Problem 1. (May § 10.7 Problem 2)

a. Let $f: X \sim Y$ be a weak homotopy equivalence. Assuming $X$ is a CW-complex and $Y$ has the homotopy type of a CW-complex, show that $f$ is a homotopy equivalence.

**Solution.** Let $g: Y \sim K$ be a homotopy equivalence to a CW complex $K$. Then the composite $gf: X \rightarrow K$ is a weak homotopy equivalence, hence a homotopy equivalence by Whitehead. Therefore $f$ is a homotopy equivalence, since $g$ and $gf$ are (2-out-of-3 property).

b. Show that the space $A := \{0\} \cup \{\frac{1}{n} | n \in \mathbb{N}\} \subset \mathbb{R}$ does not have the homotopy type of a CW-complex.

**Solution.** By part (a), it suffices to produce a CW-complex $W$ and a weak homotopy equivalence $f: W \sim A$ which is not a homotopy equivalence.

Consider the discrete space $W = \bigsqcup_{n \leq \mathbb{N} \cup \{0\}} \{x_n\}$ and the obvious bijection $f: W \rightarrow A$, which is continuous since $W$ is discrete. Note that $W$ is a (0-dimensional) CW-complex.

* $f$ is a weak homotopy equivalence. Since both $W$ and $A$ are totally disconnected, $f$ induces a bijection $f_*: \pi_0(W) \rightarrow \pi_0(A)$ and all higher homotopy groups $\pi_n(W)$ and $\pi_n(A)$ are trivial at any basepoint.

* $f$ is not a homotopy equivalence. Let $g: A \rightarrow W$ be any continuous map and consider $g(0) \in W$. Since the singleton $\{g(0)\}$ is open in $W$, there is a neighborhood $U \subseteq A$ of 0 satisfying $g(U) \subseteq \{g(0)\}$. Since $U$ contains more than one point (in fact infinitely many), $g$ is not bijective and therefore does not induce an isomorphism on $\pi_0(A) \rightarrow \pi_0(W)$. □
Problem 2. Consider the “equatorial” embeddings

\[ S^0 \subset S^1 \subset S^2 \subset \ldots \]

of spheres, and define the infinite-dimensional sphere \( S^\infty := \text{colim}_n S^n \). Show that \( S^\infty \) is contractible.

Solution. The embeddings \( S^0 \subset S^1 \subset \ldots \) define the standard CW-structure on \( S^\infty \), with two cells in each dimension \( 0, 1, \ldots \), where the \( n \)-skeleton is \( S^n \). Note that \( S^\infty \) is path-connected (since its 1-skeleton \( S^1 \) is) and its homotopy groups are trivial:

\[ \pi_k(S^\infty) = \text{colim}_n \pi_k(S^n) = 0 \]

as \( \pi_k(S^n) = 0 \) for \( n > k \). Therefore the inclusion \(* \hookrightarrow S^\infty\) is a weak homotopy equivalence, and thus a homotopy equivalence, by Whitehead.

Remark. More is true: The inclusion \(* \hookrightarrow S^\infty\) of one of the two points in \( S^0 \) is the inclusion of a subcomplex of \( S^\infty \), and thus a strong deformation retract.
Problem 3. (Hatcher § 4.1 Exercise 14 and more)

a. Let $X$ and $Y$ be homotopy equivalent spaces. Assuming that $X$ and $Y$ admit CW-structures without $(n + 1)$-cells (for some $n \geq 0$), show that the $n$-skeleta $X_n$ and $Y_n$ are homotopy equivalent.

Solution. Let $f: X \xrightarrow{\sim} Y$ be a homotopy equivalence with homotopy inverse $g: Y \xrightarrow{\sim} X$. By cellular approximation, WLOG $f$ and $g$ are cellular maps. Consider their restrictions to the $n$-skeleta $f|_{X_n}: X_n \rightarrow Y_n$ and $g|_{Y_n}: Y_n \rightarrow X_n$.

Let $H: X \times I \rightarrow X$ be a homotopy from $gf$ to $\text{id}_X$. By cellular approximation, $H$ is homotopic rel $X \times \partial I$ to a cellular map $H': X \times I \rightarrow X$, which in particular restricts to

$$H'|_{X_n \times I}: X_n \times I \rightarrow X_{n+1} = X_n$$

using the fact that $X$ has no $(n+1)$-cells. Thus the composite $g|_{Y_n} \circ f|_{X_n}: X_n \rightarrow X_n$ is homotopic to the identity. Likewise, since $Y$ has no $(n+1)$-cells, the composite $f|_{X_n} \circ g|_{Y_n}: Y_n \rightarrow Y_n$ is homotopic to the identity. \qed

b. Find an example of homotopy equivalent spaces $X$ and $Y$, and CW-structures on $X$ and $Y$ such that for all $n \geq 0$, the $n$-skeleta $X_n$ and $Y_n$ are not homotopy equivalent.

Solution. Take the space $*$ with a single 0-cell, and the infinite-dimensional sphere $S^\infty$. By problem 2, the inclusion $* \hookrightarrow S^\infty$ is a homotopy equivalence. However, their respective $n$-skeleta $(*)_n = *$ and $(S^\infty)_n = S^n$ are not homotopy equivalent, for any $n \geq 0$. \qed
Problem 4. (Hatcher § 4.1 Exercise 16 and more)

a. Let \((X, x_0)\) be a pointed space. Show that the summand inclusion \(\iota: X \hookrightarrow X \vee S^n\) induces isomorphisms on homotopy groups \(\pi_i\) (based at any point) for all \(i < n\).

**Solution.** Since \(\iota: X \hookrightarrow X \vee S^n\) is obtained by attaching an \(n\)-cell (via the constant attaching map \(S^{n-1} \rightarrow X\) to the basepoint), it is an \((n - 1)\)-connected map. Moreover, \(\iota\) admits a retraction \(X \vee S^n \twoheadrightarrow X\) sending the second summand \(S^n\) to the basepoint. Therefore the induced map \(\iota_*: \pi_i(X) \rightarrow \pi_i(X \vee S^n)\) is injective for all \(i\), in particular for \(i = n - 1\). \(\square\)
b. Let $X$ and $Y$ be connected CW-complexes. Show that any map $f: X \to Y$ factors as a composite $X \xrightarrow{g} Z \xrightarrow{h} Y$ where $g: X \to Z$ induces isomorphisms on $\pi_i$ for $i \leq n$ and $h: Z \to Y$ induces isomorphisms on $\pi_i$ for $i \geq n + 1$.

**Solution.** Start with the factorization $X \xrightarrow{id} X \xrightarrow{f} Y$. Clearly the first map $\text{id}: X \to X$ induces isomorphisms on homotopy groups $\pi_i$ for $i \leq n$. The second map $f: X \to Y$ need not be surjective on $\pi_{n+1}$. Attach $(n+1)$-cells to $X$ as follows:

$$X \leftarrow \xrightarrow{\eta_{(1)}} X \vee \bigvee_{\alpha \in \pi_{n+1}(Y)} S^{n+1} \xrightarrow{(f, \eta_{(1)})} Y$$

where $\eta_{(1)}: S^{n+1} \to Y$ is a chosen representative of the class $\alpha \in \pi_{n+1}(Y)$. Rename the middle term $\eta_{(1)}$. By part (a), the map $\eta_{(1)}: X \to Z_{(1)}$ still induces isomorphisms on $\pi_i$ for $i \leq n$. By construction, the map $h_{(1)}: Z_{(1)} \to Y$ induces a surjection on $\pi_{n+1}$.

**Inductive step.** For $k \geq 1$, assume we have a factorization of $f$

$$X \xrightarrow{g_{(k)}} Z_{(k)} \xrightarrow{h_{(k)}} Y$$

such that $g_{(k)}$ induces isomorphisms on $\pi_i$ for $i \leq n$ and $h_{(k)}: Z_{(k)} \to Y$ induces isomorphisms on $\pi_i$ for $n < i < n + k$ and a surjection on $\pi_{n+k}$.

**Injectivity on $\pi_{n+k}$.** For each $\alpha \in \ker (h_{(k)*}) = \pi_{n+k}(Z_{(k)}) \to \pi_{n+k}(Y))$, pick a representative $\theta_{\alpha}: S^{n+k} \to Z_{(k)}$ and a null-homotopy $H_{\alpha}: D^{n+k+1} \to Y$ of $h_{(k)}\theta_{\alpha}$. Attach a $(n + k + 1)$-cell to $Z_{(k)}$ via the attaching map $\theta_{\alpha}$ and map this new cell to $Y$ via $H_{\alpha}$.

**Surjectivity on $\pi_{n+k+1}$.** For each $\alpha \in \pi_{n+k+1}(Y)$, pick a representative $\theta_{\alpha}: S^{n+k+1} \to Y$ and attach a $(n + k + 1)$-cell to $Z_{(k)}$ via the constant attaching map, and map this new cell to $Y$ via $\theta_{\alpha}$.

After all those cell attachments, we obtain a new factorization of $f$

$$X \xrightarrow{g_{(k+1)}} Z_{(k+1)} \xrightarrow{h_{(k+1)}} Y$$

where $g_{(k+1)}$ still induces isomorphisms on $\pi_i$ for $i \leq n$, since attaching $(n + k + 1)$-cells does not affect homotopy groups below dimension $n + k$. Moreover, $h_{(k+1)}$ still induces isomorphisms on $\pi_i$ for $n < i < n + k$ and now induces an isomorphism on $\pi_{n+k}$ and a surjection on $\pi_{n+k+1}$.

Repeating this process inductively, we obtain a factorization of $f$

$$X \xrightarrow{g} Z \xrightarrow{h} Y$$

with the desired properties. □