Problem 3. Let $X$ be a topological space and consider the set of all continuous functions on $X$ with values in $[0,1]$

$$C := \{ f : X \to [0,1] \mid f \text{ is continuous} \}.$$ 

Consider the set $[0,1]^C \cong \prod_{f \in C} [0,1]$ of all functions from $C$ to $[0,1]$, endowed with the product topology. Consider the map

$$\varphi : X \to [0,1]^C$$

$$x \mapsto (f(x))_{f \in C}$$

so that $\varphi(x)$ is “evaluation at $x$”.

a. Show that $\varphi$ is continuous.

b. Show that the closure of the image $\overline{\varphi(X)} \subset [0,1]^C$ is a compact Hausdorff space. (Feel free to assume the axiom of choice!)

c. Show that $\varphi$ is injective if and only if points of $X$ can be separated by functions, i.e. for any distinct points $x, y \in X$, there is a continuous function $f : X \to [0,1]$ satisfying $f(x) = 0$ and $f(y) = 1$.

This property is sometimes called functionally Hausdorff or completely Hausdorff.

d. While we’re at it, show that a functionally Hausdorff space is always Hausdorff.

Problem 4. (Willard Exercise 17F.1) A topological space $X$ is countably compact if every countable open cover of $X$ admits a finite subcover. (In particular, compact always implies countably compact, but not the other way around in general.)

Show that $X$ is countably compact if and only if every sequence in $X$ has a cluster point.

Hint: Recall that compactness can be described in terms of closed sets. Countable compactness has a very similar description in terms of closed sets, which could be useful here.