Problem 3. Let $X$ be a topological space.

a. Let $w, x, y, z \in X$, $\alpha : [0, 1] \to X$ a path from $w$ to $x$, $\beta : [0, 1] \to X$ a path from $x$ to $y$, and $\gamma : [0, 1] \to X$ a path from $y$ to $z$. Show that concatenation of paths is associative up to homotopy, in the following sense:

$$(\alpha \ast \beta) \ast \gamma \simeq \alpha \ast (\beta \ast \gamma) \text{ rel } \{0, 1\}.$$

b. Let $\alpha : [0, 1] \to X$ be a path in $X$ from $x$ to $y$. Denote by $\overline{\alpha} : [0, 1] \to X$ the reverse path of $\alpha$, defined by

$$\overline{\alpha}(s) = \alpha(1 - s).$$

Show that $\overline{\alpha}$ is inverse to $\alpha$ up to homotopy, in the following sense:

$$\alpha \ast \overline{\alpha} \simeq 1_x \text{ rel } \{0, 1\}$$

where $1_x : [0, 1] \to X$ denotes the constant path at $x$.

Remark. No need to check the condition $\overline{\alpha} \ast \alpha \simeq 1_y \text{ rel } \{0, 1\}$, which follows from part (b) applied to the path $\overline{\alpha}$ and observing $\overline{\alpha} = \alpha$.

Remark. We have earned the right to adopt the notation $\overline{\alpha} = \alpha^{-1}$.

Definition. Let $A \subseteq X$ be a subspace of $X$, and denote by $i : A \to X$ the inclusion. Then $A$ is called...

- a retract of $X$ if there is a continuous map $r : X \to A$ satisfying $r \circ i = \text{id}_A$, in other words $r(a) = a$ for all $a \in A$. Such a map $r$ is called a retraction from $X$ to $A$.

- a deformation retract of $X$ if there is a retraction $r : X \to A$ which is moreover a homotopy equivalence, i.e. satisfying $i \circ r \simeq \text{id}_X$.

Explicitly: There is a homotopy $H : X \times [0, 1] \to X$ satisfying $H(x, 0) = x$ for all $x \in X$, $H(x, 1) \in A$ for all $x \in X$, and $H(a, 1) = a$ for all $a \in A$.

- a strong deformation retract of $X$ if there is a retraction $r : X \to A$ which moreover satisfies

$$i \circ r \simeq \text{id}_X \text{ rel } A.$$ 

Explicitly: There is a homotopy $H : X \times [0, 1] \to X$ satisfying $H(x, 0) = x$ for all $x \in X$, $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A$ and all $t \in [0, 1]$.

Problem 4. Consider the 2-simplex

$$\Delta^2 := \{(x, y) \in \mathbb{R}^2 \mid x + y \leq 1, x \geq 0, y \geq 0\}$$

and consider the subspace of $\Delta^2$ consisting of points on the coordinate axes

$$A = \{(x, y) \in \Delta^2 \mid x = 0 \text{ or } y = 0\} = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{0\}).$$

Show that $A$ is a strong deformation retract of $\Delta^2$. 

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