1 Group objects

**Definition 1.1.** Let $\mathcal{C}$ be a category with finite products, including a terminal object $1$. A **group object** in $\mathcal{C}$ is an object $G$ of $\mathcal{C}$ together with structure maps

- $\mu: G \times G \to G$ “multiplication”
- $e: 1 \to G$ “unit”
- $i: G \to G$ “inverse”

such that the following diagrams commute:

- **Associativity**
  \[
  
  \begin{array}{ccc}
  G \times G \times G & \xrightarrow{id \times \mu} & G \times G \\
  \mu \times \text{id} & \downarrow & \downarrow \mu \\
  G \times G & \xrightarrow{\mu} & G
  \end{array}
  \]

- **Left unit**
  \[
  
  \begin{array}{ccc}
  1 \times G & \xrightarrow{e \times \text{id}} & G \times G \\
  \cong & \downarrow \mu & \downarrow \mu \\
  & G & G
  \end{array}
  \]

- **Right unit**
  \[
  
  \begin{array}{ccc}
  G \times 1 & \xrightarrow{id \times e} & G \times G \\
  \cong & \downarrow \mu & \downarrow \mu \\
  & G & G
  \end{array}
  \]

- **Left inverse**
  \[
  
  \begin{array}{ccc}
  G & \xrightarrow{(i, \text{id})} & G \times G \\
  e_G & \downarrow \mu & \downarrow \mu \\
  & G & G
  \end{array}
  \]
where $e_G : G \to G$ is the composite $X \to 1 \xrightarrow{\varepsilon} X$.

**Example 1.2.** A group object in the category **Set** is just a group.

**Notation 1.3.** The category of group objects in $\mathcal{C}$ is denoted $\mathbf{Gp}(\mathcal{C})$. Morphisms of group objects are morphisms in $\mathcal{C}$ that commute with the structure maps.

There is the forgetful functor $U : \mathbf{Gp}(\mathcal{C}) \to \mathcal{C}$ which remembers the underlying object but forgets the structure maps.

**Proposition 1.4.** Let $\mathcal{C}$ be a locally small category with finite products, including a terminal object. Let $G$ be a group object in $\mathcal{C}$. Then for any object $X$ of $\mathcal{C}$, the hom-set $\text{Hom}_\mathcal{C}(X, G)$ is naturally a group.

In other words, the structure maps of $G$ induce a group structure on $\text{Hom}_\mathcal{C}(X, G)$, and this assignment $\text{Hom}_\mathcal{C}(\cdot, G) : \mathcal{C}^{\text{op}} \to \mathbf{Gp}$ is a functor.

**Proof.** Homework 1 Problem 2. \qed

**Remark 1.5.** Several authors **define** a group object in an arbitrary locally small category $\mathcal{C}$ as an object $G$ of $\mathcal{C}$ together with a lift of the functor $\text{Hom}_\mathcal{C}(\cdot, G) : \mathcal{C}^{\text{op}} \to \textbf{Set}$ to groups, as illustrated in the diagram

$$
\begin{array}{ccc}
\mathbf{Gp} & \xrightarrow{U} & \mathbf{Set} \\
\downarrow \text{Hom}_\mathcal{C}(\cdot, G) & & \\
\mathcal{C}^{\text{op}} & \xleftarrow{U} & \text{Set}.
\end{array}
$$

This definition becomes equivalent to Definition 1.1 when $\mathcal{C}$ has finite limits.

## 2 Cogroup objects

**Definition 2.1.** Let $\mathcal{C}$ be a category with finite coproducts, including an initial object $\emptyset$. A **cogroup object** in $\mathcal{C}$ is a group object in the opposite category $\mathcal{C}^{\text{op}}$.

More explicitly, it consists of an object $C$ of $\mathcal{C}$ equipped with a comultiplication $C \to C \amalg C$, counit $C \to \emptyset$, and coinverse $C \to C$, satisfying coassociativity, etc.

**Example 2.2.** The only cogroup object in **Set** (or in **Top**) is the empty set $\emptyset$, because it is the only object $C$ admitting a map $C \to \emptyset$ to the empty set, which is the initial object.
Definition 2.3. A homotopy group object in $\mathcal{C} = \text{Top}$ or $\text{Top}_*$ (or any category with a good notion of homotopy between maps) is defined like a group object, except that the diagrams are only required to commute up to homotopy.

In particular, a homotopy group object in $\mathcal{C}$ becomes a group object in the homotopy category $\text{Ho}(\mathcal{C})$.

A homotopy cogroup object in $\mathcal{C}$ is defined similarly.