1 Evaluation map

**Proposition 1.1.** Let $X$ be a locally compact Hausdorff space and $Y$ an arbitrary topological space. Then the evaluation map

$$e: X \times C(X,Y) \to Y$$

given by $e(x,f) = f(x)$ is continuous.

**Remark 1.2.** The evaluation map is not always continuous.

**Exercise 1.3 (Munkres 46.8).** Let $T$ be a topology on the set $C(X,Y)$ making the evaluation map $e: X \times C(X,Y) \to Y$ continuous. Then $T$ contains the compact-open topology.

Now let $X$, $Y$, and $T$ be topological spaces, and let $F(X,Y)$ denote the set of all functions from $X$ to $Y$. There is a natural bijection of sets:

$$\varphi: F(X \times T, Y) \xrightarrow{\sim} F(T, F(X,Y))$$

sending a function $H: X \times T \to Y$ to the function $\varphi(H): T \to F(X,Y)$ defined by $\varphi(H)(t) = H(-,t) =: h_t$.

**Proposition 1.4.** (a) If a function $H: X \times T \to Y$ is continuous, then the following two conditions hold:

1. $h_t: X \to Y$ is continuous for all $t \in T$;
2. The corresponding function $\varphi(H): T \to C(X,Y)$ is continuous.

(b) Assuming $X$ is locally compact Hausdorff, the converse holds as well. In other words, if conditions 1. and 2. hold, then the corresponding function $H: X \times T \to Y$ is continuous.

**Proof.** (a) Homework 13 Problem 2.

(b) Rewriting $H(x,t)$ as

$$H(x,t) = H(-,t)(x)$$

$$= (\varphi(H)(t))(x)$$

$$= e(x, \varphi(H)(t))$$
we see that the function $H: X \times T \to Y$ corresponding to $\varphi(H): T \to C(X,Y)$ is the composite

\[
\begin{array}{ccc}
X \times T & \xrightarrow{H} & Y \\
\downarrow{\text{id}_X \times \varphi(H)} & & \downarrow{e} \\
X \times C(X,Y) & & 
\end{array}
\]

The map $\varphi(H): T \to C(X,Y)$ is continuous by assumption, and so is $\text{id}_X \times \varphi(H)$. Since $X$ is locally compact Hausdorff, the evaluation map $e$ is continuous (by [1.1]), and so is the composite $H = e \circ (\text{id}_X \times \varphi(H))$. 

**Interpretation.** For any spaces $X$, $Y$, $T$, part (a) ensures that the bijection $\varphi$ from (1) restricts to a map

\[
\varphi: C(X \times T, Y) \to C(T, C(X,Y))
\]

sometimes called the adjunction map. This latter $\varphi$ is always injective, since the original $\varphi$ was injective.

However, the latter $\varphi$ is not always surjective. In other words, a family $\{h_t: X \to Y\}_{t \in T}$ of continuous maps that vary continuously in the parameter $t \in T$ do not always yield a map $H: X \times T \to Y$ which is jointly continuous in both arguments.

Part (b) says that if $X$ is nice enough (e.g. locally compact Hausdorff), then $\varphi$ is indeed surjective.

**Remark 1.5.** Proposition [1.4] is useful when trying to show that a map $T \to C(X,Y)$ into a mapping space is continuous. By part (a), it suffices that the corresponding map $X \times T \to Y$ be continuous.