Practice Problems: Chapter 9 except 9.7
Problem 1 Let \( f(t) \) be the periodic function with period 4 defined by
\[
 f(t) = \begin{cases} 
 0 & \text{if } -2 < x < 0 \\
 2 & \text{if } 0 < x < 2.
\end{cases}
\]
Find its Fourier series.
Problem 2 Let $f(t) = t$ for $0 < t < \pi$. Sketch the even extension of $f(t)$, and find the Fourier cosine series of $f(t)$. 
Problem 3 Let \( f(t) \) be the function of period \( 2\pi \) defined by

\[
f(t) = \begin{cases} 
0, & -\pi < t < -\frac{\pi}{2} \\
t, & -\frac{\pi}{2} < t < \frac{\pi}{2} \\
0, & \frac{\pi}{2} < t < \pi.
\end{cases}
\]

We know that its Fourier series is

\[
\sum_{n \text{ odd}} \frac{2}{n^2 \pi} \sin\left(\frac{n\pi}{2}\right) \sin nt + \sum_{n \text{ even}} -\frac{1}{n} \cos\left(\frac{n\pi}{2}\right) \sin nt.
\]

a) Sketch the graph. To what value does the series converge at \( t = \frac{\pi}{2} \)?

b) Show that \( \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8} \).
Problem 4 Find the formal Fourier series solution of

\[ y'' + 4y = \sum_{n \text{ odd}} \frac{40}{n\pi} \sin nt. \]
Problem 5 Consider the following boundary value problem
\[
\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2,
\]
\[
\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(2,t) = 0, \quad \text{(endpoint conditions)}
\]
\[
u(x,0) = x. \quad \text{(initial condition)}
\]
(a) Assume that \( u(x,t) = X(x)T(t) \). Rewriting the differential equation and the endpoint conditions in terms of \( X(x) \) and \( T(t) \), find separate equations and endpoints conditions (if any) for \( X(t) \) and \( T(t) \).

(b) Find the solution satisfying endpoints conditions assuming that eigenvalues are non negative.
(c) Find the solution satisfying both endpoints conditions and the initial condition.
Problem 6 Consider the following boundary value problem

\[ u_{tt} = u_{xx}, \quad 0 < x < 1, \ t > 0, \]
\[ u(0,t) = u(1,t) = 0 \]
\[ u(x,0) = f(x), \quad u_t(x,0) = g(x). \]

(a) Show that the solution \( u(x,t) \) can be written as \( u(x,t) = v(x,t) + w(x,t) \), where \( v(x,t) \) is the solution of the same problem with \( g(x) = 0 \), and \( w(x,t) \) is the solution of the same problem with \( f(x) = 0 \).

(b) Solve the above boundary value problem with \( f(x) = 0 \), and \( g(x) = 4 \sin 2\pi x + 4 \sin 4\pi x \).