < 0$?
e) What if one, but not both, of $A$ and $C$ equals 0?

f) What if both equal 0?

6. **Eccentricity**

Let $P$ be a fixed point and $l$ a fixed line not containing the point $P$. We are interested in the graph of all points whose distance from $P$ is a constant multiple $e > 0$ of their distance to the line $l$. For this reason, we will call $P$ the **focus** and $l$ the **directrix**. The constant multiple $e$ is called the **eccentricity**.

a) What kind of graph do you get if the eccentricity is $e = 1$?

Suppose $P$ is the origin and $l$ is the line $x = d$ for $d > 0$.

b) For $(x, y)$ a point on the graph, what is its distance from $P$?

c) For $(x, y)$ a point on the graph, what is its distance from $l$?
d) Using the definition of the eccentricity and your answers to parts b) and c), find an equation for this graph of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

e) Using your answers to part d) and to Problem 5, what kind of graph do we get if $e < 1$?

f) What kind of answer do we get if $e > 1$?

g) Take your answers to parts b) and c) and convert them into polar coordinates. Find a polar equation $r = f(\theta)$ that describes this graph.
This formula needs to be slightly modified for graphs with directrix not \( x = d > 0 \). A summary of these results is below:

The conic section with eccentricity \( e > 0 \), focus at \((0,0)\) has the formula corresponding to its directrix below:

- If the directrix is \( x = d > 0 \) then the polar equation is: \( r = \frac{ed}{e \cos(\theta) + 1} \).
- If the directrix is \( x = d < 0 \) then the polar equation is: \( r = \frac{ed}{e \cos(\theta) - 1} \).
- If the directrix is \( y = d > 0 \) then the polar equation is: \( r = \frac{ed}{e \sin(\theta) + 1} \).
- If the directrix is \( y = d < 0 \) then the polar equation is: \( r = \frac{ed}{e \sin(\theta) - 1} \).

7. Changing Eccentricity

Find the polar equation and graph the conic section with directrix \( x = 4 \) and the given eccentricity.

a) \( e = 0.5 \)

b) \( e = 0.9 \)

c) \( e = 1 \)
d) $e = 1.2$

e) $e = 2$