Math 535 - General Topology
Fall 2012
Final Exam

Name: Solutions

• Allowed: The four textbooks, notebooks, printouts of course material, and your own homework.

• Not allowed: Other documents, and any electronic devices (laptops, iPads, smartphones, cell phones, etc.).

• Choose 5 of the 6 problems for grading. Please indicate clearly which problem you do not want graded.

• Show your work. No credit for answers without justification.

• Good luck!

1. ________/20
2. ________/20
3. ________/20
4. ________/20
5. ________/20
6. ________/20

Total: ________/100
Problem 1. Show that an uncountable product of non-trivial spaces is not first-countable.

Solution. Let $X = \prod_{i \in I} X_i$ be a product of non-trivial spaces indexed by an uncountable set $I$. For each $i \in I$, pick a non-trivial open subset $U_i \subset X_i$ (i.e. $U_i \neq \emptyset$ and $U_i \neq X_i$), and pick a point $x_i \in U_i$. Consider the point $x = (x_i)_{i \in I} \in X$. We will show that $x$ has no countable neighborhood basis.

Let $\{V^{(n)}\}_{n \in \mathbb{N}}$ be a countable collection of neighborhoods of $x$ in $X$. WLOG the $V^{(n)}$ are basic open, so that each can be written as a product

$$V^{(n)} = \prod_{i \in I} V^{(n)}_i$$

with $V^{(n)}_i \subseteq X_i$ open, and there is a finite set of indices $F^{(n)} \subset I$ such that $V^{(n)}_i = X_i$ for all $i \not\in F^{(n)}$.

The union $J := \bigcup_{n \in \mathbb{N}} F^{(n)} \subset I$ is a countable union of finite sets, hence countable. Since $I$ is uncountable, we have $J \neq I$ and we can pick an index $i_0 \in I \setminus J$. Consider the neighborhood $W = p^{-1}_{i_0}(U_{i_0})$ of $x$, which can be written explicitly as the product $W = \prod_{i \in I} W_i$ with

$$W_i = \begin{cases} U_{i_0} & \text{if } i = i_0 \\ X_i & \text{if } i \neq i_0. \end{cases}$$

For all $n \in \mathbb{N}$, we have $V^{(n)} \not\subseteq W$. Indeed, their projections onto the factor $X_{i_0}$ are

$$p_{i_0}(V^{(n)}) = V^{(n)}_{i_0} = X_{i_0}$$

since $i_0 \not\in F^{(n)}$ whereas

$$p_{i_0}(W) = U_{i_0} \subsetneq X_{i_0}.$$

Therefore $\{V^{(n)}\}_{n \in \mathbb{N}}$ is not a neighborhood basis at $x$. \qed
Problem 2. Show that the Stone-Čech compactification of the natural numbers $\beta \mathbb{N}$ is not sequentially compact.

Solution. Consider the evaluation map $e: \mathbb{N} \to \beta \mathbb{N}$ and the sequence $(x_n)_{n \in \mathbb{N}}$ in $\beta \mathbb{N}$ defined by $x_n = e(n)$. We will show that this sequence has no convergent subsequence in $\beta \mathbb{N}$.

Let $(x_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(x_n)_{n \in \mathbb{N}}$. To prove that $(x_{n_k})_{k \in \mathbb{N}}$ does not converge in $\beta \mathbb{N}$, it suffices to find a continuous map $f: \beta \mathbb{N} \to Y$ such that $(f(x_{n_k}))_{k \in \mathbb{N}}$ does not converge in $Y$.

Take the discrete space $Y = \{0, 1\}$, which is compact (since finite) and Hausdorff (since discrete). Since the indices $(n_k)_{k \in \mathbb{N}}$ are distinct, we can define a map $h: \mathbb{N} \to \{0, 1\}$ by

$$
\begin{cases}
  h(n_k) = 0 & \text{if } k \text{ is even} \\
  h(n_k) = 1 & \text{if } k \text{ is odd} \\
  h(n) = \text{whatever} & \text{if } n \notin \{n_k \mid k \in \mathbb{N}\}.
\end{cases}
$$

Then $h: \mathbb{N} \to \{0, 1\}$ is automatically continuous since $\mathbb{N}$ is discrete. Since $\{0, 1\}$ is compact Hausdorff, there is a unique continuous map $f: \beta \mathbb{N} \to \{0, 1\}$ satisfying $h = f \circ e$, i.e. making the diagram

$$
\begin{array}{ccc}
\mathbb{N} & \xrightarrow{e} & \beta \mathbb{N} \\
\downarrow{h} & & \downarrow{f} \\
\{0, 1\} & \quad \exists f
\end{array}
$$

commute. Then the values $f(x_{n_k})$ are

$$
f(x_{n_k}) = f(e(n_k)) = h(n_k) = \begin{cases}
  0 & \text{if } k \text{ is even} \\
  1 & \text{if } k \text{ is odd}
\end{cases}
$$

so that the sequence $(f(x_{n_k}))_{k \in \mathbb{N}}$ does not converge in $\{0, 1\}$. \qed
**Alternate solution.** We will use the explicit model for $\beta\mathbb{N}$. Consider the set $C := C(\mathbb{N}, [0, 1])$ and the evaluation map

$$e: \mathbb{N} \to [0, 1]^C$$

$$n \mapsto (f(n))_{f \in C}$$

so that $\beta\mathbb{N} = \overline{e(\mathbb{N})}$.

Consider the sequence $(x_n)_{n \in \mathbb{N}}$ in $\beta\mathbb{N}$ defined by $x_n = e(n)$. We will show that this sequence has no convergent subsequence in $\beta\mathbb{N}$.

Let $(x_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(x_n)_{n \in \mathbb{N}}$. For any $f \in C$, projection onto the $f^{\text{th}}$ factor $p_f: \beta\mathbb{N} \to [0, 1]$ is continuous. To prove that $(x_{n_k})_{k \in \mathbb{N}}$ does not converge in $\beta\mathbb{N}$, it suffices to find an $f \in C$ such that the sequence of values $p_f(x_{n_k}) = f(n_k)$ does not converge in $[0, 1]$.

Since the indices $(n_k)_{k \in \mathbb{N}}$ are distinct, we can define a map $f: \mathbb{N} \to [0, 1]$ by

$$\begin{cases} 
  f(n_k) = 0 & \text{if } k \text{ is even} \\
  f(n_k) = 1 & \text{if } k \text{ is odd} \\
  f(n) = \text{whatever} & \text{if } n \notin \{n_k \mid k \in \mathbb{N}\}.
\end{cases}$$

Then $f: \mathbb{N} \to [0, 1]$ is automatically continuous since $\mathbb{N}$ is discrete. Moreover, the sequence $(f(n_k))_{k \in \mathbb{N}}$ does not converge in $[0, 1]$. \qed
**Problem 3.** Show that connectedness is a homotopy invariant. In other words, given homotopy equivalent spaces $X$ and $Y$, then $X$ is connected if and only if $Y$ is connected.

**Solution.** By symmetry of the relation of homotopy equivalence, it suffices to prove that $X$ being connected implies that $Y$ is connected.

Let $f : X \to Y$ be a homotopy equivalence, and let $g : Y \to X$ be a homotopy inverse of $f$. Let $y, y' \in Y$ be arbitrary points. We want to show that $y$ and $y'$ lie in the same connected component in $Y$.

Consider $g(y), g(y') \in X$ and recall that $X$ is connected. Thus $f(X) \subseteq Y$ is connected, and contains the points $f(g(y))$ and $f(g(y'))$, which are therefore in the same connected component in $Y$.

Since $f \circ g$ is homotopic to the identity of $Y$, there is a path from $f(g(y))$ to $\text{id}_Y(y) = y$. Therefore $f(g(y))$ and $y$ lie in the same path component in $Y$, in particular in the same connected component, and likewise for $f(g(y'))$ and $y'$. By transitivity:

$$y \sim f(g(y)) \sim f(g(y')) \sim y'$$

it follows that $y$ and $y'$ lie in the same connected component in $Y$. □
Alternate solution. By symmetry of the relation of homotopy equivalence, if suffices to prove that $X$ being connected implies that $Y$ is connected.

Let $f: X \to Y$ be a homotopy equivalence, and let $g: Y \to X$ be a homotopy inverse of $f$. Let $h: Y \to \{0,1\}$ be a continuous map. We want to show that $h$ is constant.

The composite $h \circ f: X \to \{0,1\}$ is continuous and therefore constant, since $X$ is connected. Therefore the composite $h \circ f \circ g: Y \to \{0,1\}$ is constant. But we have a homotopy

$$h \circ f \circ g \simeq h \circ (\text{id}_Y) = h$$

of maps $Y \to \{0,1\}$. Since $\{0,1\}$ is discrete, the only paths in $\{0,1\}$ are constant paths, so that homotopic maps into $\{0,1\}$ must be equal. It follows that $h = h \circ f \circ g$ is constant. \qed
Problem 4. Let \( X \subseteq \mathbb{R}^n \) be the subspace defined by

\[
X = \{ x \in \mathbb{R}^n \mid 7 \leq x_n < 8 \} = \mathbb{R}^{n-1} \times [7,8).
\]

Let \( p_1, p_2, p_3, \ldots \) be a countable collection of non-constant polynomials in \( n \) variables with real coefficients. Show that the subset \( A \subseteq X \) defined by

\[
A = \{ x \in X \mid p_j(x) \neq 0 \text{ for all } j \in \mathbb{N} \}
\]
is dense in \( X \).

Here we denote by \( p_j(x) = p_j(x_1, \ldots, x_n) \in \mathbb{R} \) the value of \( p_j \) at the point \( x = (x_1, \ldots, x_n) \).

Solution. For all \( j \in \mathbb{N} \), consider the subsets \( U_j \subseteq \mathbb{R}^n \) defined by

\[
U_j := \{ x \in \mathbb{R}^n \mid p_j(x) \neq 0 \} = p_j^{-1}(\mathbb{R} \setminus \{0\}).
\]

Then \( U_j \) is open in \( \mathbb{R}^n \), since a polynomial function \( p_j : \mathbb{R}^n \to \mathbb{R} \) is continuous, and \( \mathbb{R} \setminus \{0\} \) is open in \( \mathbb{R} \).

Moreover, \( U_j \) is dense in \( \mathbb{R}^n \). Indeed every non-empty open subset \( U \subseteq \mathbb{R}^n \) contains points \( x \) where \( p_j(x) \neq 0 \) since \( p_j \) is not the constant zero polynomial. (See lemma below.)

By the Baire category theorem, \( \mathbb{R}^n \) is a Baire space, since it is a complete metric space. [Alternatively: since \( \mathbb{R}^n \) is locally compact Hausdorff.]

Therefore the countable intersection \( B := \bigcap_{j \in \mathbb{N}} U_j \) is dense in \( \mathbb{R}^n \).

Therefore \( B \cap (\mathbb{R}^{n-1} \times (7,8)) \) is dense in \( \mathbb{R}^{n-1} \times (7,8) \), since \( \mathbb{R}^{n-1} \times (7,8) \) is open in \( \mathbb{R}^n \).

Therefore \( A = B \cap X \) is dense in \( X \), since \( \mathbb{R}^{n-1} \times (7,8) \) is dense in \( X = \mathbb{R}^{n-1} \times [7,8) \). \[\square\]

Lemma. If a polynomial \( p \) in \( n \) variables with real coefficients vanishes on a non-empty open subset \( U \) of \( \mathbb{R}^n \), then \( p \) is the constant zero polynomial.

Analytic proof. The polynomial function \( p : \mathbb{R}^n \to \mathbb{R} \) is analytic everywhere on \( \mathbb{R}^n \), with infinite radius of convergence. If \( p \) vanishes on the open subset \( U \), then for any point \( x \in U \), the Taylor series of \( p \) at \( x \) is identically zero. Therefore \( p \) is identically zero on \( \mathbb{R}^n \). \[\square\]

Algebraic proof. We proceed by induction on the number of variables \( n \).

For \( n = 1 \), the open subset \( U \subseteq \mathbb{R} \) is infinite, and a polynomial \( p(x_1) \) of one variable \( x_1 \) with infinitely many zeros must be the constant zero polynomial.

Assume that the statement holds for \( n - 1 \) variables. The polynomial function \( p : \mathbb{R}^n \to \mathbb{R} \) vanishes on an “open box” \( \prod_{i=1}^n (a_i, b_i) \subseteq \mathbb{R}^n \). For any fixed value \( z_n \in (a_n, b_n) \), the polynomial function of \( n - 1 \) variables

\[
p(x_1, \ldots, x_{n-1}, z_n)
\]
vanishes on the open box \( \prod_{i=1}^{n-1} (a_i, b_i) \subseteq \mathbb{R}^{n-1} \), and is therefore identically zero, by the induction hypothesis.

Now for any fixed values \( (z_1, \ldots, z_{n-1}) \in \mathbb{R}^{n-1} \), the polynomial function of one variable \( x_n \)

\[
p(z_1, \ldots, z_{n-1}, x_n)
\]
vanishes whenever \( x_n \in (a_n, b_n) \), and therefore vanishes identically. \[\square\]
Problem 5. Let \((X, d)\) be a pseudometric space. Consider the relation \(\sim\) on \(X\) defined by
\[
x \sim y \text{ if } d(x, y) = 0.
\]
Then \(\sim\) is an equivalence relation. (Do not show this.)
On the quotient set \(X/\sim\), the formula
\[
d([x], [y]) := d(x, y)
\]
is well-defined (i.e. does not depend on the choice of representatives) and is a metric. (Do not show this.)

Show that the topology induced by the metric \(d\) on \(X/\sim\) makes \(X/\sim\) into the Kolmogorov quotient \(KQ(X)\), i.e. the quotient space of \(X\) by the relation of topological indistinguishability.

Solution.

1. The relation \(\sim\) is topological indistinguishability. Consider the equivalent conditions for points \(x, y \in X\):
   
   \(x\) and \(y\) are topologically distinguishable, i.e. there is an open subset \(U \subset X\) containing one of \(x, y\) but not the other.
   \[
   \iff \text{There is an open ball } B_r(z) \subset X \text{ centered at one of } x, y \text{ not containing the other point.}
   \]
   \[
   \iff d(x, y) > 0, \text{ i.e. } x \nolimits \sim y.
   \]

2. The metric topology on \(X/\sim\) is the quotient topology from \(X\). Denote by \(\pi: X \to X/\sim\) the quotient function and consider the equivalent conditions for a subset \(V \subseteq X/\sim\):
   
   \(V\) is open in the metric topology.
   \[
   \iff \text{For all } [x] \in V, \text{ there is a radius } r > 0 \text{ satisfying } B_r([x]) \subseteq V, \text{ or equivalently } d([x], [y]) < r \Rightarrow [y] \in V.
   \]
   \[
   \iff \text{For all } x \in \pi^{-1}(V), \text{ there is a radius } r > 0 \text{ satisfying } d(\pi(x), \pi(y)) < r \Rightarrow \pi(y) \in V.
   \]
   \[
   \iff \text{For all } x \in \pi^{-1}(V), \text{ there is a radius } r > 0 \text{ satisfying } d(x, y) < r \Rightarrow y \in \pi^{-1}(V).
   \]
   \[
   \iff \pi^{-1}(V) \text{ is open in } X \text{ (in the pseudometric topology).}
   \]
   \[
   \iff V \text{ is open in the quotient topology on } X/\sim.
   \]
\(\square\)
Alternate solution. We will show that $X/\sim$ with its metric topology, along with the map $\pi: X \to X/\sim$, satisfies the universal property of the Kolmogorov quotient of $X$.

1. $X/\sim$ is $T_0$, since it is a metric space.

2. The map $\pi: X \to X/\sim$ is continuous, since it is Lipschitz continuous (with Lipschitz constant 1).

3. Let $Z$ be a $T_0$ space and $f: X \to Z$ a continuous map. We want to show that there exists a unique continuous map $g: X/\sim \to Z$ satisfying $f = g \circ \pi$, i.e. making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\pi} & X/\sim \\
\downarrow{f} & & \downarrow{g} \\
& Z &
\end{array}
$$

commute.

Let $x, y \in X$ be points satisfying $f(x) \neq f(y)$. Since $Z$ is $T_0$, there is an open subset $U \subset Z$ containing one of $f(x)$ and $f(y)$ but not the other, say WLOG $f(x) \in U$ and $f(y) \notin U$. Since $f$ is continuous at $x$, there is a radius $\delta > 0$ guaranteeing $d(x, x') < \delta \Rightarrow f(x') \in U$. The condition $f(y) \notin U$ implies $d(x, y) \geq \delta$ and in particular $x \sim y$. Taking the contrapositive, $f$ is constant on equivalence classes, i.e. the condition $x \sim y$ implies $f(x) = f(y)$.

Therefore there exists a unique function $g: X/\sim \to Z$ satisfying $f = g \circ \pi$. It remains to show that $g$ is continuous.

Let $[x] \in X/\sim$ and pick a representative $x \in [x]$. Let $U \subseteq Z$ be a neighborhood of $g([x]) = f(x)$. Since $f: X \to Z$ is continuous at $x$, there is a radius $\delta > 0$ guaranteeing $d(x, y) < \delta \Rightarrow f(y) \in U$. Let us check that the same radius $\delta$ works to prove continuity of $g$ at $[x]$.

Let $[y] \in X/\sim$ satisfy $d([x], [y]) < \delta$. For any representative $y \in [y]$, we have $d(x, y) = d([x], [y]) < \delta$ and therefore $f(y) \in U$. But note $f(y) = g([y])$, which yields $g([y]) \in U$. $\square$
Problem 6. Let \( X \) be a \( T_{\frac{3}{2}} \) space, and consider the function space \( C(X, \mathbb{R}) \) endowed with the compact-open topology. If \( C(X, \mathbb{R}) \) is first-countable, show that \( X \) is hemicompact.

Hint: Consider neighborhoods of a constant function \( f \in C(X, \mathbb{R}) \). First show that in any neighborhood basis of \( f \), the neighborhoods can be assumed of the form \( V(K, U) \) for some \( K \subseteq X \) compact and \( U \subseteq \mathbb{R} \) open.

Solution. Consider the constant function \( f \equiv 0 \) on \( X \). Note that the condition \( f \in V(K, U) \), which means \( f(K) \subseteq U \), is equivalent to \( 0 \in U \), i.e. \( U \) is an open neighborhood of \( 0 \) in \( \mathbb{R} \).

Given any basic open neighborhood \( V(K_1, U_1) \cap \ldots \cap V(K_n, U_n) \) of \( f \), we have \( 0 \in U_i \) for all \( i \), so that \( U := U_1 \cap \ldots \cap U_n \) is an open neighborhood of \( 0 \). Taking the compact subspace \( K := K_1 \cup \ldots \cup K_n \) yields
\[
f \in V(K, U) \subseteq V(K_1, U_1) \cap \ldots \cap V(K_n, U_n).
\]

Thus neighborhoods of the form \( V(K, U) \) are a neighborhood basis at \( f \).

Since \( C(X, \mathbb{R}) \) is first-countable, there is a countable neighborhood basis \( \{ B_n \mid n \in \mathbb{N} \} \) at \( f \in C(X, \mathbb{R}) \). By the previous argument, each \( B_n \) can be assumed of the form \( B_n = V(K_n, U_n) \).

We will show that the collection of compact subspaces \( \{ K_n \mid n \in \mathbb{N} \} \) exhibits \( X \) as being hemicompact.

Let \( K \subseteq X \) be compact. We want to show that there is an \( n \in \mathbb{N} \) satisfying \( K \subseteq K_n \). Take \( U = B_1(0) = (-1, 1) \subset \mathbb{R} \) the unit open ball around \( 0 \in \mathbb{R} \). Then \( V(K, U) \) is an open neighborhood of \( f \) in \( C(X, \mathbb{R}) \). Hence there is an \( n \in \mathbb{N} \) satisfying
\[
f \in V(K_n, U_n) \subseteq V(K, U).
\]

Claim: \( K \subseteq K_n \).

Assume to the contrary that there is a point \( x \in K \setminus K_n \). Note that \( K_n \subset X \) is compact and thus closed in \( X \) (since \( X \) is Hausdorff). Since \( X \) is completely regular and \( x \notin K_n \), there is a continuous function \( g: X \to \mathbb{R} \) satisfying \( g|_{K_n} \equiv 0 \) and \( g(x) = 5 \). Such a function satisfies \( g \in V(K_n, U_n) \) but \( g \notin V(K, U) \) since \( x \in K \) and \( g(x) = 5 \notin U \). This contradicts the inclusion \( V(K_n, U_n) \subseteq V(K, U) \). \( \square \)