MATH 285 Exam 2

Problem 1 (6 points) Find the general solutions of the following differential equations.

a) \( y'' - y = 0. \)
   
   solution:
   \[
   y = c_1 e^x + c_2 e^{-x}, \quad \text{or} \quad c_3 \cosh x + c_4 \sinh x.
   \]

b) \( y'' + y = 0. \)
   
   solution:
   \[
   y = c_1 \cos x + c_2 \sin x.
   \]

c) \( y'' + 2y' + y = 0. \)
   
   solution:
   \[
   y = c_1 e^{-x} + c_2 xe^{-x}.
   \]

Problem 2 (9 points) Find the appropriate form of a particular solution of the equation
   
   \[
   (D - 3)^2(D^2 - 6D + 13)y = 5e^{3x} \cos 2x + x^2 e^{3x}.
   \]
   
   Do not evaluate the coefficients. Explain your answer.

   solution: Characteristic equation is
   \[
   (r - 3)^2(r^2 - 6r + 13) = 0.
   \]
   
   \( r = 3 \) (multiplicity 2), \( 3 \pm 2i \).
   
   Four linearly independent solutions of \( y_c \) are
   \[
   e^{3x}, xe^{3x}, e^{3x} \cos 2x, e^{3x} \sin 2x.
   \]
   
   Hence,
   \[
   y_p = xe^{3x}(A \cos 2x + B \sin 2x) + x^2 e^{3x}(Cx^2 + Dx + E).
   \]
   
   Note that we multiplied \( x \) and \( x^2 \) to remove the duplications.

Problem 3 (7 points) Find the solution of the following initial value problem when the two linearly independent solutions \( y_1, y_2 \) of the associated homogeneous equation and a particular solution \( y_p \) is given.

   \[
   y'' - 3y' - 4y = 3e^{2x}, \quad y(0) = 1, \quad y'(0) = -1.
   \]
   
   \[
   y_1 = e^{-x}, \quad y_2 = e^{4x}, \quad y_p = -\frac{1}{2} e^{2x}.
   \]

   solution:
   \[
   y = y_c + y_p,
   \]
   
   \[
   y = c_1 e^{-x} + c_2 e^{4x} - \frac{1}{2} e^{2x}.
   \]
   
   \[
   y' = -c_1 e^{-x} + 4c_2 e^{4x} - e^{2x}.
   \]
Since \( y(0) = 1 \) and \( y'(0) = -1 \), we have

\[
\begin{align*}
c_1 + c_2 - \frac{1}{2} &= 1, \\
-c_1 + 4c_2 - 1 &= -1.
\end{align*}
\]

Hence \( c_1 = \frac{6}{5} \) and \( c_2 = \frac{3}{10} \). So,

\[
y = \frac{6}{5}e^{-x} + \frac{3}{10}e^{4x} - \frac{1}{2}e^{2x}.
\]

**Problem 4** Consider the undamped forced oscillation

\[
3x'' + 75x = 2 \cos 5t.
\]

The complementary function is \( x_c(t) = c_1 \cos 5t + c_2 \sin 5t \).

a) (18 points) Find a particular solution.

**solution:**

\[
x_p = t(A \cos 5t + B \sin 5t).
\]

Note that we multiplied \( t \) to eliminate the duplication.

\[
x_p'' = 2(-5A \sin 5t + 5B \cos 5t) + t(-25A \cos 5t - 25B \sin 5t).
\]

\[
3x_p'' + 75x_p = -30A \sin 5t + 30B \cos 5t = 2 \cos 5t.
\]

\[
-30A = 0, \quad 30B = 2.
\]

Hence,

\[
x_p = \frac{1}{15}t \sin 5t.
\]

b) (6 points) Explain the phenomenon which occurs in this motion by briefly sketching \( x_p \).

**solution:** \( x_p \) is not bounded as \( t \to \infty \) and it grows linearly. So the resonance occurs.

See the other part for the graph.

**Problem 5** Consider the nonhomogeneous equation

\[
x^2y'' + 7xy' + 5y = x.
\]

a) (8 points) Find two solutions of the homogeneous equation \( x^2y'' + 7xy' + 5y = 0 \).

**solution:** Note that this is an equidimensional equation (or Euler equation). So, put \( y = x^r \). Then,

\[
y' = rx^{r-1}, \quad y'' = r(r - 1)x^{r-2}.
\]

Hence we have

\[
x^r (r(r - 1) + 7r + 5) = 0.
\]

\[
r^2 + 6r + 5 = 0.
\]

\[
r = -1, \quad -5.
\]
Thus the solutions are
\[ y_1 = x^{-1}, \quad y_2 = x^{-5}. \]

b) (16 points) Find a particular solution \( y_p \).

**solution:** Since the coefficients are not constants, we need to use the variations of parameter method.

\[ y_p = u_1 x^{-1} + u_2 x^{-5}, \]

where
\[ u_1 = -\int \frac{y_2 f}{W(y_1, y_2)}, \quad u_2 = \int \frac{y_1 f}{W(y_1, y_2)}. \]

Dividing each side of the equation by \( x^2 \), we have
\[ y'' + 7x^{-1}y' + 5x^{-2}y = x^{-1}. \]

So, \( f(x) = x^{-1} \).

Now
\[ W(y_1, y_2) = x^{-1} - x^{-2} - 5x^{-6} = -4x^{-7}. \]

\[ u_1 = -\int \frac{x^{-5} \cdot x^{-1}}{-4x^{-7}} dx = \frac{1}{4} \int x dx = \frac{1}{8} x^2, \]
\[ u_2 = \int \frac{x^{-1} \cdot x^{-1}}{-4x^{-7}} dx = -\frac{1}{4} \int x^5 dx = -\frac{1}{24} x^6. \]

Thus,
\[ y_p = \frac{1}{8} x - \frac{1}{24} x = \frac{1}{12} x. \]

**Problem 6** Consider the damped forced oscillation
\[ x'' + \frac{3}{2} x' + x = 18 \cos 2t. \]

a) (5 points) Find \( x_c(t) \).

**solution:** Characteristic equation:
\[ r^2 + \frac{3}{2} r + 1 = 0, \ i.e., \ 2r^2 + 3r + 2 = 0, \quad r = -\frac{3}{4} \pm \frac{\sqrt{7}}{4}i. \]

So
\[ x_c = e^{-\frac{3}{4}t} \left( A \cos \frac{\sqrt{7}}{4}t + B \sin \frac{\sqrt{7}}{4}t \right). \]

b) (5 points) \( x_p \) is given by \( x_p = -3 \cos 2t + 3 \sin 2t \). Write \( x_p \) in the form of \( x_p(t) = C \cos(\omega t - \alpha) \) with \( C > 0 \).

**solution:**
\[ x_p = \sqrt{9 + 9} \cos \left( 2t - \frac{3\pi}{4} \right) = 3\sqrt{2} \cos \left( 2t - \frac{3\pi}{4} \right). \]

c) (5 points) Explain why \( x(t) \rightarrow x_p(t) \) as \( t \rightarrow \infty \).
solution: Since both $e^{-\frac{3}{4}t} \cos \sqrt{7}t$ and $e^{-\frac{3}{4}t} \sin \sqrt{7}t$ approaches 0 as $t \to \infty$, $x(t) = x_c(t) + x_p(t) \to x_p(t)$ as $t \to \infty$.

Problem 7 (15 points) Find all the positive eigenvalues and associated eigenfunctions of 

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$ 

solution: 

$$\lambda = \alpha^2, \quad \alpha > 0.$$ 

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x.$$ 

$$y(0) = c_1 = 0.$$ 

$$y(\pi) = c_2 \sin \alpha \pi = 0.$$ 

So, $\sin \alpha \pi = 0$, which implies $\alpha = n \in \mathbb{Z}$.

Hence,

Eigenvalues: $\lambda_n = n^2$, $n \in \mathbb{Z}$,

Eigenfunctions: $y_n = \sin nx$, $n \in \mathbb{Z}$. 