Problem (7 points) Consider a free undamped mass-spring system with \( m = 3 \) kg and \( k = 75 \) N/m. It is set in motion with initial position \( x(0) = 1 \) m and initial velocity \( x'(0) = -5 \) m/s. Find its position function \( x(t) \), and write in the form \( C \cos(\omega_0 - \alpha) \) with \( \alpha \in (0, 2\pi) \). Also, find the amplitude and the period.

solution)

\[
3x'' + 75x = 0.
\]

Characteristic equation: \( 3r^2 + 75 = 0 \Rightarrow r = \pm 5i \).

\[
x(t) = c_1 \cos 5t + c_2 \sin 5t.
\]

\[
x'(t) = -5c_1 \sin 5t + 5c_2 \cos 5t.
\]

\[
x(0) = 0: \quad c_1 = 1.
\]

\[
x'(0) = 0: \quad 5c_2 = -5 \Rightarrow c_2 = -1.
\]

Hence, \( x(t) = \cos 5t - \sin 5t \)

\[
= \sqrt{1^2 + 1^2} \cos(5t - \frac{7\pi}{4}) = \sqrt{2} \cos(5t - \frac{7\pi}{4}).
\]

The amplitude is \( \sqrt{2} \), and the period is \( \frac{2\pi}{5} \).
Problem 2 Find an appropriate form of a particular solution. Do not evaluate the coefficients.
(a) (3 points) \(4y'' + 4y' + y = 3xe^x\)

**Solution**

Characteristic equation: \(4r^2 + 4r + 1 = 0\)

\((2r + 1)^2 = 0 \Rightarrow r = -\frac{1}{2}, -\frac{1}{2}.\)

Hence, two linearly independent solutions in \(y_c\) are \(e^{-\frac{1}{2}x}\) and \(xe^{-\frac{1}{2}x}\).

Since \(f(x) = 3xe^x\), we can set up \(y_p = (Ax + B)e^x\).

You do not multiply \((Ax + B)e^x\) by \(x^4\), since there is no duplication.

Hence,

\[y_p = (Ax + B)e^x.\]

(b) (5 points) \(y^{(5)} - y^{(3)} = e^x + 2x^2 - 5 + \sin x\)

**Solution**

Characteristic equation: \(r^5 - r^3 = 0\)

\(r^3(r + 1)(r - 1) = 0 \Rightarrow r = 0, 0, 0, -1, 1.\)

Hence, linearly independent solutions in \(y_c\) are \(e^{0x} = 1\), \(x\), \(x^2\), \(e^{-x}\), and \(e^x\).

Set up a particular solution for each \(e^x\), \(2x^2 - 5\), \(\sin x\) separately.

\(e^x: y_{p1} = Axe^x\): because of the duplication \(e^x\) in \(y_c\), we need to multiply by \(x\).

\(2x^2 - 5: y_{p2} = x^3(Bx^2 + Cx + D)\): we need to multiply by \(x^3\) to eliminate duplication \(1, x, x^2\) in \(y_c\).

\(\sin x: y_{p2} = E\sin x + F\cos x\): there is no duplication in \(y_c\).

Hence,

\[y_p = Axe^x + x^3(Bx^2 + Cx + D) + E\sin x + F\cos x.\]