Section 14.5 The Chain Rule

Chain Rule
Review: Let \( y = f(x) \) and \( x = x(t) \). Then, we can think \( y \) as a function of \( t \) and we can consider \( \frac{dy}{dt} \).

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}.
\]

Example 1 Let \( y = e^{x^2+1} \) and \( x(t) = \sin t + t + 1 \). Find \( \frac{dy}{dt} |_{t=0} \).

Chain Rule in several variable
- **Case 1:** Let \( z = f(x,y) \) and \( x = x(t) \) and \( y = y(t) \). Then, we can think \( z \) as a function of \( t \).

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.
\]

Example 2 If \( z = x^2 + y^2 \), where \( x = t \) and \( y = t^2 \), find \( \frac{dz}{dt} \).

Remark: You can interpret \( \frac{dz}{dt} \) in the example 2 as the rate of change of \( z \) along the curve \( \vec{r}(t) = \langle t, t^2 \rangle \).

Example 3 If \( w = x^2 y + z \cos x \), where \( x(t) = t \), \( y(t) = t^2 \), \( z(t) = t^3 \), find \( \frac{dw}{dt} \).

- **Case 2:** Let \( z = f(x,y) \) and \( x = x(s,t) \) and \( y = y(s,t) \). Then, you can think \( z \) as a function of \( s \) and \( t \).

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s},
\]
\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}.
\]

**Remark:** See the tree diagram on page 920, 921 in the textbook.

**Example 4** Let \( z = x^2 - 2xy + y^3 \) and \( x = s^2 \ln t \) and \( y = 2st^3 \). Find \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \).

**Example 5** Let \( w = xy + yz + zx \) and \( x = st, y = e^{st} \) and \( z = t^2 \). Find \( \frac{\partial w}{\partial s} \) and \( \frac{\partial w}{\partial t} \) when \( s = 0, t = 1 \).

**Example** (if time permits) Let \( z = f(x,y) \) and \( x = r^2 + s^2 \) and \( y = 2rs \). Find \( \frac{\partial z}{\partial r} \) and \( \frac{\partial^2 z}{\partial r^2} \).
• **Explicit and Implicit function**

Explicit function is of the form \( z = f(x,y) \).

Implicit function is of the form \( g(x,y,z) = c \) (Relation is given by an equation)

• **Implicit differentiation:**

Let \( F(x,y) = 0 \) defines \( y \) implicitly as a differentiable function of \( x \). Then,

\[
\frac{dy}{dx} = -\frac{F_x}{F_y}
\]

**Example 6** Find \( \frac{dy}{dx} \) if \( \cos(x - y) = xe^y \).

Let \( F = F(x,y,z) \) is continuously differentiable, and \( z = f(x,y) \) is given implicitly by \( F(x,y,z) = 0 \), then at the points \( (x,y,z) \) where \( \frac{\partial u}{\partial z} \neq 0 \)

\[
\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} , \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.
\]

**Example 7** Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).

a. \( z^3 + xz + y^2 + xy = 2 \)

b. \( \cos xyz + \ln(x^2 + y^2 + z^2) = 0 \)
HW: 3, 7, 11, 15, 19, 21, 25, 29, 31, 43

hint: 43: Let $u = x - y$. Then $\frac{\partial z}{\partial x} = \frac{dz}{du} \frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \frac{dz}{du} \frac{\partial u}{\partial y}$.