Definition. Let \((X, \mathcal{T})\) be a topological space. A subset \(C \subseteq X\) is **closed** (with respect to \(\mathcal{T}\)) if its complement \(C^c := X \setminus C\) is open (with respect to \(\mathcal{T}\)).

**Problem 5.** Show that the collection of closed subsets of \(X\) satisfies the following properties.

1. The empty subset \(\emptyset\) and \(X\) itself are closed.
2. An arbitrary intersection of closed subsets is closed: \(C_\alpha\) closed for all \(\alpha\) implies \(\bigcap_\alpha C_\alpha\) is closed.
3. A finite union of closed subsets is closed: \(C, C'\) closed implies \(C \cup C'\) is closed.

**Remark.** In fact, a collection of subsets satisfies these properties if and only if their complements form a topology. Moreover, open subsets and closed subsets determine each other.

**Upshot:** One might as well define a topology via a collection of “closed subsets” satisfying the three properties above. Their complements then form the topology in question.

**Problem 6.** Let \(X\) be a set. Consider the collection of **cofinite** subsets of \(X\) together with the empty subset:

\[
\mathcal{T}_{\text{cofin}} := \{U \subseteq X \mid X \setminus U\text{ is finite}\} \cup \{\emptyset\}.
\]

**a.** Show that \(\mathcal{T}_{\text{cofin}}\) is a topology on \(X\), called the cofinite topology.

**b.** Assuming \(X\) is infinite, show that the cofinite topology on \(X\) cannot be induced by a metric on \(X\).

**Definition.** Let \(X\) be a set.

- The **discrete** topology on \(X\) is the one where all subsets are open:
  \[
  \mathcal{T}_{\text{disc}} = \mathcal{P}(X) = \{U \subseteq X\}.
  \]

- The **anti-discrete** (or **trivial**) topology on \(X\) is the one where only the empty subset and \(X\) itself are open:
  \[
  \mathcal{T}_{\text{anti}} = \{\emptyset, X\}.
  \]

**Problem 7.** Let \(D\) be a discrete topological space and \(A\) an anti-discrete topological space.

**a.** Describe all continuous maps \(f: D \rightarrow X\), where \(X\) is an arbitrary topological space.

**b.** Describe all continuous maps \(f: X \rightarrow A\), where \(X\) is an arbitrary topological space.

**c.** Describe all continuous maps \(f: A \rightarrow X\), where \(X\) is a metric space.

**Remark.** We will come back to the question of mapping into a discrete space when discussing the notion of connectedness.