Problem 1. (Brown Exercise 2.4.5) Consider $X = [0, 2] \setminus \{1\}$ as a subspace of the real line $\mathbb{R}$. Show that the subset $[0, 1) \subset X$ is both open and closed in $X$.

Solution. $[0, 1)$ is open in $X$ because we can write

$$[0, 1) = (-8, 1) \cap X$$

and $(-8, 1)$ is open in $\mathbb{R}$.

On the other hand, $[0, 1)$ is closed in $X$ because we can write

$$[0, 1) = [0, 1] \cap X$$

and $[0, 1]$ is closed in $\mathbb{R}$. \hfill \Box$
Problem 2. (Bredon Exercise I.3.8) Let $X$ be a topological space that can be written as a union $X = A \cup B$ where $A$ and $B$ are closed subsets of $X$. Let $f: X \to Y$ be a function, where $Y$ is any topological space. Assume that the restrictions of $f$ to $A$ and to $B$ are both continuous. Show that $f$ is continuous.

Solution.

Lemma. Let $A \subseteq X$ be a closed subset. If $C \subseteq A$ is closed in $A$, then $C$ is also closed in $X$.

Proof. Since $C$ is closed in $A$, it can be written as $C = \tilde{C} \cap A$ for some closed subset $\tilde{C} \subseteq X$. Therefore $C$ is an intersection of closed subsets of $X$, and thus is closed in $X$. \qed

Let $C \subset Y$ be a closed subset. Its preimage under $f$ is the union

$$f^{-1}(C) = (f^{-1}(C) \cap A) \cup (f^{-1}(C) \cap B)$$

$$= (f|_A)^{-1}(C) \cup (f|_B)^{-1}(C).$$

Since the restriction $f|_A: A \to Y$ is continuous, $(f|_A)^{-1}(C)$ is closed in $A$, and thus closed in $X$ by the lemma. Likewise, $(f|_B)^{-1}(C)$ is closed in $X$. Therefore their union

$$f^{-1}(C) = (f|_A)^{-1}(C) \cup (f|_B)^{-1}(C).$$

is closed in $X$, so that $f$ in continuous. \qed

Remark. The same proof shows that the statement still holds if $A$ and $B$ are both open in $X$.  

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Problem 3. A map between topological spaces \( f : X \to Y \) is called an **open** map if for every open subset \( U \subseteq X \), its image \( f(U) \subseteq Y \) is open in \( Y \).

**a.** (Munkres Exercise 2.16.4) Let \( X \) and \( Y \) be topological spaces. Show that the projection maps \( p_X : X \times Y \to X \) and \( p_Y : X \times Y \to Y \) are open maps.

**Solution.**

**Lemma.** A map \( f : X \to Y \) is open if and only if \( f(B) \subseteq Y \) is open in \( Y \) for every \( B \in \mathcal{B} \) belonging to some basis \( \mathcal{B} \) of the topology on \( X \).

**Proof.** \((\Rightarrow)\) Each member \( B \in \mathcal{B} \) is open in \( X \).

\((\Leftarrow)\) Let \( U \subseteq X \) be open in \( X \). Then \( U \) is a union \( U = \bigcup_\alpha B_\alpha \) of basic open subsets \( B_\alpha \in \mathcal{B} \). Its image under \( f \) is

\[
  f(U) = f\left( \bigcup_\alpha B_\alpha \right) = \bigcup_\alpha f(B_\alpha)
\]

where each \( f(B_\alpha) \) is open in \( Y \) by assumption. Thus \( f(U) \) is a union of open subsets and hence open. \( \square \)

Take an “open box” \( U \times V \subseteq X \times Y \), where \( U \subseteq X \) is open and \( V \subseteq Y \) is open. Its projection onto the first factor is

\[
  p_X(U \times V) = U \subseteq X
\]

which is open in \( X \). Since open boxes form a basis of the topology on \( X \times Y \), the lemma guarantees that \( p_X \) is an open map, and likewise for \( p_Y \). \( \square \)
b. Find an example of metric spaces \( X \) and \( Y \), and a closed subset \( C \subseteq X \times Y \) such that the projection \( p_X(C) \subseteq X \) is not closed in \( X \).

In other words, the projection maps are (usually) not closed maps.

**Solution.** Take \( X = Y = \mathbb{R} \) and consider the hyperbola in \( \mathbb{R} \times \mathbb{R} \)

\[
C = \{(x, \frac{1}{x}) \mid x \neq 0\} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid xy = 1\}.
\]

Its projection onto the first factor is

\[
p_X(C) = \mathbb{R} \setminus \{0\}
\]

which is *not* closed in \( \mathbb{R} \).

To show that \( C \) is closed in \( \mathbb{R} \times \mathbb{R} \), note that the function \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) defined by \( f(x, y) = xy \) is continuous, and \( C \) is the preimage \( C = f^{-1}(\{1\}) \). Since the singleton \( \{1\} \) is closed in \( \mathbb{R} \), \( C \) is closed in \( \mathbb{R} \times \mathbb{R} \).
Problem 4. (Munkres Exercise 2.19.7) Consider the set of sequences of real numbers
\[ \mathbb{R}^\mathbb{N} = \{ (x_1, x_2, \ldots) \mid x_n \in \mathbb{R} \text{ for all } n \in \mathbb{N} \} \cong \prod_{n \in \mathbb{N}} \mathbb{R} \]
and consider the subset of sequences that are “eventually zero”
\[ \mathbb{R}^\infty := \{ x \in \mathbb{R}^\mathbb{N} \mid x_n \neq 0 \text{ for at most finitely many } n \} \].

a. In the box topology on \( \mathbb{R}^\mathbb{N} \), is \( \mathbb{R}^\infty \) a closed subset?

Solution. Yes, \( \mathbb{R}^\infty \) is closed in the box topology.

Let \( x \in \mathbb{R}^\mathbb{N} \setminus \mathbb{R}^\infty \), which means that the sequence \( x \) has infinitely many non-zero entries \( x_n \neq 0 \). For all those indices \( n \), pick an open neighborhood \( U_n \) of \( x_n \in \mathbb{R} \) which does not contain 0. For other values of \( n \), take \( U_n = \mathbb{R} \). Then the open box \( \prod_n U_n \) is an open neighborhood of \( x \) which does not intersect \( \mathbb{R}^\infty \).

Indeed, for any \( y \in \prod_n U_n \) and every index \( n \) such that \( x_n \neq 0 \), we have \( y_n \in U_n \) so that \( y_n \neq 0 \) by construction. Because there are infinitely many such indices, we conclude \( y \notin \mathbb{R}^\infty \).

b. In the product topology on \( \mathbb{R}^\mathbb{N} \), is \( \mathbb{R}^\infty \) a closed subset?

Solution. No, \( \mathbb{R}^\infty \) is not closed in the product topology.

Let \( x \in \mathbb{R}^\mathbb{N} \setminus \mathbb{R}^\infty \) and consider any open neighborhood \( U = \prod_n U_n \) of \( x \) which is a “large box”, i.e. \( U_n \subseteq \mathbb{R} \) is open for all \( n \) and \( U_n = \mathbb{R} \) except for finitely many \( n \). In particular, there is a number \( N \) such that \( U_n = \mathbb{R} \) for all \( n \geq N \). Consider a sequence \( y \) with \( y_n = 0 \) for all \( n \geq N \) and \( y_n \in U_n \) for \( 1 \leq n < N \). Then we have \( y \in U \cap \mathbb{R}^\infty \).

Because “large boxes” form a basis of the product topology, every open neighborhood of \( x \) intersects \( \mathbb{R}^\infty \). Therefore \( \mathbb{R}^\infty \) is not closed.

Remark. In fact, the argument shows that \( x \) is not an interior point of \( \mathbb{R}^\mathbb{N} \setminus \mathbb{R}^\infty \), so that the interior of \( \mathbb{R}^\mathbb{N} \setminus \mathbb{R}^\infty \) is empty. Equivalently, the closure of \( \mathbb{R}^\infty \) is all of \( \mathbb{R}^\mathbb{N} \), i.e. \( \mathbb{R}^\infty \) is dense in \( \mathbb{R}^\mathbb{N} \).
Problem 5. Let $X$ be a topological space, $S$ a set, and $f : X \to S$ a function. Consider the collection of subsets of $S$

$$\mathcal{T} := \{ U \subseteq S \mid f^{-1}(U) \text{ is open in } X \}.$$ 

a. Show that $\mathcal{T}$ is a topology on $S$.

Solution.

1. The preimage $f^{-1}(S) = X$ is open in $X$, so that the entire set $S$ is in $\mathcal{T}$. Likewise, $f^{-1}(\emptyset) = \emptyset$ is open in $X$, so that the empty set $\emptyset$ is in $\mathcal{T}$.

2. Let $U_\alpha$ be a family of members of $\mathcal{T}$. Then we have

$$f^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha f^{-1}(U_\alpha)$$

where each $f^{-1}(U_\alpha)$ is open in $X$ by assumption. Thus $f^{-1}\left(\bigcup_\alpha U_\alpha\right)$ is also open in $X$, so that the union $\bigcup_\alpha U_\alpha$ is in $\mathcal{T}$.

3. Let $U$ and $U'$ be members of $\mathcal{T}$. Then we have

$$f^{-1}(U \cap U') = f^{-1}(U) \cap f^{-1}(U')$$

where $f^{-1}(U)$ and $f^{-1}(U')$ are open in $X$ by assumption. Thus $f^{-1}(U \cap U')$ is also open in $X$, so that the finite intersection $U \cap U'$ is in $\mathcal{T}$.

b. Show that $\mathcal{T}$ is the largest topology on $S$ making $f$ continuous.

Solution. Note that $\mathcal{T}$ makes $f$ continuous by construction: for all $U \in \mathcal{T}$, the preimage $f^{-1}(U) \subseteq X$ is open in $X$.

Let $\mathcal{T}'$ be a topology on $S$ making $f$ continuous. Then for every $U \in \mathcal{T}'$, the preimage $f^{-1}(U)$ is open in $X$, which means $U \in \mathcal{T}$. This proves $\mathcal{T}' \subseteq \mathcal{T}$.

c. Let $Y$ be a topological space. Show that a map $g : S \to Y$ is continuous if and only if the composite $g \circ f : X \to Y$ is continuous.

Solution. ($\Rightarrow$) The maps $f$ and $g$ are continuous, hence so is their composite $g \circ f$.

($\Leftarrow$) Assume $g \circ f$ is continuous; we want to show that $g$ is continuous. Let $U \subseteq Y$ be open and take its preimage $g^{-1}(U) \subseteq S$. To check that this subset is open, consider its preimage

$$f^{-1}\left(\left(g^{-1}(U)\right)\right) = (g \circ f)^{-1}(U) \subseteq X$$

which is open in $X$ since $g \circ f$ is continuous. By definition of $\mathcal{T}$, $g^{-1}(U)$ is indeed open in $S$. 

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d. Show that $\mathcal{T}$ is the smallest topology on $S$ with the property that a map $g: S \to Y$ is continuous whenever $g \circ f$ is continuous.

Solution. Let $\mathcal{T}'$ be a topology on $S$ with said property. We know that $f: X \to (S, \mathcal{T})$ is continuous, but it can be written as the composite

$$X \xrightarrow{f} (S, \mathcal{T}') \xrightarrow{id} (S, \mathcal{T}).$$

By the property of $\mathcal{T}'$, the composite $\text{id} \circ f$ being continuous guarantees that the identity $\text{id}: (S, \mathcal{T'}) \to (S, \mathcal{T})$ is continuous, i.e. $\mathcal{T} \leq \mathcal{T}'$. \qed
Problem 6. Consider the subset \( X = \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \subset \mathbb{R} \) viewed as a subspace of the real line \( \mathbb{R} \). As a set, \( X \) is the disjoint union of the singletons \( \{0\} \) and \( \left\{ \frac{1}{n} \right\} \) for all \( n \in \mathbb{N} \). However, show that \( X \) does not have the coproduct topology on \( \{0\} \amalg \bigcup_{n \in \mathbb{N}} \left\{ \frac{1}{n} \right\} \).

Solution. In the coproduct topology on \( \{0\} \amalg \bigcup_{n \in \mathbb{N}} \left\{ \frac{1}{n} \right\} \) (which happens to be the discrete topology), the summand \( \{0\} \) is open.

However, in the subspace topology on \( X \), the singleton \( \{0\} \) is not open. Indeed, any open ball \( B_r(0) \) around 0 will contain other points \( \frac{1}{n} \in B_r(0) \), for all \( n \) such that \( \frac{1}{n} < r \). \( \square \)