This is a practice exam. The real exam will consist of at most 4 problems. 
In the real exam, no calculators, electronic devices, books, or notes may be used.
Show your work. No credit for answers without justification.
You may omit units for ease of reading, except in final answers which require units.
Good luck!

1. _______/15
2. _______/15
3. _______/10
4. _______/10
5. _______/10
6. _______/10

Total: _______/70
Problem 1a. (12 pts) A car is moving at constant speed $v$ on a road which becomes bumpy. The equation of the surface is $y(s) = a \sin\left(\frac{2\pi s}{L}\right) = 0.1 \sin\left(\frac{2\pi s}{6}\right)$ where $a = 0.1$ m is the amplitude of the bumps and $L = 6$ m is their wavelength (figure 1).

Assume the car behaves like a mass of 1,000 kg on a spring with spring constant 100,000 N/m, without shock absorber. We want to know how the bumpy road will affect the car. Let $x(t)$ be the upward displacement of the car, relative to its static equilibrium position – i.e. the equilibrium position once the spring has been compressed by the weight of the car.

Find $x(t)$, assuming the car was always at its vertical equilibrium position before reaching the first bump (at time $t = 0$). Assume a generic value of the speed $v$. Hint: The spring is stretched by $x - y$ and the spring force is the only force to take into account.
b. (3 pts) At what speed $v$ do resonance vibrations occur?
Problem 2a. (9 pts) A mass-spring-dashpot system with mass 1 kg, spring constant 4.5 N/m, and damping coefficient 1 N/(m/s) is subject to an external force $F(t) = 10 \cos \omega t$. Find the steady periodic solution $x_{sp}(t)$. 
b. (6 pts) Determine whether practical resonance can occur. If so, at what frequency (in Hertz), and what is the (maximal) amplitude of the steady periodic solution then?
Problem 3a. (2 pts)  Check that $y_1(x) = x + 1$ and $y_2(x) = e^x$ are solutions of the equation:

$$xy'' - (x + 1)y' + y = 0.$$ 

b. (7 pts)  Find a particular solution $y_p(x)$ of the equation:

$$xy'' - (x + 1)y' + y = x^2.$$ 

c. (1 pt)  Find the general solution of the equation in (b).
Problem 4a. (8 pts) Find a periodic solution of the equation $x'' + 5x = f(t)$ where $f(t) = t$ on the interval $-\pi < t < \pi$ and $f$ is $2\pi$-periodic. You can express the answer in exact form or as a Fourier series – or both, for practice.

b. (2 pts) Find all possible initial conditions $x(0), x'(0)$ such that the solution to the above equation is periodic. Explain.
Problem 5. (10 pts) A piece with mass $m$ inside a mechanical device is moving along an axis and is subject to a force $f(t)$, where $f(t) = 23|t|$ (in Newtons) for $-2 \leq t \leq 2$ and $f$ is periodic with period 4 s. The piece is also constrained by what is effectively a spring with spring constant 10 N/m. Friction is negligible.

Find all values of the mass $m$ such that the piece could enter into resonance.
Problem 6. (10 pts) Find all eigenvalues and corresponding eigenfunctions $y(x)$ of the problem:

\[ y'' + \lambda y = 0 \]
\[ y'(-1) = 0, \ y'(1) = 0. \]