Problem 1. An H-space (named after Hopf) is a pointed space \((X, e)\) equipped with a “multiplication” map \(\mu: X \times X \to X\) such that the basepoint \(e\) is a two-sided unit up to pointed homotopy. In other words, both maps
\[
\mu(e, -): X \to X
\]
\[
\mu(-, e): X \to X
\]
and pointed-homotopic to the identity map \(\text{id}_X\). Note that \(\mu\) is not assumed to be associative, not even up to homotopy.

Show that the fundamental group \(\pi_1(X, e)\) of an H-space is abelian.

Solution. Let us show a stronger result: The \(\pi_1(X, e)\)-action on \(\pi_n(X, e)\) is trivial for all \(n \geq 1\).

Let \([\gamma] \in \pi_1(X, e)\) and \([\theta] \in \pi_n(X, e)\) be represented by pointed maps \(\gamma: S^1 \to X\) and \(\theta: S^n \to X\) (by abuse of notation), or equivalently, maps of pairs \(\gamma: (I, \partial I) \to (X, e)\) and \(\theta: (D^n, \partial D^n) \to (X, e)\).

Consider the continuous map \(H: D^n \times I \to X\) defined by
\[
H(z, s) = \mu(\theta(z), \gamma(s)).
\]
Then \(H\) restricted to the bottom face is
\[
H|_{D^n \times \{0\}} = \theta e
\]
whereas \(H\) restricted to the remaining faces is
\[
H|_{\partial D^n \times I \cup D^n \times \{1\}} = (e\gamma) \cdot (\theta e)
\]
so that \(H\) provides a pointed homotopy between \(\theta e\) and \((e\gamma) \cdot (\theta e)\).

Because right multiplication by \(e\) is pointed-homotopic to the identity of \(X\), the composite
\[
S^n \xrightarrow{\theta} X \xrightarrow{\mu(-, e)} X
\]
\[
\theta e
\]
is pointed-homotopic to \(\theta\), yielding the equality \([\theta e] = [\theta]\) in \(\pi_n(X, e)\). Likewise, the equality \([e\gamma] = [\gamma]\) holds in \(\pi_1(X, e)\). We obtain the equality
\[
[\gamma] \cdot [\theta] = [e\gamma] \cdot [\theta e]
\]
\[
= [\theta e]
\]
\[
= [\theta]
\]
in \(\pi_n(X, e)\).
Problem 2. Let $f : X \to Y$ be a map of spaces, and $x \in X$ any basepoint. Show that the induced map

$$\pi_n f : \pi_n(X, x) \to \pi_n(Y, f(x))$$

for $n \geq 1$ is a map of $\pi_1$-modules, in the sense that it is $\pi_1 f$-equivariant. More precisely, for any $\gamma \in \pi_1(X, x)$ and $\theta \in \pi_n(X, x)$ the equation

$$(\pi_n f)(\gamma \cdot \theta) = (\pi_1 f)(\gamma) \cdot (\pi_n f)(\theta)$$

holds in $\pi_n(Y, f(x))$.

Solution. The equation to be proved can be written as the commutative diagram

$$\pi_1(X, x) \times \pi_n(X, x) \xrightarrow{\bullet} \pi_n(X, x)$$

$\pi_1 f \times \pi_n f \downarrow \quad \pi_n f \downarrow$

$$\pi_1(Y, f(x)) \times \pi_n(Y, f(x)) \xrightarrow{\bullet} \pi_n(Y, f(x))$$

(1)

Recall that the action map $\pi_1(X, x) \times \pi_n(X, x) \xrightarrow{\bullet} \pi_n(X, x)$ is obtained by applying the functor $[-, X]_* : \mathbf{Top}_* \to \mathbf{Set}_*$ to the coaction map $c : S^n \to S^1 \vee S^n$.

The map $f : (X, x) \to (Y, f(x))$ in $\mathbf{Top}_*$ yields the postcomposition natural transformation $f_* : [-, X]_* \to [-, Y]_*$. Applying $f_*$ to the coaction map $c$ yields the commutative right-hand square of the diagram

$$\begin{array}{ccc}
[S^1, (X, x)]_* & \xrightarrow{\cong} & [S^1 \vee S^n, (X, x)]_* \\
| f_\ast \times f_\ast | & \downarrow f_\ast & \downarrow f_\ast \\
[S^1, (Y, f(x))]_* & \xrightarrow{\cong} & [S^1 \vee S^n, (Y, f(x))]_*
\end{array}$$

(2)

$$\begin{array}{ccc}
[S^n, (X, x)]_* & \xrightarrow{\cong} & [S^n, (Y, f(x))]_* \\
| c_\ast | & \downarrow f_\ast & \downarrow f_\ast \\
[S^n, (X, x)]_* & \xrightarrow{\cong} & [S^n, (Y, f(x))]_*
\end{array}$$

where the left-hand square also commutes, since the wedge is the coproduct in $\mathbf{Top}_*$ and in $\text{Ho}(\mathbf{Top}_*)$. But the outer diagram in (2) is precisely the diagram (1).
Problem 3. Let $X$ be the topologist’s sine curve:

$$X = \{0\} \times [-1, 1] \cup \{(x, \sin \frac{1}{x}) \mid 0 < x \leq 1\}.$$ 

Consider the map $f: S^0 \to X$ which picks out the points $(0, 1)$ and $(1, \sin 1)$. Show that this map $f$ is a weak homotopy equivalence but not a homotopy equivalence.

Solution. Write $A = \{0\} \times [-1, 1]$ and $B = \{(x, \sin \frac{1}{x}) \mid 0 < x \leq 1\}$ with $X = A \cup B$ where the union is disjoint. Recall that $X$ is connected (being the closure of the connected subset $B \subset \mathbb{R}^2$), but not path-connected. The two path components of $X$ are $A$ and $B$.

$f$ is a weak homotopy equivalence. Write $a := (0, 1) \in A$ and $b := (1, \sin 1) \in B$, and $S^0 = \{*, a, *_b\}$, with $f(\ast) = a$ and $f(\ast_b) = b$. The map $f: S^0 \to X$ induces a bijection on the sets of path components $\pi_0 f: \pi_0(S^0) \xrightarrow{\sim} \pi_0(X) = \{[a], [b]\}$.

Since $S^n$ is path-connected for all $n \geq 1$, any pointed map $\alpha: S^n \to (X, a)$ lands inside the path component $A \subseteq X$. But $A$ is contractible, so that $\pi_n(A, a) = 0$ and $\alpha: S^n \to (A, a)$ is pointed-null-homotopic. This proves $\pi_n(X, a) = 0$. Therefore $\pi_n f: \pi_n(S^0, \ast) \xrightarrow{\sim} \pi_n(X, a)$ is an isomorphism (between trivial groups!) for all $n \geq 1$.

Likewise, $B$ is contractible, so that $\pi_n f: \pi_n(S^0, \ast_b) \xrightarrow{\sim} \pi_n(X, b)$ is also an isomorphism (between trivial groups) for all $n \geq 1$.

$f$ is not a homotopy equivalence. Let $g: X \to S^0$ be any continuous map. Since $X$ is connected, the image $g(X)$ is connected and is therefore a singleton $\{a\}$ or $\{b\}$. Hence $g$ cannot induce a bijection on $\pi_0$, and thus $f$ has no homotopy inverse. \qed