Proposition 0.1. Let $A \subseteq X$ be a connected subspace of a topological space $X$, and $E \subseteq X$ satisfying $A \subseteq E \subseteq \overline{A}$. Then $E$ is connected.

1 Connected components

Definition 1.1. Consider the relation $\sim$ on $X$ defined by $x \sim y$ if there exists a connected subspace $A \subseteq X$ with $x, y \in A$. Then $\sim$ is an equivalence relation, and the equivalence classes are called the connected components of $X$.

Proposition 1.2. 1. Let $Z \subseteq X$ be a connected subspace. Then $Z$ lies entirely within one connected component of $X$.

2. Each connected component $C \subseteq X$ is connected.

3. Each connected component $C \subseteq X$ is closed in $X$.

Remark 1.3. In particular, the connected component $C_x$ of a point $x \in X$ is the largest connected subspace of $X$ that contains $x$.

Exercise 1.4. A topological space $X$ is totally disconnected if its only connected subspaces are singletons $\{x\}$. Show that $X$ is totally disconnected if and only if for all $x \in X$, the connected component $C_x$ of $x$ is the singleton $\{x\}$.

Exercise 1.5. Show that a topological space $X$ is the coproduct of its connected components if and only if the space $X/\sim$ of connected components (with the quotient topology) is discrete.

2 Path-connectedness

Definition 2.1. Let $X$ be a topological space and let $x, y \in X$. A path in $X$ from $x$ to $y$ is a continuous map $\gamma: [a, b] \to X$ satisfying $\gamma(a) = x$ and $\gamma(b) = y$. Here $a, b \in \mathbb{R}$ satisfy $a < b$.

Definition 2.2. A topological space is path-connected is for any $x, y \in X$, there is a path from $x$ to $y$.

Proposition 2.3. Let $X$ be a path-connected space. Then $X$ is connected.

The converse does not hold in general.
Example 2.4 (Topologist’s sine curve). The space

\[ A = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y = \sin \frac{1}{x}\} \subset \mathbb{R}^2 \]

is path-connected, and therefore connected. By Proposition 0.1, its closure

\[ \overline{A} = A \cup \{(0) \times [-1, 1]\} \]

is also connected. However, \( \overline{A} \) is not path-connected.

Proposition 2.5. Let \( f : X \to Y \) be a continuous map, where \( X \) is path-connected. Then \( f(X) \) is path-connected.

3 Path components

Definition 3.1. Consider the relation \( \sim \) on \( X \) defined by \( x \sim y \) if there exists a path from \( x \) to \( y \). Then \( \sim \) is an equivalence relation, and the equivalence classes are called the **path components** of \( X \).

Note that there exists a path \( \gamma : [a, b] \to X \) from \( x \) to \( y \) if and only if there exists a path \( \sigma : [0, 1] \to X \) from \( x \) to \( y \), taking for example

\[ \sigma(t) := \gamma(a + t(b - a)) \, . \]

We will often assume that the domain of parametrization is \([0, 1]\).

**Proof that \( \sim \) is an equivalence relation.**

1. Reflexivity: The constant path \( \gamma : [0, 1] \to X \) defined by \( \gamma(t) = x \) for all \( t \in [0, 1] \) is continuous. This proves \( x \sim x \).

2. Symmetry: Assume \( x \sim y \), i.e. there is a path \( \gamma : [0, 1] \to X \) with endpoints \( \gamma(0) = x \) and \( \gamma(1) = y \). Then \( \tilde{\gamma} : [0, 1] \to X \) defined by

\[ \tilde{\gamma}(t) = \gamma(1 - t) \]

is continuous, since the flip \( t \mapsto 1 - t \) is a homeomorphism of \([0, 1]\) onto itself. Moreover \( \tilde{\gamma} \) has endpoints \( \tilde{\gamma}(0) = \gamma(1) = y \) and \( \tilde{\gamma}(1) = \gamma(0) = x \), which proves \( y \sim x \).

3. Transitivity: Assume \( x \sim y \) and \( y \sim z \), i.e. there are paths \( \alpha, \beta : [0, 1] \to X \) from \( x \) to \( y \) and from \( y \) to \( z \) respectively. Define the **concatenation** of the two paths \( \alpha \) and \( \beta \) as the path going through \( \alpha \) at double speed, followed by \( \beta \) at double speed:

\[ (\alpha \ast \beta)(t) = \begin{cases} 
\alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\
\beta\left(2\left(t - \frac{1}{2}\right)\right) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases} \]

This formula is well defined, because for \( t = \frac{1}{2} \) we have \( \alpha(1) = y = \beta(0) \).

Moreover, \( \alpha \ast \beta \) is continuous, because its restrictions to the closed subsets \([0, \frac{1}{2}]\) and \([\frac{1}{2}, 1]\) are continuous, and we have \([0, 1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]\).

Finally, \( \alpha \ast \beta \) has endpoints \( (\alpha \ast \beta)(0) = \alpha(0) = x \) and \( (\alpha \ast \beta)(1) = \beta(1) = z \), which proves \( x \sim z \).
Example 3.2. Recall the topologist’s sine curve

\[ A = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y = \sin \frac{1}{x}\} \subset \mathbb{R}^2 \]

and its closure

\[ \overline{A} = A \cup \{0\} \times [-1, 1) \]

which is connected, and therefore has only one connected component.

However, \( \overline{A} \) has exactly two path components: the curve \( A \) and the segment \( \{0\} \times [-1, 1] \).

Note that \( A \) is not closed in \( \overline{A} \), so that path components need **NOT** be closed in general, unlike connected components.

**Proposition 3.3.** Each path component of \( X \) is entirely contained within a connected component of \( X \). In other words, each connected component is a (disjoint) union of path components.

**Proof.** If two points \( x \) and \( y \) are connected by a path \( \gamma: [a, b] \to X \), then they are both contained in the connected subspace \( \gamma([a, b]) \subseteq X \). \( \square \)

**Exercise 3.4.** Let \( \{A_i\}_{i \in I} \) be a collection of path-connected subspaces of \( X \) and \( A \subseteq X \) a path-connected subspace satisfying \( A \cap A_i \neq \emptyset \) for all \( i \in I \). Show that the union \( \bigcup_{i \in I} A_i \cup A \) is path-connected.

In particular, if \( A \) and \( B \) are two path-connected subspaces of \( X \) satisfying \( A \cap B \neq \emptyset \), then their union \( A \cup B \) is path-connected.

**Proposition 3.5.**

1. Let \( Z \subseteq X \) be a path-connected subspace. Then \( Z \) lies entirely within one path component of \( X \).

2. Each path component \( C \subseteq X \) is path-connected.

**Remark 3.6.** In particular, the path component \( C_x \) of a point \( x \in X \) is the largest path-connected subspace of \( X \) that contains \( x \).