1 Stone-Čech compactification

Let $X$ be a topological space and consider the set of all continuous functions on $X$ with values in $[0, 1]$  
\[ C := \{ f : X \to [0, 1] \mid f \text{ is continuous} \}. \]

Consider the set $[0, 1]^C \cong \prod_{f \in C} [0, 1]$ of all functions from $C$ to $[0, 1]$, endowed with the product topology. Consider the evaluation map

\[ e : X \to [0, 1]^C \]
\[ x \mapsto (f(x))_{f \in C}. \]

so that $e(x)$ is “evaluation at $x$”.

**Definition 1.1.** The **Stone-Čech construction** on $X$ is the space $\beta X := \overline{e(X)}$ together with the map $e : X \to \beta X$.

Note that $e(X)$ is dense in $\beta X$, and $\beta X$ is always compact Hausdorff. The map $e : X \to \beta X$ is an embedding if and only if $X$ is Tychonoff ($T_{3\frac{1}{2}}$), in which case $e : X \hookrightarrow \beta X$ is a compactification of $X$, called the **Stone-Čech compactification** of $X$.

**Theorem 1.2.** Let $X$ be a topological space. Then the Stone-Čech construction $e : X \to \beta X$ satisfies the following universal property. For any compact Hausdorff space $K$ and any continuous map $f : X \to K$, there is a unique continuous map $g : \beta X \to K$ satisfying $g \circ e = f$, that is, making the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{e} & \beta X \\
\downarrow{f} & & \downarrow{g} \\
K & & \\
\end{array}
\]

commute.

**Remark 1.3.** As usual, because $e : X \to \beta X$ satisfies a universal property, it is unique up to unique isomorphism. More precisely, if $e' : X \to Z$ is a continuous map to a compact Hausdorff space $Z$ that satisfies the universal property of theorem 1.2, then there is a unique
homeomorphism \( h: \beta X \xrightarrow{\cong} Z \) which is compatible with the evaluation maps, meaning \( h \circ e = e' \), i.e. making the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{e} & \beta X \\
\downarrow{e'} & & \downarrow{h} \\
Z & & \\
\end{array}
\]

commute.

“The” Stone-Čech construction may as well refer to any such \( e': X \to Z \). Definition 1.1 provided one specific construction which works.

**Corollary 1.4.** Let \( X \) be a Tychonoff space, and \( f: X \to \mathbb{R} \) a bounded continuous function. Then \( f \) admits a unique continuous extension to \( \beta X \).

**Remark 1.5.** The extension property in 1.4 characterizes the Stone-Čech compactification. If \( e': X \to Z \) is a Hausdorff compactification of \( X \) that satisfies the extension property for bounded real-valued functions, then \( e': X \to Z \) is the Stone-Čech compactification of \( X \).