Math 535 - General Topology
Additional notes

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1 Compactness

1.1 Definitions

Definition 1.1. Let $X$ be a topological space.

- A cover of $X$ is a collection $\{U_\alpha\}_{\alpha \in A}$ of subsets $U_\alpha \subseteq X$ satisfying $X = \bigcup_{\alpha \in A} U_\alpha$.
- An open cover of $X$ is a cover $\{U_\alpha\}_{\alpha \in A}$ where each $U_\alpha$ is open in $X$.
- A subcover of $\{U_\alpha\}_{\alpha \in A}$ is a subcollection $\{U_\beta\}_{\beta \in B}$ (for some $B \subseteq A$) which is still a cover, i.e. $X = \bigcup_{\beta \in B} U_\beta$.

Definition 1.2. A topological space $X$ is compact if for every open cover $\{U_\alpha\}_{\alpha \in A}$ of $X$, there is a finite subcover $\{U_\alpha_1, \ldots, U_\alpha_n\}$, i.e. $X = U_\alpha_1 \cup \ldots \cup U_\alpha_n$.

1.2 Facts about compactness

Proposition 1.3. Let $X$ be a topological space and $Y \subseteq X$ a subspace. Then $Y$ is compact if and only if for every collection $\{U_\alpha\}_{\alpha \in A}$ of open subsets $U_\alpha \subseteq X$ satisfying $Y \subseteq \bigcup_{\alpha \in A} U_\alpha$, there is a finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ satisfying $Y \subseteq U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$.

Proposition 1.4. Let $K_1, \ldots, K_n$ be compact subspaces of $X$. Then their union $K_1 \cup \ldots \cup K_n$ is compact.

Slogan: “Finite union of compact is compact”.

Proposition 1.5. Let $f : X \to Y$ be a continuous map between topological spaces, and assume $X$ is compact. Then $f(X)$ is compact.

Slogan: “Continuous image of compact is compact”.

Remark 1.6. In particular, a quotient of a compact space is always compact.

Proposition 1.7. Let $X$ be a compact topological space and $C \subseteq X$ a closed subspace. Then $C$ is compact.

Slogan: “Closed in compact is compact”.

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Proposition 1.8. Let $X$ be a Hausdorff topological space and $K \subseteq X$ a compact subspace. Then $K$ is closed in $X$.

Slogan: “compact inside Hausdorff is closed”.

Example 1.9. Let $X$ be an anti-discrete space. Then every subspace $Y \subset X$ is compact, though most of them are not closed in $X$ (only the empty set $\emptyset$ and $X$ itself are closed in $X$).

Proposition 1.10. Let $f : X \to Y$ be a continuous map between topological spaces, where $X$ is compact and $Y$ a Hausdorff. Then $f$ is a closed map.

In particular, if $f$ is a continuous bijection, then $f$ is a homeomorphism.

1.3 An important example

A basic example of compact space, yet one of the most important, is provided by the following classic theorem.

Theorem 1.11 (Bolzano-Weierstrass). The interval $[0, 1]$ is compact.

Proof. Suppose $[0, 1]$ is not compact, i.e. there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ which does not admit a finite subcover. Then either $[0, 2 \cdot \frac{1}{2}, 1]$ or $[2, 1]$ (or both) cannot be covered by a finite subcover. Call this new interval $[a_1, b_1]$, where we write $[0, b_0] := [0, 1]$.

Repeating the argument, for every $n \geq 0$, we obtain an interval $[a_n, b_n]$ which cannot be covered by a finite subcover, and each interval has length $b_n - a_n = \frac{1}{2^n}$. Moreover, the intervals are nested (decreasing):

$[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \ldots$

The sequences $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ are monotone and bounded, therefore they converge, say $a_n \to a$ and $b_n \to b$. We have

$$\lim_{n \to \infty} (b_n - a_n) = \lim_{n \to \infty} b_n - \lim_{n \to \infty} a_n$$

$$\lim_{n \to \infty} \frac{1}{2^n} = b - a = 0$$

so that $a = b$. This point $a \in [0, 1]$ is in some $U_{a_0}$, which is open, so we can find some small radius $\epsilon > 0$ such that the open ball $(a - \epsilon, a + \epsilon) \subseteq U_{a_0}$. (To be nitpicky, we should instead write $(a - \epsilon, a + \epsilon) \cap [0, 1]$, which is an open ball in $[0, 1]$.)

By the convergence $a_n \to a$ and $b_n \to a$, for $n$ large enough we have $[a_n, b_n] \subset (a - \epsilon, a + \epsilon) \subseteq U_{a_0}$. These intervals $[a_n, b_n]$ can thus be covered by a finite subcover, namely the collection $\{U_{a_0}\}$ consisting of only one member. This contradicts the construction of $[a_n, b_n]$. □

Remark 1.12. Any closed interval $[a, b] \subset \mathbb{R}$ is homeomorphic to $[0, 1]$ and thus also compact.

Example 1.13. Consider the continuous map

$$f : [0, 2\pi] \to S^1$$

$$t \mapsto (\cos t, \sin t)$$

which induces a continuous map on the quotient

$$\overline{f} : [0, 2\pi]/\sim \to S^1$$

where the equivalence relation $\sim$ identifies the endpoints of the interval, i.e. is generated by $0 \sim 2\pi$. Then $\overline{f}$ is a continuous bijection, the domain $[0, 2\pi]/\sim$ is compact, and $S^1$ is Hausdorff, therefore $\overline{f}$ is a homeomorphism.