Practice problems for section 9.7

Problem 1 Find the solution of the following problem.

1) \[ u_{xx} + u_{yy} = 0, \quad 0 < x < 2, \quad 0 < y < 2 \]

2) \[ u(x, 0) = u(x, 2) = 0 \]

\[ u(0, y) = 0, \quad u(2, y) = \sin \frac{\pi y}{2}. \]

**solution:** Set \( u(x, y) = X(x)Y(y) \). Then, (1) tells us that

\[
\frac{X''Y + XY''}{X''} = 0.
\]

\[
\frac{X''}{X} = \frac{Y''}{Y} = -\lambda.
\]

So, we have

3) \[ Y'' + \lambda Y = 0, \]

4) \[ X'' - \lambda X = 0. \]

From (2), we know \( Y(0) = Y(2) = 0 \). Using these boundary conditions and the equation (3) above, we have a familiar eigenvalue problem

\[ Y'' + \lambda Y = 0 \]

\[ Y(0) = Y(2) = 0. \]

Eigenvalues: \( \lambda_n = \frac{n^2 \pi^2}{4} \)

Eigenfunctions: \( Y_n(y) = \sin \frac{n\pi y}{2} \).

We find from \( u(0, y) = 0 \) that \( X(0) = 0 \). Now plug in \( \lambda_n = \frac{n^2 \pi^2}{4} \) into the equation (4) and consider

\[ X'' - \frac{n^2 \pi^2}{x} = 0 \]

\[ X(0) = 0. \]

We know that

\[ X(x) = c_1 e^{\frac{n\pi x}{2}} + c_2 e^{-\frac{n\pi x}{2}}. \]

Since \( X(0) = c_1 + c_2 = 0, c_2 = -c_1. \)

Hence,

\[ X(x) = c_1 \left( e^{\frac{n\pi x}{2}} - e^{-\frac{n\pi x}{2}} \right). \]

Thus,

\[ X_n(x) = e^{\frac{n\pi x}{2}} - e^{-\frac{n\pi x}{2}}. \]
So,

\[ u(x, y) = \sum_{n=1}^{\infty} c_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} \left( e^{n\pi x} - e^{-n\pi x} \right) \sin \frac{n\pi y}{2}. \]

Now let’s determine \( c_n \) by using the nonhomogeneous condition \( u(2, y) = \sin \frac{\pi y}{2} \).

\[ u(2, y) = \sum_{n=1}^{\infty} c_n (e^{n\pi} - e^{-n\pi}) \sin \frac{n\pi y}{2} = \sin \frac{\pi y}{2}. \]

The right hand side is already a sine series, so we deduce that

\[ c_1(e^{\pi} - e^{-\pi}) = 1, \quad c_n(e^{n\pi} - e^{-n\pi}) = 0 \quad \text{if } n \neq 1. \]

Hence,

\[ c_1 = \frac{1}{e^{\pi} - e^{-\pi}} \quad \text{and} \quad c_n = 0, \quad \text{otherwise}. \]

Therefore,

\[ u(x, t) = \frac{1}{e^{\pi} - e^{-\pi}} \left( e^{\frac{\pi x}{2}} - e^{-\frac{\pi x}{2}} \right) \sin \frac{\pi y}{2} \left( = \frac{\sinh \frac{\pi x}{2} \sin \frac{\pi y}{2}}{\sinh \pi} \right). \]
Problem 2 Find the solution of the following problem.

\[ u_{xx} + u_{yy} = 0, \quad 0 < x < 2, \quad 0 < y < 2 \]
\[ u(x, 0) = u(x, 2) = 0 \]
\[ u(0, y) = 1, \quad u(2, y) = 0. \]

**solution:** Set \( u(x, y) = X(x)Y(y) \) and proceed as the problem 1. Then, we have

\[ Y'' + \lambda Y = 0 \]
\[ Y(0) = 0, \quad Y(2) = 0. \]

Hence, \( \lambda_n = \frac{n^2 \pi^2}{4} \) and \( Y_n(y) = \sin \frac{n \pi y}{2} \). As in the problem 1, the equation for \( X \) is

\[ X'' - \lambda X = 0. \]

The general solution is as before

\[ X(x) = c_1 e^{\frac{n \pi x}{2}} + c_2 e^{-\frac{n \pi x}{2}}. \]

But a homogeneous condition \( u(2, y) = 0 \) tells us that \( X(2) = 0 \). So,

\[ X(2) = c_1 e^{n \pi} + c_2 e^{-n \pi} = 0. \]
\[ c_2 e^{-n \pi} = -c_1 e^{n \pi}. \]

Multiply through by \( e^{n \pi} \). Then,

\[ c_2 = -c_1 e^{2n \pi}. \]

Hence, \( X(x) = c_1 \left(e^{\frac{n \pi x}{2}} - e^{2n \pi} e^{-\frac{n \pi x}{2}}\right) \) and

\[ X_n(t) = e^{\frac{n \pi x}{2}} - e^{2n \pi} e^{-\frac{n \pi x}{2}}. \]

Therefore,

\[ u(x, t) = \sum_{n=1}^{\infty} c_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} c_n \left(e^{\frac{n \pi x}{2}} - e^{2n \pi} e^{-\frac{n \pi x}{2}}\right) \sin \frac{n \pi y}{2}. \]

Now using the nonhomogeneous condition \( u(0, y) = 1 \), determine \( c_n \).

\[ u(0, y) = \sum_{n=1}^{\infty} c_n (1 - e^{2n \pi}) \sin \frac{n \pi y}{2} = 1. \]

Fourier sine series of 1 is

\[ \sum_{n \text{ odd}} \frac{4}{n \pi} \sin \frac{n \pi y}{2}. \]

Hence,

\[ c_n (1 - e^{2n \pi}) = \begin{cases} \frac{4}{n \pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \]
Thus,
\[ u(x, y) = \sum_{n \text{ odd}} \frac{4}{n\pi(1 - e^{2n\pi})} \left( e^{\frac{n\pi x}{2}} - e^{2\pi n} e^{-\frac{n\pi x}{2}} \right) \sin \frac{n\pi y}{2}. \]

The above answer is good enough. But let us try to write the answer in terms of \( \sinh \) function.

Multiplying top and bottom by \( e^{-n\pi} \), we have
\[ \frac{e^{\frac{n\pi x}{2}} - e^{2\pi n} e^{-\frac{n\pi x}{2}}}{1 - e^{2n\pi}} = \frac{e^{-n\pi} e^{\frac{n\pi x}{2}} - e^{n\pi} e^{-\frac{n\pi x}{2}}}{e^{-n\pi} - e^{n\pi}}, \]

Multiplying top and bottom by \(-1\) and combining the exponential functions, the above function becomes
\[ \frac{e^{\frac{2n\pi - n\pi x}{2}} - e^{\left(\frac{2n\pi - n\pi x}{2}\right)}}{e^{n\pi} - e^{-n\pi}} = \frac{\sinh \left(\frac{n\pi(2-x)}{2}\right)}{\sinh n\pi}, \text{ by the definition of } \sinh x = \frac{e^x - e^{-x}}{2}. \]

Thus,
\[ u(x, t) = \sum_{n \text{ odd}} \frac{4}{n\pi} \frac{\sinh \frac{n\pi(2-x)}{2}}{\sinh n\pi} \sin \frac{n\pi y}{2}. \]