Math 535 - General Topology
Additional notes

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1 Subspaces

Definition 1.1. Let $X$ be a topological space and $A \subseteq X$ any subset. The subspace topology on $A$ is the smallest topology $T_A^{\text{sub}}$ making the inclusion map $i: A \hookrightarrow X$ continuous.

In other words, $T_A^{\text{sub}}$ is generated by subsets $V \subseteq A$ of the form

$$V = i^{-1}(U) = U \cap A$$

for any open $U \subseteq X$.

Proposition 1.2. The subspace topology on $A$ is

$$T_A^{\text{sub}} = \{ V \subseteq A \mid V = U \cap A \text{ for some open } U \subseteq X \}.$$ 

In other words, the collection of subsets of the form $U \cap A$ already forms a topology on $A$.

2 Products

Before discussing the product of spaces, let us review the notion of product of sets.

2.1 Product of sets

Let $X$ and $Y$ be sets. The Cartesian product of $X$ and $Y$ is the set of pairs

$$X \times Y = \{ (x, y) \mid x \in X, y \in Y \}.$$ 

It comes equipped with the two projection maps $p_X: X \times Y \to X$ and $p_Y: X \times Y \to Y$ onto each factor, defined by

$$p_X(x, y) = x$$

$$p_Y(x, y) = y.$$ 

This explicit description of $X \times Y$ is made more meaningful by the following proposition.
Proposition 2.1. The Cartesian product of sets satisfies the following universal property. For any set $Z$ along with maps $f_X: Z \to X$ and $f_Y: Z \to Y$, there is a unique map $f: Z \to X \times Y$ satisfying $p_X \circ f = f_X$ and $p_Y \circ f = f_Y$, in other words making the diagram commute.

Proof. Given $f_X$ and $f_Y$, define $f: Z \to X \times Y$ by

$$f(z) := (f_X(z), f_Y(z))$$

which clearly satisfies $p_X \circ f = f_X$ and $p_Y \circ f = f_Y$.

To prove uniqueness, note that any pair $(x, y) \in X \times Y$ can be written as

$$(x, y) = (p_X(x, y), p_Y(x, y))$$

i.e. the projections give us each individual component of the pair. Therefore, any function $g: Z \to X \times Y$ can be written as

$$g(z) = (p_X(g(z)), p_Y(g(z)))$$

$$= ((p_X \circ g)(z), (p_Y \circ g)(z))$$

so that $g$ is determined by its components $p_X \circ g$ and $p_Y \circ g$. \qed

In slogans: “A map into $X \times Y$ is the same data as a map into $X$ and a map into $Y$”.

Yet another slogan: “$X \times Y$ is the closest set equipped with a map to $X$ and a map to $Y$.”

As usual with universal properties, this characterizes $X \times Y$ up to unique isomorphism. This statement is made precise in the following proposition.

Proposition 2.2. Let $W$ be a set equipped with maps $\pi_X: W \to X$ and $\pi_Y: W \to Y$ satisfying the universal property of the product. Then there is a unique isomorphism $\varphi: W \to X \times Y$ commuting with the projections, i.e. making the diagrams commute.
Proof. Starting from the data of the maps $\pi_X: W \to X$ and $\pi_Y: W \to Y$, the universal property of $X \times Y$ provides a unique map $\varphi: W \to X \times Y$ commuting with the projections.

Likewise, starting from the data of the maps $p_X: X \times Y \to X$ and $p_Y: X \times Y \to Y$, the universal property of $W$ provides a unique map $\psi: X \times Y \to W$ commuting with the projections.

We claim that $\varphi$ is an isomorphism, with inverse $\psi$.

The composite $\psi \circ \varphi: W \to W$ is a map into $W$ commuting with the projections. But so is the identity map $\text{id}_W: W \to W$. By uniqueness (guaranteed in the universal property of $W$), we obtain $\psi \circ \varphi = \text{id}_W$.

Likewise, the composite $\varphi \circ \psi: X \times Y \to X \times Y$ is a map into $X \times Y$ commuting with the projections. But so is the identity map $\text{id}_{X \times Y}: X \times Y \to X \times Y$. By uniqueness (guaranteed in the universal property of $X \times Y$), we obtain $\varphi \circ \psi = \text{id}_{X \times Y}$.

2.2 Product topology

The next goal is to define the product $X \times Y$ of topological spaces $X$ and $Y$ such that it satisfies the analogous universal property in the category of topological spaces.

In other words, we want to find a topology on $X \times Y$ such that the projection maps $p_X: X \times Y \to X$ and $p_Y: X \times Y \to Y$ are continuous, and such that for any topological space $Z$ along with continuous maps $f_X: Z \to X$ and $f_Y: Z \to Y$, there is a unique continuous map $f: Z \to X \times Y$ satisfying $p_X \circ f = f_X$ and $p_Y \circ f = f_Y$.

Definition 2.3. Let $X$ and $Y$ be topological spaces. The product topology $T_{X \times Y}$ on $X \times Y$ is the smallest topology on $X \times Y$ making the projections $p_X: X \times Y \to X$ and $p_Y: X \times Y \to Y$ continuous.

In other words, $T_{X \times Y}$ is generated by “strips” of the form

\[ p_X^{-1}(U) = U \times Y \]
\[ p_Y^{-1}(V) = X \times V \]

for some open $U \subseteq X$ or some open $V \subseteq Y$.

Proposition 2.4. The collection of “rectangles”

\[ \{U \times V \mid U \subseteq X \text{ is open and } V \subseteq Y \text{ is open}\} \]

is a basis for the product topology on $X \times Y$.

Proof. Finite intersections of strips

\[ (U \times Y) \cap (X \times V) = U \times V \]

provide all rectangles. However a finite intersection of rectangles

\[ (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \]

is again a rectangle, since $U_1 \cap U_2 \subseteq X$ is open and $V_1 \cap V_2 \subseteq Y$ is open.

Proposition 2.5. The topological space $(X \times Y, T_{X \times Y})$ along with the projections $p_X: X \times Y \to X$ and $p_Y: X \times Y \to Y$ satisfies the universal property of a product.
Proof. Let $Z$ be a topological space along with continuous maps $f_X: Z \to X$ and $f_Y: Z \to Y$. In particular, these continuous maps are functions, so that there is a unique function $f: Z \to X \times Y$ satisfying $p_X \circ f = f_X$ and $p_Y \circ f = f_Y$. In other words, $f$ is given by

$$f(z) = (f_X(z), f_Y(z)).$$

It remains to check that $f$ is continuous. For any rectangle $U \times V \subseteq X \times Y$ where $U \subseteq X$ is open and $V \subseteq Y$ is open, its preimage is

$$f^{-1}(U \times V) = \{ z \in Z \mid f(z) \in U \times V \} = \{ z \in Z \mid f_X(z) \in U \text{ and } f_Y(z) \in V \} = f_X^{-1}(U) \cap f_Y^{-1}(V).$$

Since $f_X$ and $f_Y$ are continuous, the subsets $f_X^{-1}(U)$ and $f_Y^{-1}(V)$ are open in $Z$, and so is their intersection $f_X^{-1}(U) \cap f_Y^{-1}(V)$. Since those rectangles $U \times V$ form a basis for the product topology on $X \times Y$, the function $f: Z \to X \times Y$ is continuous. $\square$

Remark 2.6. Why did we choose the smallest topology making the projections $p_X$ and $p_Y$ continuous?

If there is a product topology $\mathcal{T}_{X \times Y}$ satisfying the universal property, consider any other topology $\mathcal{T}$ on $X \times Y$ making the projections $p_X$ and $p_Y$ continuous. Then the universal property of $\mathcal{T}_{X \times Y}$ provides a unique continuous map $f$ making the diagram

$$\begin{array}{ccc}
(X \times Y, \mathcal{T}) & \xrightarrow{\exists f} & (X \times Y, \mathcal{T}_{X \times Y}) \\
p_X & | & p_X \\
| & \|
\end{array}$$

$$\begin{array}{ccc}
& & \\
\downarrow p_Y & \searrow f & \downarrow p_Y \\
X & \to & Y \\
p_X & | & p_X \\
\searrow & \nearrow f & \nearrow \\
& Y & \to \\
\end{array}$$

commute. As a function, $f: X \times Y \to X \times Y$ must be the identity:

$$f(x, y) = (p_X(x, y), p_Y(x, y)) = (x, y).$$

The identity $\text{id}: (X \times Y, \mathcal{T}) \to (X \times Y, \mathcal{T}_{X \times Y})$ being continuous means precisely the inequality $\mathcal{T}_{X \times Y} \leq \mathcal{T}$. That is why $\mathcal{T}_{X \times Y}$ had to be the smallest topology making the projections continuous.

Exercise 2.7. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces.

1. For points $(x, y)$ and $(x', y')$ in $X \times Y$, define their distance as the sum

$$d((x, y), (x', y')) := d_X(x, x') + d_Y(y, y').$$

Show that $d$ is a metric on $X \times Y$.

2. Show that the metric $d$ induces the product topology on $X \times Y$. 

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