Problem 1. Let $\text{Top}$ denote the category of topological spaces and continuous maps, and let $\text{CHaus}$ denote the category of compact Hausdorff topological spaces and continuous maps. Show that the Stone-Čech construction

$$\beta : \text{Top} \to \text{CHaus}$$

is a functor.

Note: So far we know that $\beta$ sends objects of $\text{Top}$ to objects of $\text{CHaus}$. There remain three things to check.

Solution. 1) $\beta$ sends morphisms to morphisms. Let $f : X \to Y$ be a continuous map between topological spaces. Consider the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{e_X} & \beta X \\
\downarrow f && \downarrow e_Y \\
Y & \xrightarrow{e_Y} & \beta Y
\end{array}
$$

where the horizontal maps are the canonical evaluation maps. Note that $\beta Y$ is compact Hausdorff, and the composite $e_Y \circ f : X \to \beta Y$ is continuous. By the universal property of $\beta X$, there is a unique continuous map $h : \beta X \to \beta Y$ making the diagram commute, i.e. satisfying

$$h \circ e_X = e_Y \circ f.$$

Define $\beta f$ to be this map $h$.

2) $\beta$ preserves composition. Consider a composite $X \xrightarrow{f} Y \xrightarrow{g} Z$ and consider the diagram

Note that the outer rectangle commutes because the two inner squares commute:

$$
(\beta g \circ \beta f) \circ e_X = \beta g \circ (\beta f \circ e_X) = \beta g \circ (e_Y \circ f) = (\beta g \circ e_Y) \circ f = (e_Z \circ g) \circ f = e_Z \circ (g \circ f).
$$
But $\beta(g \circ f) : \beta X \to \beta Z$ is the unique continuous map making the outer rectangle commute, which proves

$$\beta(g \circ f) = \beta g \circ \beta f.$$ 

3) $\beta$ preserves identities. The diagram

\[
\begin{array}{ccc}
X & \xrightarrow{e_X} & \beta X \\ 
\downarrow \text{id}_X & & \downarrow \text{id}_{\beta X} \\
X & \xrightarrow{e_X} & \beta X
\end{array}
\]

commutes:

$$\text{id}_{\beta X} \circ e_X = e_X = e_X \circ \text{id}_X.$$ 

But $\beta(\text{id}_X) : \beta X \to \beta X$ is the unique continuous map making this diagram commute, which proves

$$\beta(\text{id}_X) = \text{id}_{\beta X}.$$
Problem 2. Let $\textbf{Top}_*$ denote the category of pointed topological spaces and pointed continuous maps. To a space $X$, one can associate the pointed space

$$X_+ := X \amalg \{\ast\}$$

(with the coproduct topology) called “$X$ with a disjoint basepoint”, where $\ast \in X_+$ is the basepoint. To a continuous map $f: X \to Y$, one can assign the pointed continuous map

$$f_+: (X_+, \ast) \to (Y_+, \ast)$$

defined by

$$\begin{align*}
  f_+(x) &= f(x) \quad \text{if } x \in X \\
  f_+(& \ast) = \ast.
\end{align*}$$

One readily checks that this assignment makes the disjoint basepoint construction $(-)_+: \textbf{Top} \to \textbf{Top}_*$ into a functor.

a. Show that for any space $X$ and pointed space $(Y, y_0)$, there is a bijection

$$\text{Hom}_{\textbf{Top}_*}((X_+, \ast), (Y, y_0)) \cong \text{Hom}_{\textbf{Top}}(X, Y).$$

Solution. Let $\varphi: \text{Hom}_{\textbf{Top}_*}((X_+, \ast), (Y, y_0)) \to \text{Hom}_{\textbf{Top}}(X, Y)$ be the restriction map defined by

$$\varphi(f) = f|_X.$$

In the other direction, consider the function $\psi: \text{Hom}_{\textbf{Top}}(X, Y) \to \text{Hom}_{\textbf{Top}_*}((X_+, \ast), (Y, y_0))$ that sends a continuous map $g: X \to Y$ to the map $\psi(g): X_+ \to Y$ defined by

$$\psi(g)(x) = \begin{cases} 
  g(x) & \text{if } x \in X \\
  y_0 & \text{if } x = \ast.
\end{cases}$$

By construction, the map $\psi(g): (X_+, \ast) \to (Y, y_0)$ is pointed.

To prove moreover that $\psi(g)$ is continuous, note that its restrictions $\psi(g)|_X = g$ and $\psi(g)|_{\{\ast\}}$ are both continuous. Thus $\psi(g)$ is continuous, since $X_+ = X \amalg \{\ast\}$ has the coproduct topology.

$\psi \circ \varphi = \text{id}$. Let $f: (X_+, \ast) \to (Y, y_0)$ be a continuous pointed map. Then the map $\psi \varphi(f) = \psi(f|_X): (X_+, \ast) \to (Y, y_0)$ is given by

$$\psi(f|_X)(x) = \begin{cases} 
  f|_X(x) = f(x) & \text{if } x \in X \\
  y_0 & \text{if } x = \ast.
\end{cases}$$

Recall that $f$ is pointed, i.e. it satisfies $f(\ast) = y_0$, so that $\psi(f|_X)$ agrees with $f$ everywhere. This proves $\psi(f|_X) = f$.

$\varphi \circ \psi = \text{id}$. Let $g: X \to Y$ be a continuous map. Then we have $\varphi \psi(f) = (\psi f)|_X = f$ by construction. □

3
b. Show that the bijection in part (a) induces a bijection

\[ [(X_+, *), (Y, y_0)]_\ast \cong [X, Y] \]

where \([[(A, a_0), (B, b_0)]_\ast := \text{Hom}_{\text{Top}_\ast}((A, a_0), (B, b_0))\]
denotes the set of pointed homotopy classes of pointed continuous maps from \((A, a_0)\) to \((B, b_0)\).

As usual, \([X, Y] := \text{Hom}_{\text{Top}}(X, Y)\]
denotes the set of homotopy classes of continuous maps from \(X\) to \(Y\).

**Solution.** Let \(f, f' : (X_+, \ast) \to (Y, y_0)\) be pointed maps. The statement is that \(f\) and \(f'\) are pointed homotopic if and only if their restrictions \(\varphi(f) = f|_X\) and \(\varphi(f') = f'|_X\) are homotopic.

\((\Rightarrow)\)

Let \(F : X_+ \times [0, 1] \to Y\) be a pointed homotopy from \(f\) to \(f'\). Then the restriction

\[ F_{X \times [0, 1]} : X \times [0, 1] \to Y \]

is a homotopy from \(f|_X\) to \(f'|_X\).

\((\Leftarrow)\)

Let \(G : X \times [0, 1] \to Y\) be a homotopy from \(f|_X\) to \(f'|_X\). Consider the map

\[ \tilde{G} : X_+ \times [0, 1] \cong (X \times [0, 1]) \amalg \{\ast\} \times [0, 1] \to Y \]

defined by

\[
\begin{cases}
\tilde{G}|_{X \times [0, 1]} = G \\
\tilde{G}(\ast, t) = y_0 \quad \text{for all } t \in [0, 1].
\end{cases}
\]

In other words, \(\tilde{G}\) is obtained from \(G\) by applying \(\psi\) at each time: \(\tilde{g}_t = \psi(g_t)\).

Then \(\tilde{G}\) is continuous since its restriction to each summand \(X \times [0, 1]\) and \(\{\ast\} \times [0, 1]\) is continuous. By construction \(\tilde{G}\) is a pointed homotopy, i.e. it satisfies \(\tilde{G}(\ast, t) = y_0\) for all \(t \in [0, 1]\). In fact, it is a pointed homotopy between the pointed maps

\[
\begin{align*}
\tilde{g}_0 &= \psi(g_0) = \psi(f|_X) = f \\
\tilde{g}_1 &= \psi(g_1) = \psi(f'|_X) = f'
\end{align*}
\]

which are therefore pointed homotopic. \(\square\)
Problem 3. Let $X$ be a topological space.

a. Let $w, x, y, z \in X$, $\alpha : [0, 1] \to X$ a path from $w$ to $x$, $\beta : [0, 1] \to X$ a path from $x$ to $y$, and $\gamma : [0, 1] \to X$ a path from $y$ to $z$. Show that concatenation of paths is associative up to homotopy, in the following sense:

$$(\alpha \ast \beta) \ast \gamma \simeq \alpha \ast (\beta \ast \gamma) \text{ rel } \{0, 1\}.$$ 

Solution. Note that both sides are paths that go along $\alpha$, $\beta$, and $\gamma$ but at varying speeds. Given $0 < \sigma_1 < \sigma_2 < 1$, consider the path $\delta(\sigma_1, \sigma_2) : [0, 1] \to X$ from $w$ to $z$ defined by

$$\delta(\sigma_1, \sigma_2)(s) = \begin{cases} 
\alpha\left(\frac{s-0}{\sigma_1-0}\right) & \text{if } 0 \leq s \leq \sigma_1 \\
\beta\left(\frac{s-\sigma_1}{\sigma_2-\sigma_1}\right) & \text{if } \sigma_1 \leq s \leq \sigma_2 \\
\gamma\left(\frac{s-\sigma_2}{1-\sigma_2}\right) & \text{if } \sigma_2 \leq s \leq 1.
\end{cases}$$

In this notation, we have:

$$(\alpha \ast \beta) \ast \gamma = \delta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$\alpha \ast (\beta \ast \gamma) = \delta\left(\frac{1}{2}, \frac{3}{4}\right).$$

Now if $\sigma_1(t)$ and $\sigma_2(t)$ are continuous functions of $t$ satisfying $0 < \sigma_1(t) < \sigma_2(t) < 1$ for all $t \in [0, 1]$, then the map $H : [0, 1] \times [0, 1] \to X$ defined by

$$H(-, t) = h_t = \delta(\sigma_1(t), \sigma_2(t))$$

is continuous and satisfies

$$H(0, t) = \delta(\sigma_1(0), \sigma_2(t))(0) = w$$

$$H(1, t) = \delta(\sigma_1(1), \sigma_2(t))(1) = z$$

for all $t \in [0, 1]$, so that $H$ is a path homotopy from $h_0$ to $h_1$.

In the case at hand, take $\sigma_1(t) = \frac{1}{4} + \frac{1}{4}t$ and $\sigma_2(t) = \frac{1}{2} + \frac{1}{4}t$ to obtain a path homotopy $H$ between

$$h_0 = \delta(\sigma_1(0), \sigma_2(0)) = \delta\left(\frac{1}{4}, \frac{1}{2}\right) = (\alpha \ast \beta) \ast \gamma$$

$$h_1 = \delta(\sigma_1(1), \sigma_2(1)) = \delta\left(\frac{1}{2}, \frac{3}{4}\right) = \alpha \ast (\beta \ast \gamma).$$

\qed
b. Let α: [0, 1] → X be a path in X from x to y. Denote by \( \overline{\alpha} : [0, 1] \to X \) the reverse path of \( \alpha \), defined by

\[
\overline{\alpha}(s) = \alpha(1 - s).
\]

Show that \( \overline{\alpha} \) is inverse to \( \alpha \) up to homotopy, in the following sense:

\[
\alpha \ast \overline{\alpha} \simeq 1_x \text{ rel } \{0, 1\}
\]

where \( 1_x : [0, 1] \to X \) denotes the constant path at \( x \).

**Solution.** The left-hand side is the path given by

\[
(\alpha \ast \overline{\alpha})(s) = \begin{cases} 
\alpha(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\
\alpha(2s - 1) = \alpha(2 - 2s) & \text{if } \frac{1}{2} \leq s \leq 1
\end{cases}
\]

which can be rewritten as \( \alpha \ast \overline{\alpha} = \alpha \circ p \) where \( p : [0, 1] \to [0, 1] \) is the “spike-shaped” function

\[
p(s) = \begin{cases} 
2s & \text{if } 0 \leq s \leq \frac{1}{2} \\
2 - 2s & \text{if } \frac{1}{2} \leq s \leq 1.
\end{cases}
\]

It suffices to show that \( p \) is homotopic rel \( \{0, 1\} \) to the constant function 0 in order to conclude

\[
\alpha \ast \overline{\alpha} = \alpha \circ p \\
\simeq \alpha \circ 0 \text{ rel } \{0, 1\} \\
= \text{ constant path at } \alpha(0) \\
= 1_x
\]

as desired.

The map \( H : [0, 1] \times [0, 1] \to [0, 1] \) defined by

\[
H(s, t) = tp(s)
\]

is continuous and satisfies

\[
H(0, t) = tp(0) = 0 \\
H(1, t) = tp(1) = 0
\]

for all \( t \in [0, 1] \). Therefore \( H \) is a homotopy rel \( \{0, 1\} \) between \( h_0 = H(-, 0) \equiv 0 \) and \( h_1 = H(-, 1) = p \) as desired. \( \square \)

**Remark.** No need to check the condition \( \overline{\alpha} \ast \alpha \simeq 1_y \text{ rel } \{0, 1\} \), which follows from part (b) applied to the path \( \overline{\alpha} \) and observing \( \overline{\alpha} = \alpha \).

**Remark.** We have earned the right to adopt the notation \( \overline{\alpha} = \alpha^{-1} \).
Definition. Let $A \subseteq X$ be a subspace of $X$, and denote by $i: A \to X$ the inclusion. Then $A$ is called...

- a **retract** of $X$ if there is a continuous map $r: X \to A$ satisfying $r \circ i = \text{id}_A$, in other words $r(a) = a$ for all $a \in A$. Such a map $r$ is called a **retraction** from $X$ to $A$.

- a **deformation retract** of $X$ if there is a retraction $r: X \to A$ which is moreover a homotopy equivalence, i.e. satisfying $i \circ r \simeq \text{id}_X$.

  Explicitly: There is a homotopy $H: X \times [0, 1] \to X$ satisfying $H(x, 0) = x$ for all $x \in X$, $H(x, 1) \in A$ for all $x \in X$, and $H(a, 1) = a$ for all $a \in A$.

- a **strong deformation retract** of $X$ if there is a retraction $r: X \to A$ which moreover satisfies

$$i \circ r \simeq \text{id}_X \text{ rel } A.$$

Explicitly: There is a homotopy $H: X \times [0, 1] \to X$ satisfying $H(x, 0) = x$ for all $x \in X$, $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A$ and all $t \in [0, 1]$. 

7
**Problem 4.** Consider the 2-simplex

\[\Delta^2 := \{(x, y) \in \mathbb{R}^2 \mid x + y \leq 1, x \geq 0, y \geq 0\}\]

and consider the subspace of \(\Delta^2\) consisting of points on the coordinate axes

\[A = \{(x, y) \in \Delta^2 \mid x = 0 \text{ or } y = 0\} = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{0\}).\]

Show that \(A\) is a strong deformation retract of \(\Delta^2\).

**Solution.** Write \(\Delta^2\) as the union \(\Delta^2 = C \cup D\) of two simplices with (ordered) vertices \((0, 0), (0, 1), (\frac{1}{2}, \frac{1}{2})\) and \((0, 0), (1, 0), (\frac{1}{2}, \frac{1}{2})\) respectively. In other words, \(C\) is the “top left” half of \(\Delta^2\) where \(y \geq x\), while \(D\) is the “bottom right” half where \(y \leq x\). Note that \(C\) and \(D\) are both closed (in \(\mathbb{R}^2\) and therefore in \(\Delta^2\)).

For \(t \in [0, 1]\), let \(C_t\) and \(D_t\) be the simplices with (ordered) vertices \((0, 0), (0, 1), t(\frac{1}{2}, \frac{1}{2})\) and \((0, 0), (1, 0), t(\frac{1}{2}, \frac{1}{2})\) respectively. Let \(H : \Delta^2 \times [0, 1] \to \Delta^2\) be the map defined as follows.

- \(H|_{C \times [0, 1]}(-, t)\) is the unique affine transformation sending \(C\) to \(C_t\).
- \(H|_{D \times [0, 1]}(-, t)\) is the unique affine transformation sending \(D\) to \(D_t\).

Note that both maps are continuous, since the vertices of \(C_t\) and \(D_t\) vary continuously as a function of \(t\).

Also note that \(C \cap D\) is the segment joining \((0, 0)\) and \((\frac{1}{2}, \frac{1}{2})\), and those two vertices are sent to \((0, 0)\) and \(t(\frac{1}{2}, \frac{1}{2})\) by \(H|_{C \times [0, 1]}(-, t)\) and by \(H|_{D \times [0, 1]}(-, t)\). Therefore the two maps agree on the intersection

\((C \times [0, 1]) \cap (D \times [0, 1]) = (C \cap D) \times [0, 1]\)

and thus define a map \(H\) on \((C \times [0, 1]) \cup (D \times [0, 1]) = \Delta^2 \times [0, 1]\). Moreover \(H\) is continuous since the restrictions \(H|_{C \times [0, 1]}^0\) and \(H|_{D \times [0, 1]}^0\) are continuous, and both subsets \(C \times [0, 1]\) and \(D \times [0, 1]\) are closed in \(\Delta^2 \times [0, 1]\).

For all \(t \in [0, 1]\), the map \(H|_{C \times [0, 1]}(-, t)\) sends \((0, 0)\) to \((0, 0)\) and \((0, 1)\) to \((0, 1)\) and therefore (since it is affine) leaves every point on the vertical segment between \((0, 0)\) to \((0, 1)\) fixed.

Likewise, the map \(H|_{D \times [0, 1]}(-, t)\) leaves every point on the horizontal segment between \((0, 0)\) to \((1, 0)\) fixed. This proves \(H(a, t) = a\) for all \(a \in A\) and \(t \in [0, 1]\).

The equalities \(C_1 = C\) and \(D_1 = D\) prove \(H(-, 1) = \text{id}_{\Delta^2}\).

The equality \(C_0 \cup D_0 = A\) proves \(H(x, 0) \in A\) for all \(x \in \Delta^2\).

Therefore \(H\) is a homotopy rel \(A\) between \(\text{id}_{\Delta^2}\) and a retraction \(\Delta^2 \to A\).

\(\square\)

**Remark.** A straightforward calculation yields the explicit formula of \(H\):

\[h_t(x, y) = \begin{cases} (tx, y - x + tx) & \text{if } y \geq x \\ (x - y + ty, ty) & \text{if } y \leq x. \end{cases}\]
Problem 5. Two objects \( X \) and \( Y \) of a category \( C \) are connected by morphisms if there is a zigzag of morphisms between them. More precisely, there is a finite sequence of objects

\[
X = X_0, X_1, \ldots, X_{n-1}, X_n = Y
\]

and for every \( 0 \leq i < n \), there is a morphism \( f_i : X_i \to X_{i+1} \) or \( f_i : X_{i+1} \to X_i \).

a. Show that two objects \( X \) and \( Y \) of a groupoid \( G \) are connected by morphisms if and only if there is a morphism \( f : X \to Y \).

Solution. \((\Leftarrow\Rightarrow)\) The morphism \( f : X \to Y \) exhibits \( X \) and \( Y \) as being connected by morphisms, i.e. \( X_0 = X, X_1 = Y, f_0 = f \).

\((\Rightarrow)\) Assume there is a zigzag of morphisms \( f_i \) from \( X = X_0 \) to \( Y = X_n \). Since \( G \) is a groupoid, every morphism has an inverse, and we can define morphisms \( g_i : X_i \to X_{i+1} \) by

\[
g_i = \begin{cases} f_i & \text{if } f_i : X_i \to X_{i+1} \\ f_i^{-1} & \text{if } f_i : X_{i+1} \to X_i.\end{cases}
\]

Then the composite

\[
X = X_0 \xrightarrow{g_0} X_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{n-1}} X_n = Y
\]

is a morphism from \( X \) to \( Y \).

Remark. In particular, two points \( x \) and \( y \) in a space \( X \) are connected by morphisms in the fundamental groupoid \( \Pi_1(X) \) if and only if they lie in the same path component of \( X \). \(\square\)
b. Find an example of category $C$ and objects $X$ and $Y$ of $C$ that are connected by morphisms, but such that there are no morphisms from $X$ to $Y$ and no morphisms from $Y$ to $X$:

$$\text{Hom}_C(X, Y) = \emptyset \quad \text{and} \quad \text{Hom}_C(Y, X) = \emptyset.$$ 

Solution. Let $C$ be the category described by the graph

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & Y \\
\downarrow{f} & & \downarrow \\
X & & \\
\end{array}
\]

More precisely, $C$ has three objects $\text{Ob}(C) = \{X, Y, Z\}$ and only two non-identity morphisms $f: Z \to X$ and $g: Z \to Y$. This automatically forms a category, since no non-identity morphisms are composable.

The objects $X$ and $Y$ are connected by the zigzag of morphisms

\[
\begin{array}{ccc}
X & \xleftarrow{f} & Z & \xrightarrow{g} & Y \\
\end{array}
\]

but by definition, there are no morphisms $X \to Y$ or $Y \to X$. \qed
c. Let $X$ and $Y$ be objects of a groupoid $\mathcal{G}$ that are connected by morphisms. Show that the vertex groups at $X$ and $Y$ are isomorphic (as groups):

$$\text{Aut}_\mathcal{G}(X) \simeq \text{Aut}_\mathcal{G}(Y).$$

**Solution.** By part (a), let $f : X \to Y$ be a morphism. Consider the map $\varphi^f : \text{Aut}_\mathcal{G}(Y) \to \text{Aut}_\mathcal{G}(X)$ defined by

$$\varphi^f(g) = f^{-1} \circ g \circ f.$$

$\varphi^f$ is a group homomorphism.

$$\varphi^f(g_1 \circ g_2) = f^{-1} \circ g_1 \circ g_2 \circ f$$

$$= f^{-1} \circ g_1 \circ f \circ f^{-1} \circ g_2 \circ f$$

$$= \varphi^f(g_1) \circ \varphi^f(g_2).$$

$\varphi^f$ is invertible. In fact its inverse is $\varphi^{-1}_f : \text{Aut}_\mathcal{G}(X) \to \text{Aut}_\mathcal{G}(Y)$.

For any $g \in \text{Aut}_\mathcal{G}(Y)$ we have:

$$\varphi^{-1}_f \varphi^f(g) = \varphi^{-1}_f(f^{-1} \circ g \circ f)$$

$$= (f^{-1})^{-1} \circ f^{-1} \circ g \circ f \circ f^{-1}$$

$$= f \circ f^{-1} \circ g \circ f \circ f^{-1}$$

$$= \text{id}_Y \circ g \circ \text{id}_Y$$

$$= g.$$

Likewise, we have $\varphi^f \varphi^{-1}_f(g) = g$ for all $g \in \text{Aut}_\mathcal{G}(X)$.

$\square$

Remark. This proves in particular that if $x$ and $y$ are two points in the same path component of a space $X$, then the fundamental groups $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic.
Problem 6. Let \( f : X \xrightarrow{\sim} Y \) be a homotopy equivalence between topological spaces. Show that for any choice of basepoint \( x_0 \in X \), the induced group homomorphism
\[
\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))
\]
is an isomorphism.

Solution. Note: We omit “homotopy classes” of paths to ease the notation, i.e. write \( \gamma \) instead of \([\gamma]\).

Lemma. Let \( f, f' : X \to Y \) be two continuous maps, and let \( H : X \times [0, 1] \to Y \) be a homotopy (unpointed) from \( f \) to \( f' \). Then for any basepoint \( x_0 \in X \), the induced maps \( \pi_1(f) \) and \( \pi_1(f') \) differ by a “change of basepoint” isomorphism
\[
\pi_1(X, x_0) \simeq \pi_1(Y, f(x_0)) \to \pi_1(Y, f'(x_0)) \xrightarrow{\varphi^\alpha}
\]
where \( \varphi^\alpha : \pi_1(Y, f'(x_0)) \xrightarrow{\sim} \pi_1(Y, f(x_0)) \) is the group isomorphism (c.f. Problem 5c)
\[
\varphi^\alpha(\gamma) = \alpha * \gamma * \alpha^{-1}
\]
induced by the path \( \alpha \) in \( Y \) from \( f(x_0) \) to \( f'(x_0) \) given by \( \alpha(t) = H(x_0, t) \).
In particular, \( \pi_1(f) \) is an isomorphism if and only if \( \pi_1(f') \) is.

Proof. Let \( \gamma : [0, 1] \to X \) be a loop based at \( x_0 \). Then the two loops in \( Y \) being compared are
\[
\pi_1(f)(\gamma) = f(\gamma) = h_0(\gamma)
\]
and
\[
\pi_1(f')(\gamma) = f'(\gamma) = h_1(\gamma)
\]
and they are based at different points: \( f(x_0) \) and \( f'(x_0) \) respectively. In fact, for any \( t \in [0, 1] \), \( h_t(\gamma) \) is a loop in \( Y \) based at \( h_t(x_0) \in Y \).

Consider the loop in \( Y \) based at \( f(x_0) \) that first runs along \( \alpha \) up to \( \alpha(t) \), then goes through the loop \( h_t(\gamma) \), then comes back to \( f(x_0) \) along \( \alpha^{-1} \):
\[
a_t : [0, 1 + 2t] \to Y \quad \text{with} \quad a_t(s) = \begin{cases} 
\alpha(s) & \text{if } 0 \leq s \leq t \\
h_t(s - t) & \text{if } t \leq s \leq 1 + t \\
\alpha(t - (s - 1 - t)) = \alpha(1 + 2t - s) & \text{if } 1 + t \leq s \leq 1 + 2t.
\end{cases}
\]
Then \( a_t \) is continuous because \( \alpha \) and \( H \) are. Reparametrizing to the interval \([0, 1]\) yields the loop \( b_t : [0, 1] \to Y \) defined by
\[
b_t(s) = a_t(s(1 + 2t))
\]
Now the map $B: [0,1] \times [0,1] \to Y$ defined by $B(s,t) = b_t(s)$ is continuous and satisfies the endpoint conditions

\[
B(0,t) = b_t(0) = a_t(0) = f(x_0)
\]
\[
B(1,t) = b_t(1) = a_t(1 + 2t) = f(x_0)
\]

for all $t \in [0,1]$. Therefore $B$ is a pointed homotopy from the loop

$$b_0 = a_0 = h_0(\gamma) = f(\gamma)$$

to the loop

$$b_1 \simeq \alpha * h_1(\gamma) * \alpha^{-1} = \varphi^\alpha(h_1(\gamma)) = \varphi^\alpha(f'(\gamma))$$

which proves $\pi_1(f) = \varphi^\alpha \circ \pi_1(f')$.

Let $g: Y \to X$ be a homotopy inverse of $f: X \to Y$. Consider the composite

\[
\pi_1(X,x_0) \xrightarrow{\pi_1(f)} \pi_1(Y,f(x_0)) \xrightarrow{\pi_1(g)} \pi_1(X,g(f(x_0))) \xrightarrow{\sim} \pi_1(g \circ f)
\]

where $\pi_1(g) \circ \pi_1(f) = \pi_1(g \circ f)$ is an isomorphism by the lemma. Indeed, $g \circ f$ is (unpointed) homotopic to $\text{id}_X$, and $\pi_1(\text{id}_X)$ is an isomorphism. Therefore $\pi_1(f)$ is injective and $\pi_1(g)$ is surjective.

Since the basepoint was arbitrary, the argument also applies to the homotopy equivalence $g: Y \xrightarrow{\sim} X$ and basepoint $f(x_0) \in Y$, so that

$$\pi_1(g): \pi_1(Y,f(x_0)) \to \pi_1(X,g(f(x_0)))$$

is also injective, and thus an isomorphism.

Therefore $\pi_1(f) = \pi_1(g)^{-1} \circ \pi_1(g \circ f)$ is an isomorphism.

\[\Box\]
Alternate proof that $\pi_1(f)$ is surjective. Consider the diagram

\[
\begin{array}{ccc}
\pi_1(Y, f(x_0)) & \xrightarrow{\pi_1(f \circ g)} & \pi_1(Y, f(g(f(x_0)))) \\
\pi_1(X, g(f(x_0))) & \xrightarrow{\pi_1(f)} & \pi_1(Y, f(g(f(x_0)))) \\
\end{array}
\]

where the top composite is an isomorphism, again by the lemma. Here $\alpha$ denotes the path in $X$ from $x_0$ to $g(f(x_0))$ given by $\alpha(t) = H(x_0, t)$ where $H$ is a homotopy from $\text{id}_X$ to $g \circ f$.

The square on the right commutes. For any $\gamma \in \pi_1(X, g(f(x_0)))$ we have

\[
\pi_1(f) \circ \varphi^\alpha(\gamma) = \pi_1(f) \left( \alpha \ast \gamma \ast \alpha^{-1} \right)
= f(\alpha \ast \gamma \ast \alpha^{-1})
= f(\alpha) \ast f(\gamma) \ast f(\alpha^{-1})
= f(\alpha) \ast f(\gamma) \ast f(\alpha)^{-1}
= \varphi^{f(\alpha)}(f(\gamma))
= \varphi^{f(\alpha)} \circ \pi_1(f)(\gamma).
\]

It follows that the composite

\[
\varphi^{f(\alpha)} \circ \pi_1(f) \circ \pi_1(g) = \pi_1(f) \circ \varphi^\alpha \circ \pi_1(g)
\]

is an isomorphism. Therefore the last step $\pi_1(f): \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is surjective. \qed