Problem 4. In this problem, we will show that a countable product of metrizable spaces is metrizable.

a. Let \((X,d)\) be a metric space. Consider the function \(\rho: X \times X \to \mathbb{R}\) defined by

\[
\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.
\]

Show that \(\rho\) is a metric on \(X\).

b. Show that the metric \(\rho\) from part (a) induces the same topology on \(X\) as the original metric \(d\).

Remark. We could also have used the formula \(\rho(x, y) = \min\{d(x, y), 1\}\). The goal was just to find a metric \(\rho\) which is topologically equivalent to \(d\) and is bounded.

c. Let \(\{(X_i, d_i)\}_{i \in \mathbb{N}}\) be a countable family of metric spaces, where each metric \(d_i\) is bounded by 1, i.e.

\[d_i(x_i, y_i) \leq 1 \text{ for all } x_i, y_i \in X_i.\]

Write \(X := \prod_{i \in \mathbb{N}} X_i\) and consider the function \(d: X \times X \to \mathbb{R}\) defined by

\[
d(x, y) = \sum_{i=1}^\infty \frac{1}{2^i} d_i(x_i, y_i).
\]

Show that \(d\) is a metric on \(X\). (First check that \(d\) is a well-defined function.)

d. Show that the metric \(d\) from part (c) induces the product topology on \(X = \prod_{i \in \mathbb{N}} X_i\).