Practice Problems for Exam 3

Problem 1 Calculate the following double integrals.

a) $\int_0^1 \int_0^x (x^2 + y^2) dy \, dx$

solution)

$$\int_0^1 \int_0^x (x^2 + y^2) dy \, dx = \int_0^1 \left[ x^2y + \frac{1}{3}y^3 \right]_{y=0}^{y=x} \, dx$$

$$= \int_0^1 \left( x^3 + \frac{1}{3}x^3 - x^4 - \frac{1}{3}x^6 \right) \, dx = \left[ \frac{1}{3}x^4 - \frac{1}{5}x^5 - \frac{1}{21}x^7 \right]_0^1 = \frac{3}{35}.$$

b) $\int_0^1 \int_0^x e^{x^2} \, dy \, dx$

solution)

$$\int_0^1 \int_0^x e^{x^2} \, dy \, dx = \int_0^1 \left[ ye^{x^2} \right]_{y=0}^{y=x} \, dx$$

$$= \int_0^1 xe^{x^2} \, dx, \quad u = x^2, \quad du = 2x \, dx$$

$$= \frac{1}{2} \int_{u=0}^{u=1} e^u \, du = \left[ \frac{1}{2} e^u \right]_{u=0}^{u=1} = \frac{1}{2} (e - 1).$$
**Problem 2** Evaluate \( \int_0^1 \int_{2y}^2 \cos(x^2) \, dx \, dy \) by interchanging the order of integration.

**solution** First, sketch the region of integration:

\[
2y \leq x \leq 2, \quad 0 \leq x \leq 1.
\]

\[
\int_0^1 \int_{2y}^2 \cos(x^2) \, dx \, dy = \int_0^2 \int_0^{\frac{x}{2}} \cos(x^2) \, dy \, dx
\]

\[
= \int_0^2 \left[ y \cos(x^2) \right]_{y=0}^{\frac{x}{2}} \, dx = \int_0^2 \frac{x}{2} \cos(x^2) \, dx, \quad u = x^2, \quad du = 2x \, dx
\]

\[
= \int_{x=0}^2 \frac{1}{4} \cos u \, du = \left[ \frac{1}{4} \sin u \right]_{x=0}^{\sqrt{x}} = \left[ \frac{1}{4} \sin x^2 \right]_{x=0}^{1} = \frac{1}{4} \sin 4.
\]

**Problem 3** Use a double integral to find the area bounded by \( y = x^2 \) and \( y = \sqrt{x} \).

**solution** Let \( D \) be the region as described.

\[
A(D) = \iint_D dA = \int_0^1 \int_{x^2}^{\sqrt{x}} dy \, dx = \cdots = \frac{1}{3}.
\]
Problem 4 Use a double integral to find the volume of the solid in the first octant bounded by the paraboloid \( z = x^2 + y^2 \) and the planes \( z = 0 \), \( x + y = 1 \).

solution) The given solid is the region under the paraboloid \( z = x^2 + y^2 \) and above the triangle with vertices \((0,0)\), \((1,0)\), and \((0,1)\).

So, the volume is

\[
\int \int_D (x^2 + y^2) \, dA = \int_0^1 \int_{0}^{1-x} (x^2 + y^2) \, dy \, dx = \cdots = \frac{1}{6}.
\]
**Problem 5** Calculate the average value of $f(x, y) = e^x$ over the region $D: 0 \leq x \leq \ln y, \ 1 \leq y \leq e$. 

(solution) Note that the average value $f_{\text{avg}} = \frac{1}{A(D)} \iint_D f(x, y) dA$. 

\[ A(D) = \iint_D dA = \int_1^e \int_0^{\ln y} dxdy = \int_0^e \int_0^y dydx = \int_0^e (e - e^x)dx = [ex - e^x]_{x=0}^1 = e - e - (0 - 0^0) = 1. \]

Or, 

\[ A(D) = \iint_D dA = \int_1^e \int_0^{\ln y} dxdy = \int_1^e \ln ydy, \quad u = \ln y, dv = dy \]

\[ = [y \ln y]_1^e - \int_1^e dy, \quad \text{integration by parts} \]

\[ = e \ln e - \ln 1 - [y]_1^e = e - (e - 1) = 1. \]

So, 

\[ f_{\text{avg}} = \int_0^1 \int_0^y e^x dydx = \int_0^1 \int_0^{e^x} y e^x dydx = \int_0^1 (e^{x+1} - e^{2x})dx \]

\[ = \left[ e^{x+1} - \frac{1}{2} e^{2x} \right]_{x=0}^{x=1} = e^2 - \frac{1}{2} e^2 - \left( e - \frac{1}{2} \right) = \frac{1}{2} e^2 - e + \frac{1}{2}. \]

Or, 

\[ f_{\text{avg}} = \int_1^e \int_0^{\ln y} e^x dydx = \int_1^e [e^x]_{x=0}^{x=1} dy \]
\[ e^{\ln y} - 1 \, dy = e^y - 1 \, dy = \cdots = \frac{y^2}{2} - e + \frac{1}{2}. \]

Here, note that \( e^{\ln y} = y \).

**Problem 6** Use a triple integral to find the volume of the solid in the first octant bounded by the cylinder \( x^2 + y^2 = 4 \), and the planes \( z = y, z = 0 \).

**solution** Let \( E \) be the region described as above. Then we know that the volume of \( E \) is \( \iiint_E dV \). Project the region \( E \) onto the \( xy \)-plane and denote the projection by \( D \).

\[
\iiint_E dV = \iint_D \int_0^y dz \, dA = \iint_D y \, dA
\]
\[
= \int_0^{\pi/2} \int_0^2 r \sin \theta r \, dr \, d\theta
\]
\[
= \int_0^{\pi/2} \left[ \frac{r^3}{3} \sin \theta \right]_{r=0}^2 d\theta = \cdots = \frac{8}{3}.
\]

**Problem 7** Calculate \( \int_0^3 \left( \sqrt{9 - y^2} \right) \frac{1}{\sqrt{x^2 + y^2}} \, dx \, dy \) by changing to polar coordinates.

**solution** \( D : -\sqrt{9 - y^2} \leq x \leq \sqrt{9 - y^2}, \ 0 \leq y \leq 3 \)
Problem 8 Use a double integral in polar coordinates to find the area that is inside $r = 3 \sin 3\theta$.

First, find the area of one loop and multiply by 3.

\[
3 \sin 3\theta = 0.
\]

$3\theta = 0, \pi, 2\pi, \ldots, \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots$

So, the area of one loop is

\[
\int_0^{\pi/3} \int_0^{3\sin 3\theta} r\,dr\,d\theta = \int_0^{\pi/3} \left[ \frac{r^2}{2} \right]_{r=0}^{3\sin 3\theta} d\theta = \int_0^{\pi/3} \frac{9}{2} \sin^2 3\theta d\theta
\]

\[
= \frac{9}{2} \int_0^{\pi/3} \frac{1 - \cos 6\theta}{2} d\theta, \quad \text{by the double angle formula } \sin^2 u = \frac{1 - \cos 2u}{2}
\]
\[
= \frac{9}{4} \int_0^\pi \frac{\pi}{3} (1 - \cos 6\theta) d\theta = \cdots = \frac{3}{4} \pi.
\]
Hence, the area that is inside \( r = 3 \sin(3\theta) \) is \( 3 \cdot \frac{3}{4} \pi = \frac{9}{4} \pi \).

**Problem 9** Evaluate \( \int_0^1 \int_0^3 \int_0^{\sqrt{x^2}} x \, dx \, dy \, dz \).

**Solution**

\[
\int_0^1 \int_0^3 \left[ \frac{x^2}{2} \right]_{y=0}^{y=x} \, dy \, dz
= \int_0^1 \frac{y^2}{2} \, dy = \int_0^1 \left[ \frac{y^3}{4} \right]_{y=0}^{y=x} \, dy
= \int_0^1 \frac{x^3}{4} \, dx = \left[ \frac{x^4}{16} \right]_0^1 = \frac{1}{16}.
\]

**Problem 10** Set up an iterated integral for \( \iiint_T \, x \, dV \), where \( T \) is the tetrahedron bounded by \( x + y + z = 3 \) and the coordinate planes.

**Solution**

Project \( T \) onto \( xy \)-plane. We get a triangle with vertices \( (0, 0), (3, 0), \) and \( (0, 3) \).

\[
\int_0^3 \int_{3-x}^{3-x-y} x \, dz \, dx.
\]

**Problem 11** Set up a triple integral for the volume of the solid bounded by the cylinder \( y^2 = 4x \), and the planes \( z = 0, z = x, x = 4 \).

**Solution**
Let $E$ be the given solid and project it onto $xy$-plane. Then

$$V(E) = \iiint_E dV = \int_{-4}^{4} \int_{\frac{18-x^2-y^2}{2}}^{\frac{18}{2}} dz dx dy.$$ 

**Problem 12** Set up a double integral for the volume of the solid enclosed by $z = x^2 + y^2$, $z = 18 - x^2 - y^2$ in polar coordinates.

**solution**

Intersection of two surfaces is

$$x^2 + y^2 = 18 - x^2 - y^2.$$
\[ x^2 + y^2 = 9. \]

If we project the given solid onto \( xy \)-plane, we have a disk

\[ D : x^2 + y^2 = 9. \]

So, the volume of the solid is

\[ \iint_{D} (18 - x^2 - y^2 - x^2 - y^2) \, dA = \int_{0}^{2\pi} \int_{0}^{3} (18 - 2r^2) \, rdrd\theta. \]
Problem 13 Find the center of mass of the tetrahedron $T$ with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ if the density is proportional to the distance from the $yz$-plane.

solution) Note that the distance from the $yz$-plane is $|x|$. Since $x$-coordinates is positive in the given tetrahedron, the distance from the $yz$-plane is simply $x$. Hence, the density $\rho(x,y) = kx$, $k$ : constant. By the formula, the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is

\[
\bar{x} = \frac{1}{m} \int \int \int_T x \rho(x,y) dV,
\]

\[
\bar{y} = \frac{1}{m} \int \int \int_T y \rho(x,y) dV,
\]

\[
\bar{z} = \frac{1}{m} \int \int \int_T z \rho(x,y) dV,
\]

\[
m = \int \int \int_T \rho(x,y) dV.
\]

First,

\[
m = \int \int \int_E k x dA = \int_0^1 \int_0^{1-x} \int_0^{1-y} k x dz dy dx = \cdots = \frac{k}{24}.
\]

So,

\[
\bar{x} = \frac{24}{k} \int \int \int_E x x dV = 24 \int_0^1 \int_0^{1-x} \int_0^{1-y} x^2 dz dy dx = \cdots = \frac{2}{5}.
\]

\[
\bar{y} = \frac{24}{k} \int \int \int_E y x dV = 24 \int_0^1 \int_0^{1-x} \int_0^{1-y} xy dz dy dx = \cdots = \frac{1}{5}.
\]

\[
\bar{z} = \frac{24}{k} \int \int \int_E z x dV = 24 \int_0^1 \int_0^{1-x} \int_0^{1-y} xz dz dy dx = \cdots = \frac{1}{5}.
\]
Problem 14 Find the surface area of the part of the plane \(2x + 5y + z = 10\) that lies in the first octant.

solution) The domain \(D\) is the projection of this plane onto \(xy\)-plane. So, \(D\) is the triangle with vertices \((0, 0), (5, 0), \) and \((0, 2)\). Also, \(z = f(x, y) = 10 - 2x - 5y\) and \(f_x = -2, f_y = -5\). By the formula, the surface area is

\[
\int \int_D \sqrt{1 + f_x^2 + f_y^2} \, dA = \int \int_D \sqrt{1 + 4 + 25} \, dA
\]

\[
= \sqrt{30} \int \int_D \, dA = \sqrt{30} \text{Area}(D) = \sqrt{30} \frac{1}{2} \cdot 2 \cdot 5 = 5\sqrt{30}.
\]

Problem 15 Find the surface of the part of the sphere \(x^2 + y^2 + z^2 = 4\) that lies within the cylinder \(x^2 + y^2 = 2x\) and above the \(xy\)-plane.

solution) Note that since the surface is above \(xy\)-plane, \(z > 0\). Hence, we can rewrite the surface as \(z = \sqrt{4 - x^2 - y^2}\).

\(f(x, y) = \sqrt{4 - x^2 - y^2}, f_x = \frac{-x}{\sqrt{4 - x^2 - y^2}}, f_y = \frac{-y}{\sqrt{4 - x^2 - y^2}}\).

Since the surface lies within the cylinder \(x^2 + y^2 = 2x\), the domain \(D : x^2 + y^2 = 2x\).

Writing \(D\) in polar coordinates, we have \(r^2 = 2r \cos \theta\), \(r = 2 \cos \theta\).

By the formula, the surface area over \(D\) is

\[
\int \int_D \sqrt{1 + \frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2}} \, dA
\]

\[
= \int \int_D \frac{2}{\sqrt{4 - x^2 - y^2}} \, dA, \quad \text{converting to polar coordinates}
\]

\[
= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2 \cos \theta} \frac{2}{\sqrt{4 - r^2}} r \, dr \, d\theta, \quad u = 4 - r^2, \quad du = -2r \, dr
\]

\[
= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2 \cos \theta} -\frac{1}{\sqrt{u}} \, du = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left[ -2 \sqrt{4 - r^2} \right]_{r=0}^{r=2 \cos \theta} \, d\theta
\]

\[
= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left( -2\sqrt{4 - 4 \cos^2 \theta} + 4 \right) \, d\theta
\]

\[
= 2 \int_{0}^{\frac{\pi}{2}} (-4 \sin \theta + 4) \, d\theta = \cdots = 4(\pi - 2).
\]
Note that $\sqrt{\sin^2 \theta} = \begin{cases} 
\sin \theta, & \text{if } 0 \leq \theta \leq \frac{\pi}{2} 
-\sin \theta, & \text{if } -\frac{\pi}{2} \leq \theta \leq 0.
\end{cases}$
So,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\sin^2 \theta} \ d\theta = 2 \int_0^{\frac{\pi}{2}} \sin \theta \ d\theta.$$