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CONSTRUCTION OF UNIVERSAL BUNDLES, II

By John Milnor

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1. Introduction

It is well known that any compact Lie group can serve as the group of an n-universal bundle (that is a principal bundle with $(n - 1)$-connected bundle space). The main result of this paper is that a completely arbitrary topological group G can serve as the group of an n-universal bundle, $n \leqq \infty$. The construction is based on the $(n + 1)$-fold join $G \circ \cdots \circ G$.

Section 2 contains some preliminary material on the topology of joins: in particular the result that every $(n + 1)$-fold join is $(n - 1)$-connected. Section 3 contains the main construction. In Section 4 this construction is used to obtain some information about the homology of the classifying space of G. (We say that X is a *classifying space* for G if X is the base space of some ∞-universal bundle with group G.)

If the group G is suitably restricted (a countable CW-complex with cellular product map and inverse map), then it is shown in Section 5 that a countable CW-complex can be chosen as classifying space for G. (This construction is quite explicit, and could be used for homology computations.) The final theorem (5.2) summarizes the relationships between "countable CW-groups" and their classifying spaces.

In general this paper is independent of "Construction of universal bundles I" which considered the construction from a given base space, rather than from a given group.

2. Topology of joins

The join $A_1 \circ \cdots \circ A_n$ of n topological spaces $A_1, \cdots, A_n$ can be defined as follows. A point of the join is specified by

(1) n real numbers $t_1, \cdots, t_n$ satisfying $t_i \geqq 0$, $t_1 + \cdots + t_n = 1$, and

(2) a point $a_i \in A_i$ for each i such that $t_i \neq 0$. Such a point in $A_1 \circ \cdots \circ A_n$ will be denoted by the symbol $t_1a_1 \oplus \cdots \oplus t_na_n$, where the element a_i may be chosen arbitrarily or omitted whenever the corresponding t_i vanishes.

By the *strong topology* in $A_1 \circ \cdots \circ A_n$ we mean the strongest topology such that the coordinate functions

$$t_i: A_1 \circ \cdots \circ A_n \to [0, 1] \quad \text{and} \quad a_i: t_i^{-1}(0, 1] \to A_i$$

are continuous. Thus a sub-basis for the open sets is given by the sets of the following two types

(1) the set of all $t_1a_1 \oplus \cdots \oplus t_na_n$ such that $\alpha < t_i < \beta$,

(2) the set of all $t_1a_1 \oplus \cdots \oplus t_na_n$ such that $t_i \neq 0$ and $a_i \in U$, where U is an arbitrary open subset of A_i.

The join of infinitely many topological spaces in the strong topology can be defined in exactly the same manner, with the restriction that all but a finite number of the t_i should vanish. It is clear from the definition that the formation of finite or infinite joins in the strong topology is an associative, commutative operation.

The strong topology is not the same as the more conventional *weak topology*, in which $A_1 \circ \cdots \circ A_n$ is considered as an identification space of the product of $A_1 \times \cdots \times A_n$ with an $(n - 1)$-simplex. However the three lemmas of this section will be true for either topology.

LEMMA 2.1. *The reduced singular homology groups of the join $A \circ B$ with coefficients in a principal ideal domain are given by*

$$\tilde{H}_{r+1}(A \circ B) \cong \sum_{i+j=r} \tilde{H}_i(A) \otimes \tilde{H}_j(B) + \sum_{i+j=r-1} \operatorname{Tor}(\tilde{H}_i(A), \tilde{H}_j(B)).$$

(The symbol $\tilde{H}_i$ is supposed to stand for H_i whenever $i \neq 0$.) Consider the triad $(A \circ B, \bar{A}, \bar{B})$ where $\bar{A}$ is the set of points $ta \oplus (1 - t)b$ with $t \geqq \frac{1}{2}$, and $\bar{B}$ is the set of $ta \oplus (1 - t)b$ with $t \leqq \frac{1}{2}$. It is easily verified that this is a proper triad, so that its reduced Mayer-Vietoris sequence

$$\cdots \leftarrow \tilde{H}_r(A \circ B) \xleftarrow{\phi} \tilde{H}_r(\bar{A}) + \tilde{H}_r(\bar{B}) \xleftarrow{\psi} \tilde{H}_r(\bar{A} \cap \bar{B}) \leftarrow \tilde{H}_{r+1}(A \circ B) \xleftarrow{\phi} \cdots$$

is defined and exact.

Identify the spaces A, B, and $A \times B$ with the subsets of $A \circ B$ consisting of all $ta \oplus (1 - t)b$ with $t = 1$, $t = 0$, and $t = \frac{1}{2}$ respectively. Then A is a deformation retract of $\bar{A}$, B is a deformation retract of $\bar{B}$, and $A \times B = \bar{A} \cap \bar{B}$. Since the inclusion maps $A \to A \circ B$ and $B \to A \circ B$ are homotopic to constants, it follows that the homomorphism ϕ is always trivial. Thus the above exact sequence reduces to the following.

$$0 \leftarrow \tilde{H}_r(A) + \tilde{H}_r(B) \xleftarrow{\psi'} \tilde{H}_r(A \times B) \leftarrow \tilde{H}_{r+1}(A \circ B) \leftarrow 0.$$

According to a theorem of Eilenberg and Zilber (Amer. J. Math., 75 (1953) pp. 200–204) the singular complex $S_*(A \times B)$ has the same homology groups as the complex $S_*(A) \otimes S_*(B)$. By the Künneth formulas, the latter complex has homology groups

$$H_r(S_*(A) \otimes S_*(B)) \cong \sum_{i+j=r} H_i(A) \otimes H_j(B) + \sum_{i+j=r-1} \operatorname{Tor}(H_i(A), H_j(B)).$$

Substituting this expression for $H_r(A \times B)$ in the above exact sequence, and computing the homomorphism ψ', we easily arrive at the required expression for kernel $\psi' \cong H_{r+1}(A \circ B)$.

LEMMA 2.2. *If B is arc-wise connected and A is any non-vacuous space, then $A \circ B$ is simply connected.*

PROOF. Let S be the circle. Any map $f: S \to A \circ B$ can be described by the formula

$$f(s) = t(s)a(s) \oplus (1 - t(s))b(s)$$

(where $b(s)$ is defined whenever $t(s) \neq 1$, and $a(s)$ whenever $t(s) \neq 0$). Construct a map $b': S \to B$, defined for all s, so that $b'(s) = b(s)$ whenever $t(s) \leqq \frac{1}{2}$. This is possible since B is arc-wise connected. Let a_0 be some point in A. A homotopy

$$h: S \times [0, 3] \to A \circ B$$

can now be defined as follows. Let

$$t(s, u) = \begin{cases} \operatorname{Min}(1, (1 + u)t(s)) & \text{for } 0 \leqq u \leqq 1 \\ (2 - u)t(s, 1) & \phantom{\text{for }} 1 \leqq u \leqq 2. \end{cases}$$

Define

$$h(s, u) = \begin{cases} t(s, u)a(s) \oplus (1 - t(s, u))b(s) & \text{for } 0 \leqq u \leqq 1 \\ t(s, u)a(s) \oplus (1 - t(s, u))b'(s) & \phantom{\text{for }} 1 \leqq u \leqq 2 \\ (u - 2)a_0 \oplus (3 - u)b'(s) & \phantom{\text{for }} 2 \leqq u \leqq 3. \end{cases}$$

Then it is easily verified that h is a well defined homotopy satisfying $h(s, 0) = f(s)$, $h(s, 3) =$ constant. This completes the proof.

By definition, every non-vacuous space will be called (-1)-connected.

LEMMA 2.3. *The join of $n + 1$ non-vacuous spaces is always $(n - 1)$-connected. In fact if each space A_i is $(c_i - 1)$-connected, then $A_0 \circ A_1 \circ \cdots \circ A_n$ is $(c_0 + c_1 + \cdots + c_n + n - 1)$-connected.*

If we can prove this for the case $n = 1$, then the general case will clearly follow by induction. Since the join of any two non-vacuous spaces is connected, this lemma is true for the case $c_0 = c_1 = 0$. If A_0 is $(c_0 - 1)$-connected and A_1 is $(c_1 - 1)$-connected where either c_0 or c_1 is positive, then 2.2 implies that $A_0 \circ A_1$ is simply connected and 2.1 implies that $H_r(A_0 \circ A_1) = 0$ for $r \leqq c_0 + c_1$. Therefore $A_0 \circ A_1$ is $(c_0 + c_1)$-connected. This completes the proof.

It follows from 2.3 that the join of infinitely many non-vacuous spaces is always ∞-connected.

3. The construction

For an arbitrary topological group G let $E_n = G \circ \cdots \circ G$ be the join of $n + 1$ copies of G in the strong topology. Define the right translation $R: E_n \times G \to E_n$ by

$$R(t_0g_0 \oplus \cdots \oplus t_ng_n, g) = t_0(g_0g) \oplus \cdots \oplus t_n(g_ng).$$

Let X_n be the identification space of E_n produced by identifying e with e' if and only if $e' = R(e, g)$ for some $g \in G$. Let $p: E_n \to X_n$ be the identification map.

THEOREM 3.1. *G is the group of an n-universal bundle having bundle space E_n, base space X_n, and projection p.*

The space E_n is $(n - 1)$-connected by Lemma 2.3. (In fact if G is $(c - 1)$-connected then E_n is $((n + 1)c + n - 1)$-connected.) The bundle structure is defined as follows. Let the coordinate neighborhood V_j be the set of all points

$p(t_0g_0 \oplus \cdots \oplus t_ng_n)$ in X_n such that $t_j \neq 0$. Define the coordinate functions $\phi_j: V_j \times G \to p^{-1}(V_j)$ by

$$\phi_j(p(t_0g_0 \oplus \cdots \oplus t_ng_n), g) = t_0(g_0g_j^{-1}g) \oplus \cdots \oplus t_n(g_ng_j^{-1}g).$$

It is easily verified that ϕ_j is well defined. (Continuity will be proved later.) Define

$$p_j: p^{-1}(V_j) \to G \quad \text{by} \quad p_j(t_0g_0 \oplus \cdots \oplus t_ng_n) = g_j .$$

The identities $p\phi_j(x, g) = x$, $p_j\phi_j(x, g) = g$, and $\phi_j(p(e), p_j(e)) = e$ show that (p, p_j) is an inverse to ϕ_j.

The coordinate transformations $g_{ij}: V_i \cap V_j \to G$ are defined by

$$g_{ij}(p(t_0g_0 \oplus \cdots \oplus t_ng_n)) = g_ig_j^{-1},$$

and satisfy the identity $p_i\phi_j(x, g) = g_{ij}(x)\cdot g$.

It is now necessary to prove that all of these functions are continuous. Starting directly from the definition of the strong topology in the join, it is proved that R and p_j are continuous. (I do not know if R would be continuous for the weak topology.) The identification map p is certainly continuous.

Let ε be the identity element of G. The identity $\phi_j(p(e), \varepsilon) = R(e, p_j(e)^{-1})$ shows that $\phi_j(p(e), \varepsilon)$ is a continuous function of e. By the definition of the identification topology, this means that $\phi_j(x, \varepsilon)$ is a continuous function of x.

Now the identity

$$\phi_j(x, g) = R(\phi_j(x, \varepsilon), g)$$

implies that ϕ_j is continuous as a function of two variables.

Finally the identity $g_{ij}(x) = p_i\phi_j(x, \varepsilon)$ shows that g_{ij} is continuous. This completes the proof.

An ∞-universal bundle for G may be constructed in exactly the same way, using the join of infinitely many copies of G in the strong topology. This bundle will be denoted by $p: E_\infty \to X_\infty$.

4. Homology in the universal bundle

The preceding construction can be used to study relations between the homology groups of G and the homology groups of the classifying space X_∞.

Identify X_n, E_n with the subspaces of X_∞, E_∞ for which $t_{n+1} = t_{n+2} = \cdots = 0$. Then we have a sequence

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_\infty$$

of spaces (where X_0 is a single point, X_1 is the suspension of G, etc.).

LEMMA 4.1. *The singular homology group $H_k(X_n, X_{n-1})$ is isomorphic to the reduced singular homology group $\tilde{H}_{k-1}(E_{n-1})$, for $n, k > 0$.*

The proof will be given later. Since E_{n-1} is the n-fold join $G_{(1)} \circ \cdots \circ G_{(n)}$, its homology (with coefficients in a principal ideal domain) may be computed by Lemma 2.1. In particular if the homology of G is torsion free (for example if the coefficient group is a field) then

$$\tilde{H}_{k-1}(E_{n-1}) = \sum_{i_1+\cdots+i_n=k-n} \tilde{H}_{i_1}(G) \otimes \cdots \otimes \tilde{H}_{i_n}(G).$$

This lemma can be used to prove the following.

THEOREM 4.2. *There is a spectral sequence* $\{E^i_{n,q}\}$ *whose limit term* E^∞ *is the graded group corresponding to* $H_*(X_\infty)$ *under a suitable filtration,*[1] *for which*

$$E^1_{n,q} = H_{n+q}(X_n, X_{n-1}) \cong \tilde{H}_{n+q-1}(G_{(1)} \circ \cdots \circ G_{(n)}), \quad \text{for} \quad n > 0.$$

In particular if $H_*(G)$ *is torsion free then*

$$E^1_{n,q} = \sum_{i_1+\cdots+i_n=q} \tilde{H}_{i_1}(G) \otimes \cdots \otimes \tilde{H}_{i_n}(G).$$

PROOF OF LEMMA 4.1. Consider the homomorphisms

$$H_{k-1}(E_{n-1}) \xleftarrow{\partial} H_k(E_{n-1} \circ \varepsilon, E_{n-1}) \xrightarrow{p'_*} H_k(X_n, X_{n-1})$$

where the cone $E_{n-1} \circ \varepsilon$ is considered as a subset of $E_n = E_{n-1} \circ G$, and where the map p' is induced by the projection $p: E_n \to X_n$. Since $E_{n-1} \circ \varepsilon$ is contractible, the boundary homomorphism ∂ is an isomorphism.

The map $p': (E_{n-1} \circ \varepsilon, E_{n-1}) \to (X_n, X_{n-1})$ is a relative homeomorphism. In fact an inverse map $X_n - X_{n-1} \to E_{n-1} \circ \varepsilon - E_{n-1}$ is given by $x \to \phi_n(x, \varepsilon)$. Since E_{n-1} is a deformation retract of a neighborhood in $E_{n-1} \circ \varepsilon$, and since X_{n-1} is a deformation retract of the corresponding neighborhood in X_n, it follows that p'_* is an isomorphism. This completes the proof.

PROOF OF THEOREM 4.2. Consider the singular chain groups

$$S_*(X_0) \subset S_*(X_1) \subset \cdots \subset S_*(X_\infty).$$

Let $\bar{S}_*$ denote the union of the $S_*(X_n)$, $n < \infty$. Then the groups $S_*(X_n)$ form a filtration of $\bar{S}_*$. This filtered, graded, differential group $\bar{S}_*$ gives rise to a spectral sequence $\{E^i_{n,q}\}$, where E^∞ is the graded group corresponding to $H(\bar{S}_*)$ under the induced filtration, and where

$$E^1_{n,q} = H_{n+q}(S_*(X_n)/S_*(X_{n-1})) = H_{n+q}(X_n, X_{n-1}).$$

Thus to complete the proof it is only necessary to show the following.

LEMMA 4.3. *The inclusion* $\bar{S}_* \subset S_*(X_\infty)$ *induces isomorphisms* $H(\bar{S}_*) \to H_*(X_\infty)$.

First observe that the inclusion $X_n \subset X_\infty$ induces isomorphisms $H_k(X_n) \to H_k(X_\infty)$ for $k < n$. To prove this it is sufficient to prove that the corresponding homomorphisms $\pi_k(X_n) \to \pi_k(X_\infty)$ are isomorphisms for $k < n$. But this follows easily from consideration of the homotopy sequences of the two universal bundles.

Since $H_k(\bar{S}_*)$ is the direct limit of the groups $H_k(X_n)$ as $n \to \infty$, this completes the proof.

In conclusion we remark that the homomorphism $d^1: E^1_{n,q} \to E^1_{n-1,q}$ can be

[1] A similar spectral sequence for an arbitrary space X has been studied by G. Whitehead, the group G being replaced by the space Ω of loops on X.

explicitly computed, at least if the homology of G is torsion free. The result, stated without proof, is as follows. Let D be the diagonal map of G into the $(n-1)$-fold product $G \times \cdots \times G$. Define $\gamma : G \times G \to G$ by $\gamma(g_1, g_2) = g_1 g_2^{-1}$.

If $z_1 \circ \cdots \circ z_n$ is an element of $\tilde{H}_{n+q-1}(G \circ \cdots \circ G) \cong E^1_{n,q}$ such that $D_*(z_n) = \sum_j y_1^{(j)} \times \cdots \times y_{n-1}^{(j)}$ then

$$d^1(z_1 \circ \cdots \circ z_n) = \sum_j (-)^{\alpha_j} \gamma_*(z_1 \times y_1^{(j)}) \circ \cdots \circ \gamma_*(z_{n-1} \times y_{n-1}^{(j)})$$

where α_j is an appropriate sign.

5. Countable CW-groups

A topological group G will be called a *countable CW-group* if G is a countable CW-complex such that the map $g \to g^{-1}$ of G into itself and the product map $G \times G \to G$ are both cellular (that is, carry the k-skeleton into the k-skeleton). For example the groups $\tilde{G}$ constructed in Part I of this paper are clearly countable CW-groups.

THEOREM 5.1. *Every countable CW-group G is the group of an ∞-universal bundle for which the base space X_∞ is a countable CW-complex.*

Let E_n be the join of $(n + 1)$ copies of G in the weak topology. Then an n-universal bundle $p : E_n \to X_n$ can be constructed just as in Section 3. (It is necessary to use the fact that E_n is a countable CW-complex in order to prove the continuity of the right translation $R : E_n \times G \to E_n$. Otherwise the proofs are the same as in Section 3.)

A special CW-structure for E_n is constructed as follows by induction on n. The cells for $E_0 = G$ are just those of G. As cells for $E_n = E_{n-1} \circ G$ we take all cells of the form $\tau \circ \phi$, $\phi \circ \sigma$, and $(\tau \circ \varepsilon)\sigma$, where τ is a generic cell of E_{n-1}, σ is a generic cell of G, and ϕ is the empty set. The multiplication is understood in the sense of right translation, so that $(\tau \circ \varepsilon)\sigma$ is the set of all $R(te \oplus (1 - t)\varepsilon, g)$ with $e \in \tau$, $g \in \sigma$, and $0 < t < 1$.

The cells of the base space $X_n = p(E_{n-1} \circ G)$ can now be given as (1) the cells τ' of X_{n-1}, (2) the point $p(\phi \circ \varepsilon)$, and (3) the cells $p(\tau \circ \varepsilon)$ where τ is a cell of E_{n-1}. It is easily verified that X_n is a countable CW-complex with respect to this subdivision. The unions E_∞, X_∞ of the complexes E_n, X_n can be topologized as CW-complexes, and clearly the map $p : E_\infty \to X_\infty$ is the projection map of an ∞-universal bundle.

The main results of this paper can be summarized in the following omnibus theorem. We will say that two countable CW-groups G_1, G_2 are *equivalent* if there is a third countable CW-group G_3 which can be mapped homomorphically into both G_1 and G_2 in maps which are also homotopy equivalences. (This is clearly a reflexive, symmetric relation. Transitivity will follow from 5.2.)

THEOREM 5.2. (1) *Any countable CW-group G has a countable CW-complex X_1 as classifying space.* (2) *A second CW-complex X_2 is also a classifying space for G if and only if it has the same homotopy type as X_1.*

(3) *Any countable, connected CW-complex X is the classifying space for some countable CW-group G_1.* (4) *A second countable CW-group G_2 has the same classifying space X if and only if G_2 is equivalent to G_1.*

Part (1) is a restatement of 5.1. Part (2) is well known for finite complexes, and the proof for CW-complexes is identical. Part (3) follows from Corollary 3.7 of Part I of this paper.

Proof of (4). If X is a classifying space for both G_1 and G_2, then Corollary 5.4 of Part I implies that G_1 and G_2 are equivalent. If X is a classifying space for G_1 and if there exist continuous homomorphisms $G_3 \to G_1$, $G_3 \to G_2$ which are also homotopy equivalences, then we will construct an ∞-universal bundle with base space X and group G_3. The homomorphism $G_3 \to G_2$ will then induce the required bundle over X with group G_2.

Let the CW-complex X_3 be a classifying space for G_3. Then the homomorphism $G_3 \to G_1$ induces an ∞-universal bundle over X_3 with group G_1. Since X and X_3 are both classifying spaces for G_1, they have the same homotopy type. Since X_3 is a classifying space for G_3, this implies that X is also a classifying space for G_3. This completes the proof.

Princeton University