1 Separation axioms

Definition 1.1. A topological space $X$ is called:

- **$T_0$ or Kolmogorov** if any distinct points are topologically distinguishable: For $x, y \in X$ with $x \neq y$, there is an open subset $U \subset X$ containing one of the two points but not the other.

- **$T_1$** if any distinct points are separated (i.e. not in the closure of the other): For $x, y \in X$ with $x \neq y$, there are open subsets $U_x, U_y \subset X$ satisfying $x \in U_x$ but $y \notin U_x$, whereas $y \in U_y$ but $x \notin U_y$.

- **$T_2$ or Hausdorff** if any distinct points can be separated by neighborhoods: For $x, y \in X$ with $x \neq y$, there are open subsets $U_x, U_y \subset X$ satisfying $x \in U_x, y \in U_y$, and $U_x \cap U_y = \emptyset$.

- **regular** if points and closed sets can be separated by neighborhoods: For $x \in X$ and $C \subset X$ closed with $x \notin C$, there are open subsets $U_x, U_C \subset X$ satisfying $x \in U_x, C \subset U_C$, and $U_x \cap U_C = \emptyset$.

- **$T_3$** if it is $T_1$ and regular.

- **completely regular** if points and closed sets can be separated by functions: For $x \in X$ and $C \subset X$ closed with $x \notin C$, there is a continuous function $f: X \to [0, 1]$ satisfying $f(x) = 0$ and $f|_C \equiv 1$.

- **$T_{3\frac{1}{2}}$ or Tychonoff** if it is $T_1$ and completely regular.

- **normal** if closed sets can be separated by neighborhoods: For $A, B \subset X$ closed and disjoint, there are open subsets $U, V \subset X$ satisfying $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

- **$T_4$** if it is $T_1$ and normal.

- **perfectly normal** if closed sets can be precisely separated by functions: For $A, B \subset X$ closed and disjoint, there is a continuous function $f: X \to [0, 1]$ satisfying $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

- **$T_6$** if it is $T_1$ and perfectly normal.
2 Compactness

Definition 2.1. A topological space $X$ is called:
- **compact** if every open cover of $X$ admits a finite subcover.
- **countably compact** if every countable open cover of $X$ admits a finite subcover.
- **sequentially compact** if every sequence in $X$ has a convergent subsequence.
- **Lindelöf** if every open cover of $X$ admits a countable subcover.
- **locally compact** if every point $x \in X$ has a compact neighborhood.
- **$\sigma$-compact** if $X$ is a countable union of compact subspaces.
- **paracompact** if every open cover of $X$ admits a locally finite refinement.
- **hemicompact** if there is a countable collection of compact subspaces $K_n \subseteq X$ such that for any compact subspace $K \subseteq X$, there is an $n \in \mathbb{N}$ satisfying $K \subseteq K_n$.

3 Countability axioms

Definition 3.1. A topological space $X$ is called:
- **first-countable** if every point $x \in X$ has a countable neighborhood basis.
- **second-countable** if the topology on $X$ has a countable basis.
- **separable** if $X$ has a countable dense subset.

4 Connectedness

Definition 4.1. A topological space $X$ is called:
- **connected** if $X$ is not a disjoint union of non-empty open subsets.
- **locally connected** if for all $x \in X$ and neighborhood $U$ of $x$, there is a connected neighborhood $V$ of $x$ satisfying $V \subseteq U$.
- **path-connected** if any two points of $X$ can be joined by a path.
- **locally path-connected** if for all $x \in X$ and neighborhood $U$ of $x$, there is a path-connected neighborhood $V$ of $x$ satisfying $V \subseteq U$. 

5 Properties of maps

Definition 5.1. A function \( f : X \to Y \) between topological spaces is called:

- **continuous** if for any open \( U \subseteq Y \), the preimage \( f^{-1}(U) \subseteq X \) is open in \( X \).
- **open** if for any open \( U \subseteq X \), the image \( f(U) \subseteq Y \) is open in \( Y \).
- **closed** if for any closed \( C \subseteq X \), the image \( f(C) \subseteq Y \) is closed in \( Y \).
- a **homeomorphism** if it is bijective and its inverse \( f^{-1} : Y \to X \) is continuous.
- an **embedding** if it is injective and a homeomorphism onto its image \( f(X) \).
- a **quotient map** or **identification map** if it is surjective and \( Y \) has the quotient topology induced by \( f \).
- **proper** if for any compact subspace \( K \subseteq Y \), the preimage \( f^{-1}(K) \subseteq X \) is compact.

Definition 5.2. A function \( f : X \to Y \) between metric spaces is called:

- **uniformly continuous** if for any \( \epsilon > 0 \), there is a \( \delta > 0 \) satisfying \( f(B_\delta(x)) \subseteq B_\epsilon(f(x)) \) for all \( x \in X \).
- **Lipschitz continuous** with Lipschitz constant \( K \geq 0 \) it satisfies the inequality
  \[
d(f(x), f(x')) \leq K d(x, x')
\]
  for all \( x, x' \in X \).