1 Homeomorphisms

Definition 1.1. A map \( f: X \to Y \) between topological spaces is a \textbf{homeomorphism} if it is continuous, invertible (i.e. bijective), and its inverse \( f^{-1}: Y \to X \) is also continuous.

2 Neighborhoods

Definition 2.1. Let \( X \) be a topological space. A \textbf{neighborhood} of a point \( x \in X \) is a subset \( N \subseteq X \) such that there is an open \( U \) satisfying \( x \in U \subseteq N \).

3 Bases and subbases

Definition 3.1. Let \( (X, \mathcal{T}) \) be a topological space. A \textbf{basis} for the topology \( \mathcal{T} \) of \( X \) is a collection \( \mathcal{B} \) of subsets of \( X \) satisfying

\[
\mathcal{T} = \left\{ \bigcup_{\alpha} B_\alpha \mid B_\alpha \in \mathcal{B} \right\}
\]

i.e. open sets are precisely unions of members of \( \mathcal{B} \).

Exercise 3.2. Let \( X \) be a set. Show that a collection \( \mathcal{B} \) of subsets of \( X \) is a basis for some topology on \( X \) if and only if \( \mathcal{B} \) satisfies the following conditions:

1. \( \mathcal{B} \) covers \( X \), i.e. \( \bigcup_{B \in \mathcal{B}} B = X \).

2. Finite intersections are unions: For any \( B, B' \in \mathcal{B} \), we have \( B \cap B' = \bigcup_{\alpha} B_\alpha \) for some family \( \{B_\alpha\} \) of members of \( \mathcal{B} \).

Definition 3.3. Let \( (X, \mathcal{T}) \) be a topological space. A \textbf{subbasis} for the topology \( \mathcal{T} \) of \( X \) is a collection \( \mathcal{S} \) of subsets of \( X \) satisfying

\[
\mathcal{T} := \left\{ \bigcap_{\alpha} S_{\alpha,i} \mid S_{\alpha,i} \in \mathcal{S} \right\}
\]

i.e. finite intersections of members of \( \mathcal{S} \) form a basis for the topology.
4 Comparing topologies

For a given set $X$, topologies on $X$ can be partially ordered by inclusion.

**Definition 4.1.** Let $X$ be a set, and $\mathcal{T}_1$ and $\mathcal{T}_2$ two topologies on $X$. We say $\mathcal{T}_1$ is smaller than $\mathcal{T}_2$, denoted $\mathcal{T}_1 \leq \mathcal{T}_2$, if the inclusion $\mathcal{T}_1 \subseteq \mathcal{T}_2$ holds, viewed as subsets of the power set $\mathcal{P}(X)$. In other words, every $\mathcal{T}_1$-open is also $\mathcal{T}_2$-open.

One can also say that $\mathcal{T}_2$ is larger than $\mathcal{T}_1$.

Some references say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$, while $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.

**Remark 4.2.** The anti-discrete topology $\mathcal{T}_{\text{anti}} = \{\emptyset, X\}$ is the least element in that partial order, whereas the discrete topology $\mathcal{T}_{\text{dis}} = \mathcal{P}(X)$ is the greatest element. In other words, the inequalities

$$\mathcal{T}_{\text{anti}} \leq \mathcal{T} \leq \mathcal{T}_{\text{dis}}$$

hold for any topology $\mathcal{T}$ on $X$.

**Remark 4.3.** By definition, the inequality $\mathcal{T}_1 \leq \mathcal{T}_2$ holds if and only if the identity function

$$\text{id}: (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$$

is continuous. Note the reversal, mapping “from fine to coarse”.

The poset of topologies on $X$ has arbitrary meets (infima), described explicitly in the following proposition.

**Proposition 4.4.** Let $\{\mathcal{T}_\beta\}$ be a family of topologies on $X$. Then the intersection $\bigcap_\beta \mathcal{T}_\beta$ is a topology on $X$, and therefore the infimum of the family $\{\mathcal{T}_\beta\}$.

**Proof.** Exercise. □

**Remark 4.5.** If we consider an empty family of topologies, then their intersection is

$$\bigcap \mathcal{T}_\beta = \mathcal{P}(X) = \mathcal{T}_{\text{dis}}$$

which is a topology on $X$. Thus the proposition also holds in that case.

**Definition 4.6.** Let $X$ be a set and $\mathcal{S}$ be a collection of subsets of $X$. The topology generated by $\mathcal{S}$ (if it exists) is the smallest topology $\mathcal{T}_\mathcal{S}$ containing $\mathcal{S}$. In other words, it satisfies $\mathcal{S} \subseteq \mathcal{T}_\mathcal{S}$ and for any other topology $\mathcal{T}'$ containing $\mathcal{S}$, we have $\mathcal{T}_\mathcal{S} \leq \mathcal{T}'$.

Note that this universal property makes $\mathcal{T}_\mathcal{S}$ unique, if it exists.

**Proposition 4.7.** For any collection of subsets $\mathcal{S}$, the topology $\mathcal{T}_\mathcal{S}$ exists.

**Proof.** The topology

$$\mathcal{T}_\mathcal{S} = \bigcap_{\text{topologies } \mathcal{T} \text{ such that } \mathcal{S} \subseteq \mathcal{T}} \mathcal{T}$$

has the required properties. □

The following proposition provides an explicit description of $\mathcal{T}_\mathcal{S}$. 

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Proposition 4.8. The topology generated by $\mathcal{S}$ is

$$\mathcal{T}_{\mathcal{S}} = \left\{ \bigcup_{\alpha}^{n_{\alpha}} \bigcap_{i=1}^{n_{\alpha}} S_{\alpha,i} \mid S_{\alpha,i} \in \mathcal{S} \right\}$$

i.e. the topology for which $\mathcal{S}$ is a subbasis.

Proof. Homework 1 Problem 10. \qed