1 Quotient spaces

Definition 1.1. Let $X$ be a topological space and $\sim$ an equivalence relation on $X$, along with the canonical projection $\pi: X \to X/\sim$. The quotient topology on $X/\sim$ is the largest topology making $\pi$ continuous.

Explicitly, a subset $U \subseteq X/\sim$ is open if and only if its preimage $\pi^{-1}(U) \subseteq X$ is open in $X$.

Proposition 1.2. With the quotient topology on $X/\sim$, a map $g: X/\sim \to Z$ is continuous if and only if the composite $g \circ \pi: X \to Z$ is continuous.

Proof. Homework 2 Problem 5. \qed

Proposition 1.3. The space $X/\sim$ endowed with the quotient topology satisfies the universal property of a quotient. More precisely, the projection $\pi: X \to X/\sim$ is continuous, and for any continuous map $f: X \to Z$ which is constant on equivalence classes, there is a unique continuous map $\overline{f}: X/\sim \to Z$ such that $f = \overline{f} \circ \pi$, i.e. making the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
\pi \downarrow & & \downarrow \exists \overline{f} \\
X/\sim
\end{array}
\]

commute.

Proof. By the universal property of the projection map in sets, there is a unique function $\overline{f}: X/\sim \to Z$ such that $f = \overline{f} \circ \pi$. It remains to check that $\overline{f}$ is continuous. By proposition 1.2, the fact that $\overline{f} \circ \pi$ is continuous guarantees that $\overline{f}$ is continuous. \qed

Definition 1.4. Let $X$ and $Y$ be topological spaces. A map $q: X \to Y$ is called a quotient map or identification map if it is, up to homeomorphism, of the form $\pi: X \to X/\sim$ where $X/\sim$ is endowed with the quotient topology. More precisely, $q$ is a quotient map if there exists
an equivalence relation $\sim$ on $X$ and a homeomorphism $\varphi: X/\sim \cong Y$ making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{q} & Y \\
\downarrow \pi & & \downarrow \varphi \\
X/\sim & \xrightarrow{\cong} & Y
\end{array}
$$

commute.

Note that the definition implies that $q$ must be continuous and surjective, and that the equivalence relation $\sim$ on $X$ must be the one induced by $q$, namely $x \sim x'$ if and only if $q(x) = q(x')$.

How to recognize quotient maps? In sets, a quotient map is the same as a surjection. However, in topological spaces, being continuous and surjective is not enough to be a quotient map. The crucial property of a quotient map is that open sets $U \subseteq X/\sim$ can be “detected” by looking at their preimage $\pi^{-1}(U) \subseteq X$.

**Proposition 1.5.** Let $q: X \twoheadrightarrow Y$ be a surjective continuous map satisfying that $U \subseteq Y$ is open if and only if its preimage $q^{-1}(U) \subseteq X$ is open. Then $q$ is a quotient map.

**Proof.** Let $\sim$ be the equivalence relation on $X$ induced by $q$, i.e. $x \sim x'$ if and only if $q(x) = q(x')$. By definition, $q: X \twoheadrightarrow Y$ is constant on equivalence classes. By the universal property of the quotient space $X/\sim$, there is a unique continuous map $\overline{q}: X/\sim \twoheadrightarrow Y$ such that $\overline{q} \circ \pi = q$, i.e. making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{q} & Y \\
\downarrow \pi & & \downarrow \overline{q} \\
X/\sim & \xrightarrow{\exists \overline{q}} & Y
\end{array}
$$

commute. By construction, $\overline{q}$ is now bijective. To prove that it is a homeomorphism, it remains to show that it is an open map.

Let $U \subseteq X/\sim$ be open. We want to show that $\overline{q}(U) \subseteq Y$ is open. By assumption, $q$ has the property of “detecting” open subsets of $Y$, i.e. it suffices to check that the preimage $q^{-1}(\overline{q}(U)) \subseteq X$ is open. This preimage is

$$
q^{-1}(\overline{q}(U)) = (\overline{q} \circ \pi)^{-1}(\overline{q}(U))
= \pi^{-1}\overline{q}^{-1}(\overline{q}(U))
= \pi^{-1}(U) \text{ since } \overline{q} \text{ is injective}
$$

which is open in $X$ since $\pi: X \rightarrow X/\sim$ is continuous.