1 Separation axioms

Definition 1.1. A topological space $X$ is called:

- **$T_0$ or Kolmogorov** if any distinct points are topologically distinguishable: For $x, y \in X$ with $x \neq y$, there is an open subset $U \subset X$ containing one of the two points but not the other.

- **$T_1$** if any distinct points are separated (i.e. not in the closure of the other): For $x, y \in X$ with $x \neq y$, there are open subsets $U_x, U_y \subset X$ satisfying $x \in U_x$ but $y \notin U_x$, whereas $y \in U_y$ but $x \notin U_y$.

- **$T_2$ or Hausdorff** if any distinct points can be separated by neighborhoods: For $x, y \in X$ with $x \neq y$, there are open subsets $U_x, U_y \subset X$ satisfying $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$.

- **regular** if points and closed sets can be separated by neighborhoods: For $x \in X$ and $C \subset X$ closed with $x \notin C$, there are open subsets $U_x, U_C \subset X$ satisfying $x \in U_x$, $C \subset U_C$, and $U_x \cap U_C = \emptyset$.

- **$T_3$** if it is $T_1$ and regular.

- **completely regular** if points and closed sets can be separated by functions: For $x \in X$ and $C \subset X$ closed with $x \notin C$, there is a continuous function $f : X \to [0, 1]$ satisfying $f(x) = 0$ and $f|_C \equiv 1$.

- **$T_{3\frac{1}{2}}$ or Tychonoff** if it is $T_1$ and completely regular.

- **normal** if closed sets can be separated by neighborhoods: For $A, B \subset X$ closed and disjoint, there are open subsets $U, V \subset X$ satisfying $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

- **$T_4$** if it is $T_1$ and normal.

There are implications $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$ as well as $T_{3\frac{1}{2}} \Rightarrow T_3$. By Urysohn’s lemma (see 4.1), the implication $T_4 \Rightarrow T_{3\frac{1}{2}}$ also holds, so that the chain can be written as

$$T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$$

where each implication is strict (i.e. there are counter-examples to the reverse direction).
2 Equivalent characterizations

Proposition 2.1. The following are equivalent.

1. $X$ is $T_1$.
2. Every singleton $\{x\}$ is closed in $X$.
3. For every $x \in X$, we have
   $$\{x\} = \bigcap_{\text{all neighborhoods } N \text{ of } x} N.$$

Proposition 2.2. The following are equivalent.

1. $X$ is $T_2$.
2. The diagonal $\Delta \subseteq X \times X$ is closed in $X \times X$.
3. For every $x \in X$, we have
   $$\{x\} = \bigcap_{\text{closed neighborhoods } C \text{ of } x} C.$$

Proposition 2.3. The following are equivalent.

1. $X$ is regular.
2. For every $x \in X$, any neighborhood of $x$ contains a closed neighborhood of $x$. In other words, closed neighborhoods form a neighborhood basis of $x$.
3. Given $x \in U$ where $U$ is open, there exists an open $V \subseteq X$ satisfying
   $$x \in V \subseteq \overline{V} \subseteq U.$$

Proposition 2.4. The following are equivalent.

1. $X$ is normal.
2. For every $A \subseteq X$ closed, any neighborhood of $A$ contains a closed neighborhood of $A$.
3. Given $A \subseteq U$ where $A$ is closed and $U$ is open, there exists an open $V \subseteq X$ satisfying
   $$A \subseteq V \subseteq \overline{V} \subseteq U.$$

3 A few properties


1. A subspace of a $T_0$ space is $T_0$.
2. A subspace of a $T_1$ space is $T_1$. 

3. A subspace of a $T_2$ space is $T_2$.

4. A subspace of a regular (resp. $T_3$) space is regular (resp. $T_3$).

5. A subspace of a completely regular (resp. $T_{3\frac{1}{2}}$) space is completely regular (resp. $T_{3\frac{1}{2}}$).

6. A CLOSED subspace of a normal (resp. $T_4$) space is normal (resp. $T_4$).

Remark 3.2. A subspace of a normal space need NOT be normal in general.

**Proposition 3.3.** Behavior of (arbitrary) products.

1. A product of $T_0$ spaces is $T_0$.

2. A product of $T_1$ spaces is $T_1$.

3. A product of $T_2$ spaces is $T_2$.

4. A product of regular (resp. $T_3$) spaces is regular (resp. $T_3$).

5. A product of completely regular (resp. $T_{3\frac{1}{2}}$) spaces is completely regular (resp. $T_{3\frac{1}{2}}$).

Remark 3.4. A product of normal spaces need NOT be normal in general, even a finite product.

**Proposition 3.5.** Any compact Hausdorff space is $T_4$. See HW 4 Problem 6.

**Proposition 3.6.** Any metric space is $T_4$ (in fact $T_6$). See HW 6 Problem 3.

4 Urysohn’s lemma

**Theorem 4.1** (Urysohn’s lemma). Let $X$ be a normal space. Then closed subsets of $X$ can be separated by functions: For $A, B \subseteq X$ closed and disjoint, there is a continuous function $f: X \to [0, 1]$ satisfying $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Such a function is sometimes called an **Urysohn function** for $A$ and $B$.

**Proof. Step 1: Construction.**

Since $A$ and $B$ are disjoint, the inclusion $A \subseteq B^c =: U_1$ holds, and note that $A$ is closed and $U_1$ is open.

Since $X$ is normal, there is an open $U_\frac{1}{2}$ satisfying

$$A \subseteq U_\frac{1}{2} \subseteq \overline{U_\frac{1}{2}} \subseteq U_1.$$ 

Consider the inclusion $A \subseteq U_\frac{1}{2}$ where $A$ is closed and $U_\frac{1}{2}$ is open. There is an open $U_\frac{1}{4}$ satisfying

$$A \subseteq U_\frac{1}{4} \subseteq \overline{U_\frac{1}{4}} \subseteq U_\frac{1}{2}.$$ 

Likewise, consider $\overline{U_\frac{1}{2}} \subseteq U_1$ where $\overline{U_\frac{1}{2}}$ is closed and $U_1$ is open. There is an open $U_\frac{3}{4}$ satisfying

$$\overline{U_\frac{1}{2}} \subseteq U_\frac{3}{4} \subseteq \overline{U_\frac{3}{4}} \subseteq U_1.$$ 

Repeating the process, we obtain for every “dyadic rational” $r = \frac{k}{2^n}$ for some $n \geq 0$ and $0 < k \leq 2^n$ an open subset $U_r$ satisfying
• $A \subseteq U_r$ for all $r$;
• $U_r \subseteq U_s$ whenever $r < s$.

In particular we have $U_r \subseteq U_1 = B^c$ for all $r$, i.e. every $U_r$ is disjoint from $B$.

Define the function $f : X \to [0,1]$ by the formula

$$f(x) = \begin{cases} 1 & \text{if } x \text{ belongs to no } U_r \\ \inf \{r \mid x \in U_r\} & \text{otherwise.} \end{cases}$$

Claim: $f$ is an Urysohn function for $A$ and $B$.

**Step 2: Verification.**

First, note that the dyadic rationals in $(0,1]$ are dense in $[0,1]$.

The condition $A \subseteq U_r$ for all $r$ implies $f|_A \equiv 0$.

The condition $B \cap U_r = \emptyset$ for all $r$ implies $f|_B \equiv 1$.

It remains to show that $f$ is continuous. This follows from two facts.

**Fact A:** $x \in U_r \Rightarrow f(x) \leq r$. Indeed, the inclusion $U_r \subseteq U_s$ holds for all $s > r$, and $s$ can be made arbitrarily close to $r$.

**Fact B:** $x \notin U_r \Rightarrow f(x) \geq r$. This is because the set $\{s \mid x \in U_s\}$ is upward closed, and thus cannot contain numbers $q < r$ if $r$ is not in the set. This implies $r \leq \inf \{s \mid x \in U_s\} = f(x)$.

**Continuity where $f = 0$.**

Assume $f(x) = 0$, and let $\epsilon > 0$. Let $r$ be a dyadic rational in $(0, \epsilon)$. Then we have $x \in U_r$ (by fact B) and $f(y) \leq r < \epsilon$ for all $y \in U_r$ (by fact A). Since $U_r$ is a neighborhood of $x$, $f$ is continuous at $x$.

**Continuity where $f = 1$.**

Assume $f(x) = 1$, and let $\epsilon > 0$. Let $r$ be a dyadic rational in $(1 - \epsilon, 1)$. Then we have $x \in U_r^{c}$ (by fact A) and $f(y) \geq r > 1 - \epsilon$ for all $y \in U_r^{c}$ (by fact B). Since $U_r^{c}$ is a neighborhood of $x$, $f$ is continuous at $x$.

**Continuity where $0 < f < 1$.**

Assume $0 < f(x) < 1$, and let $\epsilon > 0$. Take $r, s$ dyadic rationals satisfying

$$f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon.$$

This implies $x \in U_s$ (by fact B) and $x \in U_r^{c}$ (by fact A), in other words $x \in U_s \setminus U_r$, which is a neighborhood of $x$.

Every $y \in U_s$ satisfies $f(y) \leq s$ (by fact A), whereas every $y \in U_r^{c}$ satisfies $f(y) \geq r$ (by fact B), so that the inequality

$$f(x) - \epsilon < r \leq f(y) \leq s < f(x) + \epsilon$$

holds for all $y \in U_s \setminus U_r$. This proves continuity of $f$ at $x$. \(\square\)

**Alternate proof of continuity.** Since intervals of the form $[0, \alpha)$ or $(\alpha, 1]$ form a subbasis for the topology of $[0,1]$, it suffices to show that their preimages $f^{-1}[0, \alpha)$ and $f^{-1}(\alpha, 1]$ are open in $X$.  

4
Consider the equivalent statements:

\[ x \in f^{-1}([0, \alpha)) \iff f(x) < \alpha \]
\[ \iff \text{There is a dyadic rational } r < \alpha \text{ satisfying } x \in U_r \]
\[ \iff x \in \bigcup_{r < \alpha} U_r. \]

This proves the equality

\[ f^{-1}([0, \alpha)) = \bigcup_{r < \alpha} U_r \]

which is open in \( X \) since each \( U_r \) is open.

Likewise, consider the equivalent statements:

\[ x \in f^{-1}(\alpha, 1] \iff f(x) > \alpha \]
\[ \iff \text{There is a dyadic rational } s > \alpha \text{ satisfying } x \notin U_s \]
\[ \iff \text{There is a dyadic rational } r > \alpha \text{ satisfying } x \notin U^c_r \]
\[ \iff x \in \bigcup_{r > \alpha} U^c_r. \]

This proves the equality

\[ f^{-1}(\alpha, 1] = \bigcup_{r > \alpha} U^c_r \]

which is open in \( X \) since each \( U^c_r \) is open.

Remark 4.2. The result is trivially true if either \( A \) or \( B \) is empty, but the proof still works!

Remark 4.3. The Urysohn function need not separate \( A \) and \( B \) precisely. In other words, there can be points \( x \notin A \) where \( f(x) = 0 \) and points \( y \notin B \) where \( f(y) = 1 \).